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INCOME INEQUALITY AND RELATIVE DEPRIVATION

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Abstract: This paper examines the implications of the ranking relation generated by two non-intersecting relative deprivation curves as developed in Kakwani (1984). It is shown that the dominance in terms of relative deprivation implies the Lorenz domination, hence welfare improvement property (that is, welfare increases under Pigou–Dalton type progressive income transfers), but the converse is not true. Next, the class of average relative deprivation indices that agrees with the deprivation dominance criterion is identified. It turns out that all such deprivation indices can be regarded as Lorenz consistent inequality indices, but the reverse implication does not follow.

Key-words: Inequality, relative deprivation, social welfare.
JEL classification numbers: D31, D63

1. INTRODUCTION

For any person in a society the feeling of *relative deprivation* arises out of the comparison of his situation with those of better off persons. Runciman (1966) used the example of promotion to describe an individual's feeling of relative deprivation: 'The more people a man sees promoted when he is not promoted himself, the more people he may compare himself with in a situation where the comparison will make himself feel relatively deprived' (op. cit, p. 19). Thus, according to Runciman the extent of deprivation felt by an individual for not being promoted is an increasing function of the number of individuals who have been promoted. Yitzhaki (1979) considered relative deprivation in terms of income and quantified a particular case of Runciman's statement. He showed that one plausible index of average relative deprivation in a society is the product of the Gini index and the mean income of the society.

An alternative derivation of Yitzhaki's result was provided by Hey and Lambert (1980). Essential to their alternative characterization is Runciman's (1966) remark:

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'The magnitude of a relative deprivation is the extent of the difference between the desired situation and that of the person desiring it' (op. cit., p. 10). Chakravarty and Chakraborty (1984) made use of such differences to develop a generalization of the Yitzhaki index. Berrebi and Silber (1985) showed that many commonly used indices of inequality can be regarded as indices of relative deprivation. Interesting variations of the Yitzhaki index have also been suggested by Paul (1991) and Chakravarty and Chattopadhyay (1994).

Sen (1973) reinterpreted the Gini index from a quite similar perspective. According to Sen, in any pairwise comparison, the individual with lower income may suffer from depression upon discovering that his income is lower. The average of all such depressions in all pairwise comparisons becomes the Gini coefficient if the extent of depressions is proportional to the differences in the incomes. Kakwani (1980) showed that if the individual's depression is proportional to the square of the income difference, we get the coefficient of variation as the average deprivation index. Kakwani (1984) plotted the sum of income share shortfalls of different individuals from richer individuals against the cumulative proportions of persons to generate the *Relative Deprivation Curve* (*RDC*) and showed that the area under this curve is the Gini index for the society.

It is evident that a particular measure of relative deprivation will generate a complete ranking of alternative income distributions. However, using more than one index we may get different rankings of the distributions. Given the diversity of numerical measures, it is, therefore, reasonable to identify the class of indices that yield a similar ordering of different income profiles. This is one of the objectives of this paper. More precisely, given any two income distributions we determine the set of average deprivation indices that will rank income distributions in exactly the same way as the ordering generated by two non-intersecting RDCs. Another objective of the paper is to look at the implications of the relative deprivation ordering in terms of Lorenz domination. Here we show that given two income distributions x and y of a fixed total over a fixed population size, if the RDC of x dominates that of y, then x Lorenz dominates y. But the converse is not true.

This paper is organized as follows. The next section discusses the ordering associated with non-intersecting RDCs and studies its implications. In section **3** we isolate the class of deprivation indices that agrees with this ordering. This section also brings out the distinguishing features between deprivation and inequality. In section **4** we provide a numerical illustration of a few relative deprivation indices. Section **5** makes some concluding remarks.

2. THE RELATIVE DEPRIVATION ORDERING

For a population of size n, the set of income distributions is denoted by D^n, with a typical element $x=(x_1, x_2, \cdots, x_n)$, where D^n is the non-negative orthant of the Euclidean n-space R^n with the origin deleted. The set of all possible income profiles is $D=\bigcup_{n\in N} D^n$, where N is the set of natural numbers. Throughout the paper we will adopt the following notation. For any function $H: D \to R^1$ we denote the restriction of H on D^n by H^n, where $n \in N$ is arbitrary. For all $n \in N$ and $x \in D^n$, we write $\lambda(x)$ for the mean of x and $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ for the illfare ranked permutation of x, that is, $\bar{x}_1 \leq \bar{x}_2 \leq \cdots \leq \bar{x}_n$. For $n=1$, the concept of relative deprivation is vacuous. We will, therefore, assume that $n \geq 2$.

Following Runciman's (1966) remark mentioned in the introduction, the deprivation d_{ij} felt by an individual with income x_i relative to j^{th} person's income x_j, where $x_j \geq x_i$, can be considered to be their income share differential (see also Kakwani, 1984). That is,

$$\begin{aligned} d_{ij} &= (x_j - x_i)/n\lambda(x) \quad &\text{if} \quad x_j \geq x_i \\ &= \quad 0 \quad &\text{if} \quad x_j < x_i \,. \end{aligned} \tag{1}$$

Thus, d_{ij} is normalized over the interval $[0, 1]$. It is continuous in x, increasing in x_j and decreasing in x_i.

