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Sub Title	
Author	LEE, Nae-Chan
Publisher	Keio Economic Society, Keio University
Publication year	1996
Jtitle	Keio economic studies Vol.33, No.1 (1996.) ,p.125- 132
JaLC DOI	
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Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19960001-0125

KEIO ECONOMIC STUDIES 33 (1), 125–132 (1996)

A BARGAINING APPROACH TO BANKRUPTCY PROBLEMS: EQUIVALENCY BETWEEN THE ASYMMETRIC NASH SOLUTION AND THE CONSTRAINED WEIGHTED AWARD SOLUTION

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First version received November 1994; final version accepted Octobeer 1995

Abstract: This paper generalizes the constrained equal award rule (or, solution), one of well-known allocation rules of a bankruptcy problem so that differences of relative importance over agents can be reflected and shows that the asymmetric Nash solution coincides with such a generalized solution called the constrained weighted award solution over a bargaining problem with the **0** disagreement point derived from a bankruptcy problem. This result is further extended to cover those problems with the non-zero disagreement point.

1. INTRODUCTION

Bankruptcy problems treat the issues of how to allocate E, the fixed amount of a perfectly divisible good to agents in accordance with their claims over the good. The most well-knowns are the proportional rule (w.r.t. claims) and the constrained equal award rule (for short, the CEA rule). The former divides the good according to relative weights generated by the claims, and the latter disemburses E in such a way that an agent receives his claim if it is less than a threshold parameter and otherwise, the threshold itself, where the parameter is so determined that the resulting allocation satiates E. On the other hand, bargaining problems deal with solution concepts resulting from negotiations among agents over a set of utility tuples satisfying some regular conditions with the disagreement point, starting from Nash's seminal paper (1950) in which what has now become known as the Nash solution is suggested.[1] There are two approaches to a bankruptcy problem. One is a coalitional approach and the other is a bargaining approach. O'Neill (1982), Aumann and Maschler (1985), Curiel, Maschler and Tijs (1988) and Driessen (1988) belong to the first category, and Dagan and Volij

Acknowledgement. The author is grateful to refree's helpful comments. Errors and shortcomings are the responsibility of the author.

[1] For an extensive survey on bargaining problems see Thomson (1992).

(1993) to the second. This paper follows the second line.

Dagan and Volij derived a specific bargaining problem with the 0-normalized disagreement point from each bankruptcy problem and probed relations between some bargaining solutions and allocation rules, among which our interest is on the inducement of the CEA rule allocation by the Nash solution. This result is generalized in the following fashion: First, unlike Dagan and Volij we interprete the CEA rule allocation as a bargaining solution. Secondly, the CEA solution is generalized so that different evalutions over agents may be considered, which we call the constrained weighted award solution (for short, the CWA solution) and it will be shown that the asymmetric Nash solution, the usual Nash solution skewed by evaluations, coincides with the CWA solution. Finally, the CWA solution is further generalized so as to incorporate the non-zero disagreement point and we show that the above result still remains effective.

Section 2 introduces the CWA rule of bankruptcy problems with some of its properties in comparison with axioms of the CEA rule. In section 3 we show that over a specific bargaining problem derived from any bankruptcy game with the $\mathbf{0}$ disagreement point the asymmetric Nash solution is equivalent to the CWA solution, of which the result can be extended to the case of the non-zero disagreement point.

2. BANKRUPTCY PROBLEMS AND THE CWA RULE

Let $N=\{1, 2, \cdots, n\}$ be a set of agents. E denotes the amount of a perfectly divisible good and $\mathbf{c}\in R^N_+$, the *claims vector* of agents. A pair $(E, \mathbf{c})$ is called a *bankruptcy problem* if $0\leq E\leq \mathbf{c}(N)$ where $\mathscr{C}$ is a set of such problems.[2] A vector $\mathbf{x}\in R^N$ is called an *allocation* if it satisfies $\mathbf{0}\leq \mathbf{x}\leq \mathbf{c}$, $\mathbf{x}(N)=E$.[3] An *allocation rule*, or simply a rule is a mapping which associates each bankruptcy problem with a unique allocation. For any vectors $\mathbf{x}, \mathbf{y}\in R^N$, let $\mathbf{x}\wedge\mathbf{y}$ be an n-dimensional vector defined by $(\mathbf{x}\wedge\mathbf{y})_i=\min\{x_i, y_i\}$, $i\in N$. Also, an *evaluation vector* is denoted by $\boldsymbol{\alpha}=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ where $\mathbf{0}\ll\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}(N)=1$, i.e., $\boldsymbol{\alpha}$ is an element of $\mathring{\Delta}^N$, the interior of the $n-1$-dimensional simplex. Note that α_i is the weight of relative importance over agent i given exogenously. For any evaluation vector $\alpha\in\mathring{\Delta}^N$, a rule ϕ^α is called the *CWA rule* if the following condition is satisfied:

$$\phi^\alpha_i(E, \mathbf{c})=c_i\wedge t^*\alpha_i, \qquad i\in N \tag{1}$$

where $t=t^*$ is a solution of $(\mathbf{c}\wedge t\boldsymbol{\alpha})(N)=E$.[4]

[2] $\mathbf{c}(N)=\sum_{i\in N} c_i$.

[3] For any vectors $\mathbf{x}, \mathbf{x}'\in R^N$, $\mathbf{x}'\leq\mathbf{x}$ implies $x'_i\leq x_i$ for any $i\in N$; $\mathbf{x}'<\mathbf{x}$ if $\mathbf{x}'\leq\mathbf{x}$ and $\mathbf{x}\neq\mathbf{x}'$; $\mathbf{x}'\ll\mathbf{x}$ if $x'_i<x_i$ for any $i\in N$. Note that for any bankruptcy problem $(E, c)\in\mathscr{C}$, if $0<E$, then there exists at least one allocation.

[4] Since t^* is non-negative, the vector $\mathbf{c}\wedge t^*\boldsymbol{\alpha}$ is an allocation. Also, note that if $E<\mathbf{c}(N)$, then t^* is unique and so does the CWA rule allocation. Otherwise, any t greater than t^* is a solution where $t^*\alpha_i=c_{i^*}$ and $i^*=\operatorname{argmax}_{i\in N} c_i$. In this case we regard $t^*=c_{i^*}/\alpha_{i^*}$ as a solution.

Note that t^* can be interpreted as a common upper bound and $t^*\alpha_i$, agent i's individual upper bound which plays a role of the threshold when E is disembursed among agents. I.e., the CWA rule divides E in such a way that an agent whose claim exceeds the inidividual threshold receives the amount of the good equivalent to it and otherwise, the amount what he claims is awarded to him. If the evaluation vector is an equi-length vector, i.e., all agents are treated equally, it is called the *CEA rule.*[5] The CEA rule, from a view point of a bankruptcy problem, is characterized by Dagan (1994) with such axioms as follows:

- Axiom ST (*Symmetric Treatment*): For any bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$ if $c_i = c_j$, then $\phi_i(E, \mathbf{c}) = \phi_j(E, \mathbf{c})$.
- Axiom IIC (*Independence of Irrelevant Claims*): For any bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$, $\phi(E, \mathbf{c}) = \phi(E, c \wedge E1)$ where $1 \in R^N$ is a unit equi-length vector.
- Axiom C (*Composition*): For any bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$ and any $0 \leq E' \leq E$, $\phi(E, \mathbf{c}) = \phi(E', \mathbf{c}) + \phi(E - E', \mathbf{c} - \phi(E', \mathbf{c}))$.
- Axiom NN (*No-claim No-pie*): For any bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$, if $c_i = 0$, then $\phi_i(E, \mathbf{c}) = 0$.

Axiom ST implies that equal awards are assigned to those agents whose claims are equivalent to each other, and Axiom IIC says that if agents claim more than there is, it is ineffective. A rule satisfies Axiom C if the original allocation generated by it can be decomposed according to any step-by-step bases. Since an allocation is non-negative, this axiom implies strong monotonicity, i.e., given any bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$, if $0 \leq E' \leq E$, then $\phi(E', \mathbf{c}) \leq \phi(E, \mathbf{c})$. The last axiom implies that whoever claims nothing receives nothing.[6] The CWA rule satisfies all the axioms referred above except Axiom ST. Instead, it satisfies *proportional treatment* (PT), i.e., for any bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$ if $\alpha_j c_i = \alpha_i c_j$, then $\alpha_j \phi_i(E, \mathbf{c}) = \alpha_i \phi_j(E, \mathbf{c})$, which implies that if claims are proportional w.r.t. relative evaluations, then so do awards.