Now, an individual with income $\bar{x}_i$ in the ordered distribution $\bar{x}$ is deprived of the incomes $\bar{x}_{i+1}, \cdots, \bar{x}_n$. Therefore, the total deprivation felt by this person is

$$d_i(x) = \sum_{j=i+1}^{n} (\bar{x}_j - \bar{x}_i)/n\lambda(x) \,. \tag{2}$$

The individual deprivation index d_i possesses many interesting properties:

(i) d_i is a decreasing function of $\bar{x}_i$.

(ii) d_i is independent of the incomes smaller than $\bar{x}_i$.

(iii) d_i decreases under a rank preserving income transfer from someone with income higher than $\bar{x}_i$ to someone with income smaller than $\bar{x}_i$.

(iv) An increase in any income higher than $\bar{x}_i$ increases d_i.

(v) An equiproportionate increase in all incomes does not alter d_i.

(vi) d_i is continuous in its arguments.

The *RDC* associated with the distribution x is defined as the plot of $d_i(x)$ against the cumulative proportion of population i/n, where $i = 0, 1, \cdots, n$ and $d_0(x) = 1$ (see Kakwani (1984)). The extension $d_0(x) = 1$ ensures that the *RDC* is a closed graph. Clearly, $d_i(x)$ is a decreasing function of i/n.

If incomes are equally distributed, then there is no feeling of deprivation by any person. In this case the *RDC* coincides with the *no deprivation line OA*. On the other hand, if the entire income is monopolized by the richest person, the RDC coincides with the line *CB*. This is the case of maximum relative deprivation.

We can rewrite $d_i(x)$ in (2) as follows

$$d_i(x) = 1 - L_i(x) - (n-i)\bar{x}_i/n\lambda(x) \,, \tag{3}$$

where $L_i(x)\left(= \sum_{j=1}^{i} \bar{x}_j/n\lambda(x)\right)$ is the cumulative proportion of the total income $n\lambda(x)$ enjoyed by the bottom i/n proportion $(0 \leq i \leq n)$ of the population. The graph of $L_i(x)$ against i/n, $i = 0, 1, \cdots, n$, where $L_0(x) = 0$ is the well-known Lorenz curve

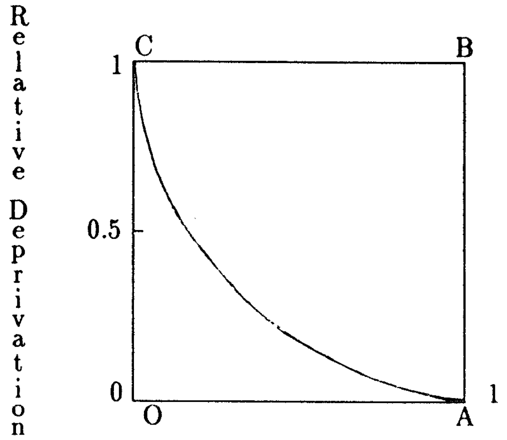

Fig. 1. The Relative Deprivation Curve.

(LC). Following Yitzhaki (1979) and Hey and Lambert (1980) we refer to the function $s_i(x) = L_i(x) + (n-i)\bar{x}_i/n\lambda(x)$ as the *relative satisfaction function* of the person with income $\bar{x}_i$. Since $d_i(x)$ is the complement (to 1) of $s_i(x)$, one can work as well with $s_i(x)$ instead of $d_i(x)$. Evidently, by augmenting $L_i(x)$ by the proportion $(n-i)\bar{x}_i/n\lambda(x)$ we get a curve, which we can refer to as the *relative satisfaction curve* (*RSC*).

Suppose that the income distribution is represented by a distribution function F: $[0, \infty] \rightarrow [0, 1]$. $F(z)$ is the cumulative proportion of persons with income less than or equal to z. $F(0) = 0$, $F(\infty) = 1$ and F is increasing. Then for any arbitrary income $z \in [0, \infty]$, the deprivation function $d_i(x)$ in (3) becomes

$$d_F(z) = 1 - F_1(z) - (1 - F(z))z/\lambda(F)\,, \tag{4}$$

where $F_1(z)$ $(= \int_0^z t dF(t)/\lambda(F))$ is the ordinate of the *LC* corresponding to the income level z for a continuum of population and $\lambda(F)$ is the mean income. We observe that $d(d_F(z))/dF(z) = (F(z) - 1)/f(z)$, where $f(z)$ is the income density function. This demonstrates rigorously that the *RDC* is monotonically decreasing. Since $d^2(d_F(z))/d^2F(z) = [(f(z))^2 + (1 - F(z))f'(z)]/(f(z))^3$, where f' is the derivative of $f(z)$, may be positive, zero or negative, no definite conclusion can be drawn regarding the curvature of the *RDC*.

Given two income distributions $x, y \in D^n$, we say that x dominates y by the *relative deprivation criterion* ($x \succeq_d y$ for short) if the *RDC* of x lies nowhere above that of y and at some places (at least) strictly inside the latter. Formally, $x \succeq_d y$ means that

$$d_i(x) \leq d_i(y) \tag{5}$$

for all $i = 1, \cdots, n$, with $<$ for at least one $i < n$. Using (3) we rewrite $x \succeq_d y$ in (5) as

$$L_i(x) + (n-i)\bar{x}_i/n\lambda(x) \geq L_i(y) + (n-i)\bar{y}_i/n\lambda(y) \tag{6}$$

for all $i=1, \cdots, n$ with $>$ for at least one $i<n$. (6) means that x dominates y by the *relative satisfaction criterion* ($x \succeq_s y$ for short). Thus, $x \succeq_d y$ and $x \succeq_s y$ are equivalent.

Before we formally state the relationship between Lorenz domination and deprivation domination, it will be worthwhile to discuss a result developed in the context of Lorenz domination. (For $x, y \in D^n$, we say that x Lorenz dominates y, $x \succeq_L y$ for short, if $L_i(x) \geq L_i(y)$ for all $i=1, 2, \cdots, n$, with strict inequality for at least one $i<n$.) Dasgupta, Sen and Starrett (1973) (see also Kolm (1969) and Atkinson (1970)) demonstrated that the relations $x \succeq_L y$, where $\lambda(x)=\lambda(y)$, is equivalent to the condition that $\bar{x}$ is obtained from $\bar{y}$ through a finite sequence of transformations transferring income from the rich to the poor. Dasgupta, Sen and Starrett also demonstrated that this condition implies and is implied by the requirement that $W^n(x) > W^n(y)$ for any Social Welfare Function (SWF) $W^n: D^n \to R^1$ that satisfy strict *S-concavity*.[1] This seems quite reasonable intuitively, since given $\lambda(x)=\lambda(y)$, $x \succeq_L y$ means that x is regarded as more equal than y by any inequality index that satisfies the Pigou–Dalton condition, a postulate that demands inequality reduction under an income transfer from a rich to a poor without making the poor person the richer one. This in turn should be equivalent to the condition that x is regarded as better than y by any equity oriented (in the sense of strictly *S-concave*) SWF.

Two interesting examples of this type of SWF are the utilitarian rule

$$W_U^n(x) = \sum_{i=1}^{n} U(x_i)\,, \tag{7}$$

where U is increasing and strictly concave and the Gini SWF

$$W_G^n(x) = \sum_{i=1}^{n} [2(n-i)+1]\bar{x}_i \tag{8}$$

$$= \sum_{i=1}^{n} (2i-1)\mathring{x}_i\,, \tag{9}$$

where $\mathring{x}=(\mathring{x}_1, \cdots, \mathring{x}_n)$ is the welfare ranked permutation of x, that is, $\mathring{x}_1 \geq \mathring{x}_2 \geq \cdots \geq \mathring{x}_n$. (See Blackorby and Donaldson (1978), Donaldson and Weymark (1980) and Chakravarty (1990) for discussions on the Gini SWF. See also Bishop, Chakrabarti and Thistle (1991).)

In the following theorem we now demonstrate the implication of the relation $x \succeq_d y$ in terms of Lorenz domination and hence social welfare.

THEOREM 1. *Let $x, y \in D^n$, where $\lambda(x)=\lambda(y)$, be arbitrary. Then $x \succeq_d y$ implies that*

(a) $x \succeq_L y$,

(b) *$W^n(x) > W^n(y)$ for all strictly $S-$concave social welfare functions $W^n: D^n \to R^1$.*

Proof. It follows from (3) that

$$d_1(x) = 1 - \frac{(n-1)\bar{x}_1}{n\lambda(x)}, \tag{10}$$

$$d_2(x) = 1 - \frac{\bar{x}_1}{n\lambda(x)} - \frac{\bar{x}_2}{\lambda(x)}, \tag{11}$$

$$d_3(x) = 1 - \frac{(\bar{x}_1 + \bar{x}_2)}{n\lambda(x)} - \frac{\bar{x}_3}{\lambda(x)}, \tag{12}$$

and so on.

Hence, when $\lambda(x) = \lambda(y)$, it follows that

$$d_1(x) \leq d_1(y) \quad \text{means that } \bar{x}_1 \geq \bar{y}_1, \tag{13}$$

$$d_2(x) \leq d_2(y) \quad \text{means that } \bar{x}_1 + n\bar{x}_2 \geq \bar{y}_1 + n\bar{y}_2, \tag{14}$$

$$d_3(x) \leq d_3(y) \quad \text{means that } \bar{x}_1 + \bar{x}_2 + n\bar{x}_3 \geq \bar{y}_1 + \bar{y}_2 + n\bar{y}_3, \tag{15}$$

and so on.

That is, in general, we have

$$\sum_{j=1}^{i-1} \bar{x}_j + n\bar{x}_i \geq \sum_{j=1}^{i-1} \bar{y}_j + n\bar{y}_i \tag{16}$$

for all $i = 1, 2, \cdots n$ with $>$ for at least one $i < n$.

From this it is clear that domination in terms of relative deprivation curve implies, but is not implied by Lorenz domination.[2]

Using part (a) of this theorem, in view of Dasgupta–Sen–Starrett's result, we can conclude that $x \succ_d y$ implies $W^n(x) > W^n(y)$. This completes the proof of the theorem. ∎

It should be evident that in the proof of Theorem 1 if for some i, say i_0, the inequality in (16) becomes strict, then all the following inequalities will also be strict. We, in particular, have $\bar{x}_n < \bar{y}_n$, which is a necessary and testable condition for the relation $x \succ_d y$ to hold. This observation along with (13) gives us the following corollary to Theorem 1.

COROLLARY 1. *Let $x, y \in D^n$, where $\lambda(x) = \lambda(y)$, be arbitrary. Then the relation $x \succ_d y$ implies that*

(a) *x is regarded as at least as good as y by the Rawlsian* (1971) *maximin criterion, that is,* $\min_i\{x_i\} \geq \min_i\{y_i\}$.