3. EQUIVALENCY BETWEEN THE ASYMMETRIC NASH SOLUTION AND THE CWA SOLUTION

Let $S \subseteq R^N$ be a set of utility tuples and $\mathbf{d} \in S$ is a disagreement point. S is $\mathbf{d}$-*comprehensive* if $\mathbf{x} \in S$ and $\mathbf{d} \leq \mathbf{y} < \mathbf{x}$, then $\mathbf{y} \in S$. Also, it is called *non-degenerate* if there exists $\mathbf{x} \in S$ such that $\mathbf{d} \ll \mathbf{x}$. A pair $(S, \mathbf{d})$ is called a *bargaining problem* if S is closed, bounded, convex, $\mathbf{d}$-comprehensive and non-degenerate, where $\mathscr{B}$ is

[5] The nomenclature originates from Aumann and Maschler (1985), of which the purpose is to find general rules behind numerical examples appeared in the Babylonian Talmud. Also, note that it was introduced by Bennasy (1982) as the rationing function for markets in disequilibria, and Sprumont (1991) named it the uniform rule and characterized it from a view point of a strategy-proof allocative schema.

[6] The last axiom is not referred in Dagan (1994), but which is a necessary and harmless assumption. Also, note that from the setting of a bankruptcy problem the CEA rule must satisfy (strong) Pareto optimality (PO), i.e., for any bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$, $\phi(E, \mathbf{c}) < \mathbf{y}$ implies that $\mathbf{y}$ is not an allocation.

a set of all bargaining problems.[7] A *bargaining solution*, or simply a solution is a function which maps each bargaining problem to a unique utility tuple in S. S can be interpreted as a set of agreements to which agents may reach possibly by a negotiation process where the disagreement point guarantees the minimum level of utilities in case that the negotiation fails. Therefore, a bargaining solution is a unique utility tuple, seeminingly desirable, selected from S. Among several well-known solutions we are interested in the *asymmetric Nash solution* defined as follows: Given a bargaining problem $(S, \mathbf{d}) \in \mathscr{B}$ and an evaluation vector $\boldsymbol{\alpha} \in \mathring{\Delta}^N$, a solution $\mathbf{f}^\alpha(S, \mathbf{d})$ is called the *asymmetric Nash solution* if

$$\mathbf{f}^\alpha(S, \mathbf{d}) = \underset{\mathbf{d} \leq \mathbf{x} \in S}{\operatorname{argmax}} \underset{i \in N}{\times} (x_i - d_i)^{\alpha_i} \tag{2}$$

This solution is a generalization of the Nash solution (Nash, 1950) and first introduced by Harasanyi and Selten (1972). We next derive a specific bargaining problem from a bankruptcy problem. For that purpose, given any bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$, a set of utility tuples is defined by $A(E, \mathbf{c}) = \{\mathbf{x} \in R^N \mid \mathbf{x} \leq \mathbf{c}, \mathbf{x}(N) \leq E\}$. This means that bargainings over all possible divisions of any amount of the good less than E are feasible as far as they satisfy the claims constraint. Note that this set satisfies all properties aforementioned except non-degeneracy, which is guaranteed whenever $0 < E$ and $\mathbf{0} \ll \mathbf{c}$. Evidently, this condition ensures that $\mathbf{0} \in A(E, \mathbf{c})$. For the time being we only consider the case of the $\mathbf{0}$ disagreement point and call a pair $(A(E,\mathbf{c}), \mathbf{0})$ $\mathbf{0}$-*associated bargaining problem* corresponding to a bankruptcy problem $(E, \mathbf{c})$. Note that essentially, the CWA rule allocation can be interpreted as a solution of $\mathbf{0}$-associated bargaining problem such as follows: A solution ψ^α is called the *CWA solution* if $\psi_i^\alpha(A(E, \mathbf{c}), \mathbf{0}) = c_i \wedge t^*\alpha_i$, $i \in N$ where $t = t^*$ is a maximum scaling factor over $\{t \in R_+ \mid \mathbf{c} \wedge t\boldsymbol{\alpha} \in A(E, \mathbf{c})\}$.