(b) *x is regarded as worse than y by the* maximax *criterion, that is,* $\max_i\{x_i\} < \max_i\{y_i\}$

To understand the reason why $x \succ_L y$ does not imply $x \succ_d y$, consider a transfer from the person with income $\bar{x}_i$ to the person with income $\bar{x}_j$, where $\bar{x}_j < \bar{x}_i < \bar{x}_n$. Evidently, the LC of the post transfer distribution shifts upwards. On the other hand, while with such a transfer the aggregate deprivation of the poorer person goes down, it increases for the richer person, thus, making the net effect ambiguous.

It may now be interesting to relate Theorem 1 to a result developed by Hey and Lambert (1980). Assuming that the relative satisfaction of an individual with income z is given by $\int_0^z (1-F(t))dt$, they demonstrated the following: 'If two distributions have the same mean and if their Lorenz curves do not cross, then under the distribution whose Lorenz curve is higher (nearer to the line of complete equality), there is more relative satisfaction at each level of income than there is at the same level under the other distribution' (Hey and Lambert, 1980, p. 569). The converse is also true (again under the assumption of equal mean income). If the extent of deprivation felt by this individual is given by $\int_z^\infty (1-F(t))dt$, it then follows that at all income levels there will be less relative deprivation under the distribution with higher Lorenz curve and vice-versa. Since in our case we have adopted the Runciman (1966)–Kakwani (1984) formulation of the deprivation function given by (4), which is different from the Hey–Lambert formulation, we have one-way implication only. Furthermore, Hey and Lambert did not develop their result in terms of *RDC*. (In fact, (4) forms the basis of *RDC*).

International comparisons of relative deprivation usually involve different population sizes and different means, as do intertemporal comparisons for the same country. For ranking profiles with the same means over different population sizes, following Dasgupta, Sen and Starrett (1973), we consider the Symmetry Axiom for Population (SAP). According to SAP, if an income distribution is replicated k times, then the aggregate welfare of the replicated distribution is simply k times the welfare of the original profile. That is, the average welfare is population replication invariant.

SYMMETRY AXIOM FOR POPULATION (SAP). For all $n \in N$, $x \in D^n$, $W^{nk}(y) = kW^n(x)$ where $y = (x^{(1)}, x^{(2)}, \cdots, x^{(k)})$, each $x^{(i)} = x$ and $W\colon D \to R^1$.

THEOREM 2. *Let $x^1 \in D^{n_1}$ and $x^2 \in D^{n_2}$, where $\lambda(x^1) = \lambda(x^2)$, be arbitrary. Then $x^1 \succeq_d x^2$ implies that*

(a) $x^1 \succeq_L x^2$,

(b) $W^{n_1}(x^1)/n_1 > W^{n_2}(x^2)/n_2$ *for all SWFs $W\colon D \to R^1$ that satisfy SAP and strict S-concavity.*

Proof. Let $x^3(x^4)$ denote the $n_2(n_1)$ fold replication of $x^1(x^2)$. Clearly, $x^3, x^4 \in D^{n_1 n_2}$. Note that $\lambda(x^3) = \lambda(x^4)$. Now, the RDC is population replication invariant. That is, $x^1 \succeq_d x^2$ is equivalent to $x^3 \succeq_d x^4$, which in view of Theorem 1 implies that $x^3 \succeq_L x^4$. Since the LC is also invariant under population replications, $x^3 \succeq_L x^4$ is same as $x^1 \succeq_L x^2$. Hence $x^1 \succeq_d x^2$ implies $x^1 \succeq_L x^2$.

By Theorem 1, $x^3 \succeq_d x^4$, that is, $x^1 \succeq_d x^2$ implies that $W^{n_1 n_2}(x^3) > W^{n_1 n_2}(x^4)$. Since W satisfies SAP, $W^{n_1 n_2}(x^3) = n_2 W^{n_1}(x^1)$ and $W^{n_1 n_2}(x^4) = n_1 W^{n_2}(x^2)$. Thus $x^1 \succeq_d x^2$ implies that $W^{n_1}(x^1)/n_1 > W^{n_2}(x^2)/n_2$. ■

3. NUMERICAL MEASURES

We begin this section by specifying some examples of a relative deprivation index. The two well-known indices of relative deprivation are the Gini coefficient

$$G^n(x)=\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i-x_j|/2n^2\lambda(x) \tag{17}$$

and the coefficient of variation

$$C^n(x)=\frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\lambda(x))^2}}{\lambda(x)}, \tag{18}$$

where $x\in D^n$ and $n\in N$ are arbitrary. Two other interesting indices are Paul's (1991) index

$$P_r^n(x)=\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n}(\bar{x}_j/\bar{x}_i)^{1/r}/n^2\right]-\left[\sum_{i=1}^{n}n_i/n^2\right], \qquad r>1, \tag{19}$$

where n_i is the number of person's richer than person i, and the ethical index suggested in Chakravarty and Chattopadhyay (1994)

$$E_\alpha^n(x)=\sum_{i=1}^{n-1}\sum_{j+i+1}^{n}[(\bar{z}_j)^\alpha-(\bar{z}_i)^\alpha]/n, \qquad 0<\alpha<1. \tag{20}$$

The vector $(\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n)$ in (20) is the income share vector $\bar{x}/n\lambda(x)$ corresponding to the distribution $\bar{x}$. In both (19) and (20), $n\in N$ and $x\in D^n$ are assumed to be arbitrary.

To develop a ranking relation involving deprivation indices, we consider some properties for an arbitrary relative deprivation index I: $D\to R^1$.

(i) Homogeneity (HM): For all $n\in N$, $x\in D^n$, $I^n(x)=I^n(cx)$, where $c>0$ is arbitrary.

(ii) Symmetry (SM): For all $n\in N$, $x\in D^n$, $I^n(x)=I^n(y)$, where y is any permutation of x.

(iii) Population Principle (PP): For all $n\in N$, $x\in D^n$, $I^n(x)=I^{mn}(y)$, where $y=(x^1, x^2, \cdots, x^m)$ and each $x^i=x$.

Condition HM means that an equiproportionate variation in all incomes does not change deprivation at the level of distribution. Thus, if income is measured in dollars instead of in pounds deprivation remains unaffected. Equivalently, we say that HM takes care of money illusion. Property SM demands symmetry of I in its arguments. It guarantees that the individuals are not distinguished by anything other than income. Clearly, PP leads us to view deprivation in average terms. Using PP, we can compare deprivation across populations.

As stated earlier, an important property of inequality indices is the Pigou–Dalton transfers principle. Kolm (1969) refers to inequality indices satisfying this property as rectifiant. To develop a similar property for deprivation indices, let us say that given $x, y \in D^n$ with $\lambda(x)=\lambda(y)$, x is obtained from y through an *adjustment program* $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ if $x=y-\alpha$, where the vector α is not indentically zero. Since $x \in D^n$, α satisfies the feasibility condition $\alpha \leq y$. Person i will be called a *donor*, *recipient* or *unaffected* according as $\alpha_i>0$, $\alpha_i<0$ or $\alpha_i=0$. Given $\lambda(x)=\lambda(y)$, we have $\sum_{i=1}^{n} \alpha_i=0$. Since α is different from the zero vector, there is a least one donor and one recipient. An adjustment program can be explained in many ways. For instance, if we say that x is obtained from y through a fiscal program that does not modify the aggregate income, then α is the corresponding tax-subsidy vector (see Fei (1981)).

We call an adjustment program to be *fair* if for each i, $1 \leq i<n$, $\alpha_i \leq \sum_{j \in S_i} \alpha_j/|S_i|$, with $<$ for at least one $i<n$, where S_i is the set of persons richer than i in y and $|S_i|$ is the number of persons in S_i. To explain this, let $\sum_{j \in S_i} \alpha_j>0$. Then *fairness* demands that person i's donation should not exceed the average donation of the persons who are richer than him. A similar explanation can be given for the case $\sum_{j \in S_j} \alpha_j \leq 0$. Our next property for a deprivation index I can now be stated.

(iv) Fair Adjustment Principle (FA): If for any $n \in N$, $y \in D^n$, x is obtained from y by a fair adjustment program, then $I^n(x)<I^n(y)$.

Thus, FA requires overall deprivation to decrease under a fair donation.

We then have

THEOREM 3. *Let $x, y \in D$ be arbitrary. Then the following statements are equivalent*:

(a) *$I(y)>I(x)$ for all relative deprivation indices I: $D \to R^1$ that satisfy* HM, SM, PP *and* FA.

(b) $x \succeq_d y$.

Theorem 3 says that an unambiguous ranking of income distributions generated by a very large class of overall deprivation indices can be obtained through the pairwise comparisons of the RDCs of the distributions.

The proof of Theorem 3 relies on the following lemma.

LEMMA 1. *Let $x, y \in D^n$, where $\lambda(x)=\lambda(y)$, be arbitrary. Then $x \succeq_d y$ holds if and only if the adjustment program $\bar{y}-\bar{x}$ is fair.*

Proof. By fairness we have $\bar{\alpha}_i \leq \sum_{j=i+1}^{n} \bar{\alpha}_j/(n-i)$ for all $i=1, 2, \cdots, n$ with $<$ for at least one $i<n$, where $\bar{\alpha}=\bar{y}-\bar{x}$. That is, $(n-i)(\bar{y}_i-\bar{x}_i) \leq \sum_{j=i+1}^{n} (\bar{y}_j-\bar{x}_i)$, which on rearrangement gives

$$\sum_{j=i+1}^{n} (\bar{x}_j-x_i) \leq \sum_{j=i+1}^{n} (\bar{y}_j-\bar{y}_i) \tag{21}$$

for all i, $1 \leq i < n$ with $<$ for at least one $i < n$. From (21), it follows that $x \succeq_d y$. The converse can be proved similarly. ∎