PROPOSITION 1. *For any $\mathbf{0}$-associated bargaining problem $(A(E, \mathbf{c}), \mathbf{0})$ and any bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$ satisfying $0 < E$ and $\mathbf{0} \ll \mathbf{c}$, the $\boldsymbol{\alpha}$-asymmetric Nash solution induces the $\boldsymbol{\alpha}$-CWA solution and vice versa where $\boldsymbol{\alpha} \in \mathring{\Delta}^N$ is an evaluation vector.*

PROOF. Consider a constrained maximization problem as follows: $\max_{\mathbf{x}} \underset{i \in N}{\times} x_i^{\alpha_i}$ s.t. $\mathbf{x} \in A(E, \mathbf{c})$. First, note that since the feasible region $A(E, \mathbf{c})$ of the above problem is compact, convex and nonempty by non-degeneracy and the objective function is strictly quasi-concave, the unique maximizer $\mathbf{x}^* \in R^N$ exists. Also, note that $W(\cdot)$ is monotonic over the n-dimensional non-negative orthant, i.e., if $\mathbf{x} \in R_+^N$ and $\mathbf{x} \leq \mathbf{x}'$, then $W(\mathbf{x}) \leq W(\mathbf{x}')$.[8] Hence, x_i^*, $i \in N$ should be positive and E should

[7] Closedness is a technical assumption, and boundedness implies that each agent's utility is finite. Convexity implies randomization of any of distinct utility tuples. S is $\mathbf{d}$-comprehensive if free disposability of utility tuples is allowed, which implies that ∂S, the boundary of S is weakly Pareto optimal, i.e., if $\mathbf{x} \in \partial S$, $\mathbf{x} \ll \mathbf{y}$, then $\mathbf{y} \notin \partial S$. If S is $\mathbf{d}$-comprehensive with the Pareto optimal ∂S, it is called *strictly* $\mathbf{d}$-*comprehensive*. The last assumption requires that the result of any bargaining be non-trivial.

[8] For notational simplicity let $W(\mathbf{x}) = \underset{i \in N}{\times} x_i^{\alpha_i}$ and $W^* = W(\mathbf{x}^*)$.

be tightly distributed over every agent, i.e., $\mathbf{x}^*(N) = E$ since otherwise, by non-degeneracy it contradicts the maximality of $\mathbf{x}^*$. Let $L(\mathbf{x}, \boldsymbol{\mu}, \lambda) = \times_{i \in N} x_i^{\alpha_i} + \sum_{i \in N} \mu_i(c_i - x_i) + \lambda(E - \mathbf{x}(N))$ be a Lagrangean function of the above problem where μ_i, $i \in N$ and λ are Lagrangean multipliers. The Kuhn-Tucker condition, given below, is a necessary condition for the existence of the solution if $0 < E$ and $\mathbf{0} \ll \mathbf{c}$.[9]

$$\text{i)} \quad x_i^*\left(\frac{\alpha_i W^*}{x_i^*} - \mu_i^* - \lambda^*\right) = 0\,,$$

$$x_i^* \geq 0\,, \quad \frac{\alpha_i W^*}{x_i^*} - \mu_i^* - \lambda^* \leq 0 \qquad \text{for all} \quad i \in N\,,$$

$$\text{ii)} \quad \mu_i^*(c_i - x_i^*) = 0\,,$$

$$\mu_i^* \geq 0\,, \quad c_i - x_i^* \geq 0 \qquad \text{for all} \quad i \in N\,,$$

$$\text{iii)} \quad \lambda^*(E - \boldsymbol{x}^*(N)) = 0\,,$$

$$\lambda^* \geq 0\,, \quad (E - \boldsymbol{x}^*(N)) \geq 0\,.$$

We first deal with the case of $x_i^* < c_i$ for all $i \in N$. Evidently, $\mu_i^* = 0$. Since $x_i^* > 0$, $\alpha_i W^*/x_i^* - \mu_i^* - \lambda^* = 0$, i.e., $x_i^* = t^* \alpha_i$ where $t^* = W^*/\lambda^* \in R_{++}$. Since $\mathbf{x}^*(N) = E$, t^* is a maximizing scaling factor of $\boldsymbol{\alpha}$ over $A(E, \mathbf{c})$ and hence $\mathbf{f}^\alpha A(E, \mathbf{c}), \mathbf{0}) = t^* \boldsymbol{\alpha} = \boldsymbol{\psi}^\alpha(A(E, \mathbf{c}), \mathbf{0})$. Next, suppose that w.l.o.g., $x_i^* < c_i$ for all $i \in K = \{1, 2, \cdots, k\}$ and $x_j^* = c_j$ for all $j \in N \backslash K$ where both sets are nonempty. Then, if $i \in K$, $x_i^* = t^* \alpha_i$ and if $j \in N \backslash K$, $x_j^* \leq t^* \alpha_j$. Also, note that t^* is a maximizing scaling factor since $\mathbf{x}^*(K) = E - \mathbf{c}(N \backslash K)$. Hence, the $\boldsymbol{\alpha}$-asymmetric Nash solution induces the $\boldsymbol{\alpha}$-CWA solution.

Next, sufficiency will be shown. Suppose that $\boldsymbol{\psi}^\alpha(A(E, \mathbf{c}), \mathbf{0}) = \mathbf{c} \wedge t^* \boldsymbol{\alpha}$, which is strictly positive, is given.[10] If $\psi_i^\alpha < c_i$ for all $i \in N$, then let $\mu_i^* = 0$, $i \in N$ and $\lambda^* = W^\alpha/t^*$. Suppose that $\psi_i^\alpha < c_i$ for all $i \in K$ and $\psi_j^\alpha = c_j$ for all $j \in N \backslash K$.[11] Then, set $\mu_i^* = 0$, $i \in K$ and $\lambda^* = W^\alpha/t^*$. Also, let $\mu_j^* = \alpha_j W^\alpha/c_j - W^\alpha/t^*, j \in N \backslash K$. Note that all the multipliers defined above are non-negative and the Kuhn–Tucker condition, which is sufficient for the maximization problem, is a satisfied. Q.E.D.

The first case deals with one in which all awards are not claims-binding and both solutions coincide with each other as the proportional solution w.r.t. the evaluation vector $\boldsymbol{\alpha}$. The second case is that awards of some agents are binding.[12] In this case the proportionality principle is applied to those agents whose awards

[9] The latter condition is called Slater's condition. Note that the constraints of the problem can be reformulated by defining functions, $g_i(\mathbf{x}) = x_i$, $i \in N$ and $g(\mathbf{x}) = \sum_{i \in N} x_i$, which are trivially convex. λ^* and μ_i^*, $i \in N$ are the optimal values of Lagrangean multipliers.

[10] For notational simplicity let $\boldsymbol{\psi}^\alpha = \boldsymbol{\psi}^\alpha(A(E, \mathbf{c}), \mathbf{0})$ and $W^\alpha = W(\boldsymbol{\psi}^\alpha)$.

[11] Note that if $i \in K$, $\psi_i^\alpha = t^* \alpha_i$.

[12] The remaining case, $\mathbf{x}^* = \mathbf{c}$ is trivial from footnote 4.

are not binding over $A(E, \mathbf{c})$, whereas the remaining agents' awards are fixed to their claims. The point is that the Pareto optimal frontier of $A(E, \mathbf{c})$ has an equi-length normal vector and so, the solution projected over the Euclidean subspace of which the dimension is equal to the number of non-binding agents is on the half-line generated by the corresponding projected evaluation vector. Also, from this proposition we can say that the asymmetric Nash solution satisfies all the properties of the CWA solution mentioned in the preceeding section. The diagrams given below illustrate these two cases of the two-agent set-up.