Proof of Theorem 3. (b)$\Rightarrow$(a). The structure of the proof parallels that of Proposition 1 of Foster (1985). Let $x \in D^m$ and $y \in D^n$. Denote the n-fold and m-fold replications of x and y by x^1 and y^1 respectively. Evidently $x^1, y^1 \in D^{mn}$. Since the RDC is population replication invariant, $\mathrm{RDC}(x^1) = \mathrm{RDC}(x)$ and $\mathrm{RDC}(y^1) = \mathrm{RDC}(y)$. Also note that $\lambda(x) = \lambda(x^1)$ and $\lambda(y) = \lambda(y^1)$. Now, let $k = \lambda(x)/\lambda(y) = \lambda(x^1)/\lambda(y^1)$. Define $x^2 = ky^1$. Since RDC is homogeneous of degree zero, $\mathrm{RDC}(x^2) = \mathrm{RDC}(y^1)$. Also, observe that $\lambda(x^2) = \lambda(ky^1) = k\lambda(y^1) = \lambda(x^1)$. Thus, x^2 and x^1 are two distributions of the given total income $mn\lambda(x^1)$ over the population size mn. Now, $x \succeq_d y$ means $x^1 \succeq_d y^1$ which is equivalent to $x^1 \succeq_d x^2$. By Lemma 1, we can then say that $\bar{x}^1$ is obtained from $\bar{x}^2$ through a fair adjustment program. Given that I satisfies FA, we have $I^{mn}(\bar{x}^2) > I^{mn}(\bar{x}^1)$. Symmetry of I shows that $I^{mn}(x^2) > I^{mn}(x^1)$. Using HM, $I^{mn}(x^2) = I^{mn}(y^1)$. Finally applying PP, we conclude that $I^n(y) = I^{mn}(y^1) > I^{mn}(x^1) = I^m(x)$.

(a)$\Rightarrow$(b). Since RDC itself is a deprivation measure satisfying HM, SM, PP and FA, this part of the theorem is true as well. ∎

If the mean income is fixed and the population size is a variable, the set of all possible income distributions is an appropriate subset D_μ of D, where $D_\mu = \{x \in D \mid \lambda(x) = \mu\}$. For indices that are consistent with the relative deprivation ordering in this case we have

Theorem 4. *Let $x, y \in D_\mu$ be arbitrary. Then the following conditions are equivalent:*

(a) $x \succeq_d y$,

(b) $I(y) > I(x)$ *for all relative deprivation indices* I: $D_\mu \to R^1$ *that satisfy SM, FA and PP.*

We can also focus our attention on fixed population, arbitrary mean income case. In such a case, the domain of the deprivation index is D^n, where $n \in N$ is fixed. Here we have

Theorem 5. *Let $x, y \in D^n$ be arbitrary. Then the relation $x \succeq_d y$ holds if and only if $I^n(y) > I^n(x)$ for all relative deprivation indices I^n: $D^n \to R^1$ that satisfy* HM, SM *and* FA.

Finally, if both the mean income and the population size are fixed, then the income space is $D^n_\mu = \{x \in D^n \mid \lambda(x) = \mu\}$, when n and μ are given. Here we have

Theorem 6. *For arbitrary $x, y \in D^n_\mu$, the relation $x \succeq_d y$ holds if and only if $I^n(y) > I^n(x)$ for all relative deprivation indices I^n: $D^n_\mu \to R^1$ that meet* SM *and* FA.

The proofs of the Theorems 4, 5 and 6 are similar to that of Theorem 3 and hence omitted.

An inequality index J defined on D is called Lorenz consistent if and only if for all x and y in D, $J(x) < J(y)$ is implied by $x \succeq_L y$. Foster (1985) has demonstrated that an inequality index $J: D \to R^1$ is Lorenz consistent if and only if it is homogeneous of degree zero, symmetric, population replication invariant and rectifiant (see also Foster and Shorrocks (1988) and Chakravarty (1990, Ch. 2)). Evidently, $x \succeq_L y$ is implied by $x \succeq_d y$ on D. Theorem 3, therefore, enables us to state the following:

THEOREM 7. *Let $x, y \in D$ be arbitrary. Then we have $J(y) > J(x)$ for all inequality indices $J: D \to R^1$ that satisfy homogeneity of degree zero, symmetry, population replication invariance and the Pigou–Dalton transfers principle whenever $I(y) > I(x)$ for all relative deprivation indices $I: D \to R^1$ that fulfill* HM, SM, PP *and* FA.

Since the converse of Theorem 1 is not true, the converse of Theorem 7 need not hold. Hence we can say that while a relative deprivation index can be regarded as a Lorenz consistent index of inequality, such an inequality index may not be an index of deprivation. The intuitive reasoning behind this is the same as one we have presented in the case of Theorem 1.

4. NUMERICAL ILLUSTRATION

The purpose of this section is to estimate the relative deprivation indices G, C, P_r, E_α using income distribution data from several countries. The dominance criterion developed in the paper has also been illustrated using the same data set. As pointed out by Kakwani (1984a), there are several problems associated with the comparablity of income distribution data from different countries owing to the differences in income units, population coverage and year of survey. In this paper, we, therefore, use data on 23 countries, selected by Kakwani (1984a), that are comparable in terms of the above three criteria.

Table 1 presents the index of GDP per capita and the values of the relative deprivation for ten decile groups for each of the 23 countries. The RDCs cross in 68 out of 253 pairwise comparisons (that is, in 26.8% cases). Out of these 68 cases, in 5.5% cases Lorenz domination was observed. This confirms that the Lorenz dominance does not imply the relative deprivation dominance.

As shown earlier, a necessary condition for the deprivation dominance is not only the Lorenz dominance, but also that the richest person in the distribution corresponding to the higher RDC must be richer than the richest person in the other distribution. Table 2 presents (i) the pair of countries for which country 1 Lorenz dominates country 2, but the RDCs intersect, and (ii) the mean income of the top decile group (treated as the income of the richest person) for each pair of countries. In 12 cases (except for Hong Kong–Costa Rica and Australia–U.K.) the condition on the richest person's income is satisfied, but the RDCs still cross. This indicates that this condition, although necessary, is far from being sufficient. In the two cases mentioned above the necessary condition itself has been violated.

In Table 3 we report estimates of the relative deprivation indices G, C, P_r and

TABLE 1. INDEX OF GDP PER CAPITA AND THE VALUES OF THE RELATIVE DEPRIVATION FOR NINE DECILE GROUPS.

Sl. No.	Country	Index of GDP per capita 1970	Decile group 1	2	3	4	5	6	7	8	9
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1	Bangladesh	4.29	0.65	0.57	0.49	0.43	0.37	0.30	0.24	0.18	0.11
2	Malawi	4.58	0.78	0.66	0.62	0.53	0.50	0.43	0.38	0.32	0.25
3	India	5.48	0.82	0.72	0.66	0.59	0.52	0.45	0.38	0.30	0.20
4	Tanzania	6.04	0.92	0.86	0.79	0.73	0.66	0.59	0.51	0.41	0.30
5	Pakistan	9.93	0.68	0.57	0.40	0.40	0.30	0.29	0.24	0.18	0.12
6	Sri Lanka	10.10	0.72	0.60	0.50	0.46	0.40	0.34	0.27	0.21	0.13
7	Philippines	10.40	0.88	0.74	0.60	0.61	0.50	0.46	0.38	0.30	0.20
8	Korea	13.70	0.69	0.61	0.50	0.48	0.40	0.34	0.27	0.20	0.12
9	Honduras	14.80	0.95	0.90	0.82	0.77	0.70	0.62	0.53	0.43	0.30
10	Turkey	18.10	0.90	0.82	0.75	0.68	0.62	0.55	0.47	0.39	0.29
11	Malaysia	18.90	0.88	0.75	0.68	0.61	0.55	0.48	0.41	0.34	0.26
12	Brazil	22.80	0.91	0.86	0.80	0.74	0.67	0.60	0.52	0.43	0.33
13	Costa Rica	25.30	0.79	0.68	0.62	0.55	0.48	0.41	0.34	0.27	0.18
14	Mexico	28.30	0.80	0.78	0.74	0.70	0.64	0.58	0.51	0.44	0.34
15	Chile	29.90	0.81	0.72	0.66	0.60	0.58	0.45	0.40	0.35	0.27
16	Hongkong	31.70	0.79	0.66	0.58	0.51	0.45	0.39	0.32	0.26	0.18
17	Japan	58.20	0.66	0.48	0.41	0.34	0.28	0.23	0.18	0.13	0.09
18	U.K.	63.50	0.77	0.59	0.50	0.42	0.35	0.28	0.20	0.14	0.08
19	New Zealand	64.30	0.66	0.52	0.45	0.38	0.32	0.26	0.21	0.15	0.10
20	Australia	75.90	0.76	0.55	0.46	0.38	0.31	0.25	0.19	0.13	0.08
21	Germany	77.80	0.78	0.64	0.56	0.49	0.42	0.35	0.28	0.21	0.13
22	Canada	89.00	0.89	0.74	0.60	0.48	0.38	0.29	0.22	0.15	0.10
23	U.S.A.	100.00	0.86	0.69	0.59	0.50	0.32	0.34	0.27	0.20	0.12

E_α. The estimation procedure of these indices from grouped data is straightforward and hence has not been discussed. From Table 3 we notice that the deprivation indices P_r and E_α are sensitive to their respective parameters. While E_α is monotonic in α, P_r is decreasing in r. Ranking of many countries produced by P_r and E_α turns out to be the same for all values of the corresponding parameters. Quite often these rankings coincide with the ranking provided by G or CV. Evidently different directional rankings are a consequence of RDC intersections.

5. CONCLUSIONS

A quantification of the extent of deprivation felt by an individual in a society is the sum of his income share shortfalls from richer individuals (Runciman, 1966). Kakwani (1984) plotted such individual deprivations against the cumulative proportions of persons to generate the relative deprivation curve. In this paper we looked at the consequence of the relative deprivation ordering, an ordering

TABLE 2. Pairs of Countries for Which the Relative Deprivation Curves Intersected but There Was Lorenz Dominance and the Mean Income of the Top Decile Groups for Each Pair.

Country 1 $\succeq_L$	Country 2	Mean incomes of the top decile group: Country 1	Country 2
(1)	(2)	(3)	(4)
Bangladesh ···	U.S.A.	1.150	29.100
India ···	Philippines	2.201	3.858
Tanzania ···	Honduras	2.821	7.045
Pakistan ···	Korea	2.691	3.836
Pakistan ···	U.S.A.	2.691	29.100
Sri Lanka ···	Germany	2.939	22.640
Philippines ···	Malaysia	3.858	7.768
Korea ···	U.S.A.	3.836	29.100
Hongkong ···	Costa Rica	10.683	8.653
Japan ···	U.K.	13.444	15.177
Japan ···	Australia	13.444	17.837
Australia ···	U.K.	17.837	15.177
U.K. ···	U.S.A.	15.177	29.100
New Zealand ···	U.S.A.	15.818	29.100

generated by two non-intersecting relative deprivation curves. It is shown that the Lorenz ordering drops out as an implication of the deprivation ordering, but the converse is not true. The class of all average relative deprivation indices consistent with the deprivation ordering is also identified. All such indices can be regarded as Lorenz consistent inequality indices, but the reverse implication is not true. The results developed in the paper are illustrated using income distribution data for 23 countries.