The above result can be extended so that the disagreement point $\mathbf{d}$ is not confined only to zero vector provided that $\mathbf{d} \ll \mathbf{c}$ and $\mathbf{d}(N) < E$, which guarantees $\mathbf{d} \in A(E, \mathbf{c})$ and non-degeneracy. For each bankruptcy problem $(E, \mathbf{c}) \in \mathscr{C}$, consider the corresponding bargaining problem $(A(E, \mathbf{c}), \mathbf{d})$ and generalize the CWA solution as follows: $\hat{\psi}_i^\alpha(A(E, \mathbf{c}), \mathbf{d}) = c_i \wedge (t^*\alpha_i + d_i)$, $i \in N$ where $t = t^*$ is a maximum scaling factor over $\{t \in R_+ \mid \mathbf{c} \wedge (t\boldsymbol{\alpha} + \mathbf{d}) \in A(E, \mathbf{c})\}$. We call $\hat{\psi}^\alpha$ the *generalized CWA solution*

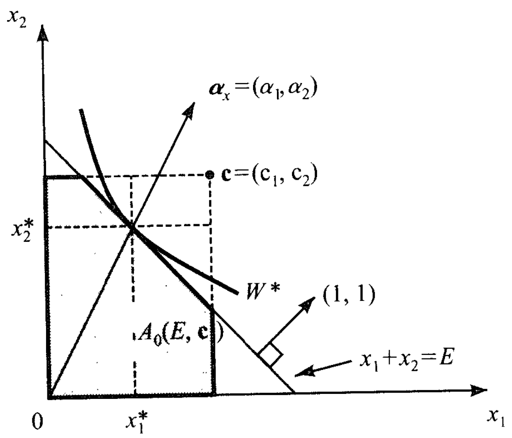

Fig. 1(a).

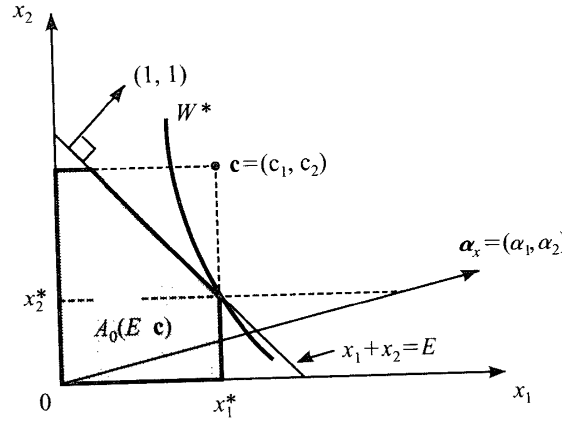

Fig. 1(b).

Fig. 1. Equivalency between two solution concepts: (a) Claims do not bound solutions, which coincide with the proportional solution w.r.t. the evaluation vector. (b) The claims-bounding case.

in a sense that the disagreement point is not necessarily equal to zero. A solution ψ satisfies *translation invariance* (TI), i.e., for any $(S, \mathbf{d}) \in \mathscr{B}$ and any vector $\mathbf{l} \in R^N$, $\psi(S+\{\mathbf{l}\}, \mathbf{d}+\mathbf{l}) = \psi(S, \mathbf{d}) + \mathbf{l}$ where $S+\{\mathbf{l}\} = \{\mathbf{y} \in R^N \mid \exists \mathbf{x} \in S; \mathbf{y} = \mathbf{x}+\mathbf{l}\}$. It is called to satisfy *independence of non-individually rational alternatives* (INRA) if $\psi(A(E, \mathbf{c}), \mathbf{d}) = \psi(A_{\mathbf{d}}(E, \mathbf{c}), \mathbf{d})$ where $A_{\mathbf{d}}(E, \mathbf{c}) = \{\mathbf{x} \in A(E, \mathbf{c}) \mid \mathbf{d} \leq \mathbf{x}\}$.[13] (See Peters, 1986) Note that the asymmetric Nash solution and the generalized CWA solution satisfy these properties.[14]

PROPOSITION 2. *For any bargaining problem* $(A(E, \mathbf{c}), \mathbf{d})$, *any bankruptcy problem* $(E, \mathbf{c}) \in \mathscr{C}$ *satisfying* $\mathbf{d} \ll \mathbf{c}$ *and* $\mathbf{d}(N) < E$, *and an evaluation vector* $\boldsymbol{\alpha} \in \Delta^N$, *the* $\boldsymbol{\alpha}$*-asymmetric Nash solution induces the generalized* $\boldsymbol{\alpha}$*-CWA solution and vice versa.*

PROOF. Suppose that a bargaining problem $(A(E, \mathbf{c}), \mathbf{d})$ is given. By Axiom TI we have $\mathbf{f}^\alpha(A(E, \mathbf{c}) + \{-\mathbf{d}\}, \mathbf{d}-\mathbf{d}) = \mathbf{f}^\alpha(A(E, \mathbf{c}) - \{\mathbf{d}\}, \mathbf{0}) - \mathbf{d}$ and so does the CWA solution. Note that $A(E, \mathbf{c}) - \{\mathbf{d}\} = \mathrm{A}(E - \mathbf{d}(N), \mathbf{c}-\mathbf{d})$, whereas $(E - \mathbf{d}(N), \mathbf{c}-\mathbf{d})$ is a bankruptcy problem. Also, note that $\hat{\psi}^\alpha(A(E, \mathbf{c}) - \{\mathbf{d}\}, \mathbf{0}) = \psi^\alpha(A(E, \mathbf{c}) - \{\mathbf{d}\}, \mathbf{0})$. Hence, by Proposition 1 $\mathbf{f}^\alpha(A(E, \mathbf{c}) - \{\mathbf{d}\}, \mathbf{0}) = \hat{\psi}^\alpha(A(E, \mathbf{c}) - \{\mathbf{d}\}, \mathbf{0})$, from which the desired result is obtained. Q.E.D.

Some remarks are referred. First, note that the set of utility tuples $A(E, \mathbf{c})$ derived from a bankruptcy problem $(E, \mathbf{c})$ is a special case of general ones since its Pareto optimal frontier is included in the hyperplane with a unit equi-length normal vector and hence it should be analyzed from a perspective of more general settings.[15] Secondly, bankruptcy problems implicitly assume that claims of agents are finite. But, there is no guarantee that it is true as far as claims are registered by agents. If claims are infinite, the CWA solution distributes E proportionate to an evaluation vector $\boldsymbol{\alpha}$.[16]

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[13] Note that Dagan and Volij define bargaining solutions over $A_{\mathbf{0}}(E, \mathbf{c})$.

[14] We can generalize and characterize the CEA rule so that it may reflect the non-zero disagreement point by considering analogues of Axiom TI and Dagan's axioms. Let $\mathscr{BC}$ be a set of triple $(E, \mathbf{d}, c)$ such that $0 \leq E \leq \mathbf{c}(N)$, $\mathbf{d} \ll \mathbf{c}$, $\mathbf{d}(N) \leq E$. The *CEA rule with the disagreement point* $\mathbf{d}$, $\hat{\phi}$ is defined by $\hat{\phi}_i(E, \mathbf{c}) = c_i \wedge (t^* + d_i)$, $i \in N$ where $t = t^*$ is a solution of $(\mathbf{c} \wedge (t\mathbf{1}+\mathbf{d}))(N) = E$. For any $(E, \mathbf{d}, c) \in \mathscr{BC}$ corresponding axioms redefined are as follows: (1) Axiom TI′: For any $\mathbf{l} \in R^N$, $\phi(E+\mathbf{l}(N), \mathbf{d}+\mathbf{l}, \mathbf{c}+\mathbf{l}) = \phi(E, \mathbf{d}, \mathbf{c}) + \mathbf{l}$, (2) Axiom ST′: If $d_i = d_j$ and $c_i = c_j$, then $\phi_i(E, \mathbf{d}, c) = \phi_j(E, \mathbf{d}, \mathbf{c})$, (3) Axiom IIC′: $\phi(E, \mathbf{d}, \mathbf{c}) = \phi(E, \mathbf{d}, \mathbf{c} \wedge ((E - \mathbf{d}(N))\mathbf{1} + \mathbf{d}))$, (4) Axiom C′: If $0 \leq E' \leq E$ and $(E', \mathbf{d}, \mathbf{c}) \in \mathscr{BC}$, $\phi(E, \mathbf{d}, \mathbf{c}) = \phi(E', \mathbf{d}, \mathbf{c}) + \phi(E - E', \mathbf{c} - \phi(E', \mathbf{d}, \mathbf{c}))$, (5) Axiom NN′: If $d_i = c_i$, then $\phi_i(E, \mathbf{d}, \mathbf{c}) = d_i$.

[15] See, e.g., Thomson and Chun (1922).

[16] The schema of Sprumont (1991) is that each agent has a uni-modal utility function and his claim corresponds to the amount of the good associated with the peak of the function.

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