The deprivation indices considered in the paper satisfy a homogeneity condition—they all remain invariant under scale transformations of incomes. An alternative assumption is translation invariance, that is, the deprivation indices should not alter under equal absolute changes in all incomes. Examples of this type of indices are the Yitzhaki (1979, 1982) index $\lambda(x)G^n(x)$ and the generalized index $I_\phi: D \rightarrow R^1$ suggested by Chakravarty and Chakraborty (1984), where for all $n \in N$, $x \in D^n$,

$$I_\phi^n(x) = \phi^{-1}\left[\frac{1}{n}\left(\sum_{i=1}^{n} \phi\left(\sum_{j=i+1}^{n} \frac{\bar{x}_j - \bar{x}_i}{n}\right)\right)\right], \tag{22}$$

with $\phi: D^1 \cup \{0\} \rightarrow R^1$ being continuous, increasing, convex and $\phi(0)=0$. As an illustration, if we choose $\phi(t)=t^k$, $k \geq 1$, then

$$I_k^n(x) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=i+1}^{n}\frac{\bar{x}_j - \bar{x}_i}{n}\right)^k\right]^{1/k}. \tag{23}$$

TABLE 3. NUMERICAL ESTIMATES OF RELATIVE DEPRIVATION INDICES.

Sl. No.	Country	G	CV	P_r				E_α			
				Values of r				Values of α			
				2	3	4	5	0.25	0.50	0.75	1.00
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
1	Bangladesh	0.343	0.661	0.217	0.131	0.094	0.073	0.453	0.508	0.434	0.334
2	Malawi	0.456	1.018	0.311	0.180	0.126	0.097	0.570	0.640	0.558	0.446
3	India	0.477	0.987	0.359	0.206	0.144	0.110	0.628	0.692	0.591	0.464
4	Tanzania	0.598	1.306	0.578	0.309	0.210	0.158	0.803	0.856	0.726	0.577
5	Pakistan	0.332	0.653	0.211	0.128	0.091	0.071	0.440	0.493	0.421	0.325
6	Sri Lanka	0.376	0.737	0.253	0.151	0.107	0.083	0.502	0.559	0.476	0.368
7	Philippines	0.493	1.008	0.417	0.234	0.162	0.124	0.676	0.732	0.617	0.481
8	Korea	0.373	0.716	0.246	0.147	0.105	0.081	0.495	0.553	0.471	0.363
9	Honduras	0.625	1.352	0.726	0.374	0.249	0.186	0.879	0.913	0.763	0.603
10	Turkey	0.564	1.234	0.504	0.275	0.188	0.143	0.749	0.807	0.687	0.547
11	Malaysia	0.510	1.110	0.421	0.235	0.162	0.124	0.676	0.737	0.629	0.498
12	Brazil	0.609	1.358	0.567	0.304	0.206	0.156	0.797	0.855	0.730	0.586
13	Costa Rica	0.443	0.902	0.320	0.186	0.131	0.101	0.585	0.648	0.553	0.432
14	Mexico	0.578	1.345	0.420	0.233	0.160	0.122	0.682	0.765	0.675	0.554
15	Chile	0.498	1.104	0.358	0.204	0.142	0.109	0.624	0.696	0.604	0.484
16	Hongkong	0.425	0.874	0.304	0.177	0.125	0.096	0.563	0.623	0.533	0.416
17	Japan	0.286	0.535	0.183	0.112	0.081	0.063	0.393	0.437	0.369	0.281
18	U.K.	0.340	0.613	0.252	0.150	0.107	0.083	0.489	0.533	0.443	0.333
19	New Zealand	0.313	0.590	0.200	0.122	0.087	0.068	0.424	0.472	0.401	0.306
20	Australia	0.316	0.578	0.231	0.138	0.098	0.076	0.457	0.498	0.414	0.311
21	Germany	0.394	0.756	0.287	0.169	0.119	0.092	0.543	0.596	0.503	0.385
22	Canada	0.389	0.707	0.383	0.216	0.150	0.115	0.613	0.642	0.520	0.385
23	U.S.A.	0.406	0.769	0.344	0.197	0.138	0.106	0.591	0.634	0.526	0.398

If $k=1$, I_k^n is the Yitzhaki index. As k increases, I_k^n becomes more sensitive to the deprivation of the poorer persons. As $k \to \infty$, $I_k^n(x) \to \max_i\left(\sum_{j=i+1}^{n} \frac{(\bar{x}_j - \bar{x}_i)}{n}\right) = \sum_{j=2}^{n} \frac{(\bar{x}_j - \bar{x}_1)}{n}$, the deprivation of the poorest person.

An interesting exercise now will be the development of a ranking criterion that will order income distributions in exactly the same way as that generated by all translation invariant deprivation indices. This is left as an open research program.

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NOTES

1. A function $h^n: D^n \rightarrow R^1$ is called S-concave if $h^n(Bx) \geq h^n(x)$ for all $x \in D^n$, where B is any bi-stochastic matrix of order n. A square matrix of order n is said to be bi-stochastic if all its entries are non-negative and each of its rows and columns sums to one. h^n is strictly S-concave if the weak inequality is replaced by a strict inequality whenever Bx is not a permutation of x.
2. We are grateful to the referee for this simple proof.

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