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EXISTENCE OF OLIGOPOLISTIC EQUILIBRIUM IN MULTIPLE MARKETS ECONOMY

Yuuji SATOH[1]

Abstract: An oligopolistic market model is considered where n firms supply a single homogeneous commodity to m different markets. In contrast to perfect competition, the model is oligopolistic in the sense that each firm decides not only the total production quantity of the commodity and its allocation to different markets, but also offering prices for individual markets. Consumers in one market would buy the commodity with the least price in that market. Furthermore, consumers may move from one market to another when the price discrepancy between the two markets is sufficiently large. The demand of one market is given by a linear combination of the least prices of m markets plus a potential demand in that market. Assuming that cost functions of all firms are strictly convex, it is shown that there exists a unique set of prices over m different markets, which achieves the optimal pricing strategy for all firms and balances the demand and supply in all individual markets.

1. INTRODUCTION

When perfect competition is present, the price is given by the market. Accordingly the market strategy for a participating firm is limited to decision of its supply quantity to the market. In a monopolistic case, the monopolist optimizes its supply quantity and price subject to the market demand. Strategic issues concerning oligopolistic competition are more complex and sophisticated. Since a limited number of firms would compete against each other, strategic decisions of one firm would affect those of others and vice versa.

This paper deals with an oligopolistic market model where n firms supply a single homogeneous commodity to m different markets. The decision problem of one firm would be to determine the total production quantity of the commodity, its allocation to m different markets, and offering prices for individual markets in order to maximize its overall profit. Since a single homogeneous commodity is supplied by all firms, consumers in one market would buy the commodity from the firm with the least price. This price is called the winning price of that market.

Each market has a potential demand for the commodity. By recognizing this demand, all firms realize upper bounds for setting their offering prices in the market. While different upper bounds may be perceived by different firms, such different upper bounds will converge to a common value in persuit of individual

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optimal pricing strategies by the firms, since only the winning price is accepted in each market. Hence, without loss of generality, we can assume that all firms recognize a common upper bound, which we call the guide price of the market. Given a set of guide prices for all markets, the firms set offering prices over m different markets with optimistic expectation that they would be a winner as long as their offering prices do not exceed the guide prices.

All markets are assumed to be substitutable to each other in the sense that consumers may move from one market to another when the price discrepancy between the two markets is sufficiently large. More specifically, we assume that the demand of one market is given by a linear combination of the winning prices of m markets plus a potential demand in that market. In this context, if cost functions of all firms are strictly convex, it will be shown that there exists a unique set of guide prices over m different markets, which achieves the optimal pricing strategy for all firms and balances the demand and supply in all individual markets. Lederer (1989) has shown a similar existence theorem but only with non-substitutable linear demand functions. The main theorem of this paper generalizes the result of his.

The structure of this paper is as follows. Section 2 introduces notation employed throughout the paper. The formal model is described in Section 3 and the main theorem proving the existence of an oligopolistic equilibrium is given in Section 4. In Section 5, some concluding remarks are given with brief discussion of possible extensions of this research.

2. NOTATION

$\mathbf{M}=\{1, 2, \cdots, i, \cdots, m\}$ is the set of markets and $\mathbf{N}=\{1, 2, \cdots, j, \cdots, n\}$ is the set of firms. Both sets are finite. $\mathbf{p}_j=(p_j^1, \cdots, p_j^m)^T$, $\mathbf{q}_j=(q_j^1, \cdots, q_j^m)^T$ are a strategic price vector and a strategic production vector of firm j respectively. Let $\mathbf{p}=(p^1, \cdots, p^m)^T$ be the market price vector. $F_j(\mathbf{q}_j)$ is a cost function of firm j and is strictly convex with respect to $\mathbf{q}_j$. We define $\mathbf{D}(\mathbf{p})=(D^1(\mathbf{p}), \cdots, D^m(\mathbf{p}))^T$ as the total demand vector and $\mathbf{d}$ as the potential demand vector when $\mathbf{p}=\mathbf{0}$. Finally, we employ $\pi_j(\mathbf{p}_j, \mathbf{q}_j)$ as a profit function of firm j.

3. THE MODEL

We shall consider an oligopolistic market model in which all firms ($j=1, 2, \cdots, n$) supply a same commodity to multi markets ($i=1, 2, \cdots, m$) simultaneously. Contrary to a competitive case, it is assumed that each firm decides not only production schedule concerning total amount of the product and its allocation to each market, but also pricing strategy at each market. It will be likely to exist some different prices offered by firms in each market. Since the same commodity is supplied, however, consumers in the market will buy one with the least price, which we say the winning market price. Moreover, if there is another market where the winning market price is much lower than that of the market, a part of consumers may flow out to such market, and if the winning market price of the market is much lower than that of other's, a part of them may flow into the market. It will be also natural to expect that on the amount of demand in a market, winning prices in other markets totally have less effects than that in the market under consideration.

For the sake of simplicity, we approximate the demand vector $\mathbf{D}(\mathbf{p})$ by a linear function of the winning market price $\mathbf{p}$ as follows:

Definition 1. Demand Function

$$\mathbf{D}(\mathbf{p}) \equiv \mathbf{A}\mathbf{p} + \mathbf{d} \tag{1}$$

where $d^i > 0$ $(i \in \mathbf{M}, \mathbf{d} \equiv (d^1, \cdots, d^m)^T)$ *denotes the potential demand of the i-th market.*

The above disscussion directly leads to

Condition 1. $a_{ii} < 0,\ a_{ij} \geqq 0\ (i \neq j)$

Condition 2.a. $|a_{ii}| > \sum_{j \neq i} a_{ij}\ (i \in \mathbf{M})$

2.b. $|a_{jj}| > \sum_{i \neq j} a_{ij}\ (j \in \mathbf{N})$

on $\mathbf{A} \equiv (a_{ij})$. First, Conditions 1 and 2.b mean that the demands between markets satisfy gross-substitutability in the same sense with that on commodities. Second, Condition 2.a slightly strengthens the substitutability in the sense that if the winning price in all markets increase $\Delta p > 0$ uniformly, then the amount of demand at each market decreases. On this point, Lederer's formulation (1989) can be interpreted as $\mathbf{A} = k\mathbf{I}$ $(k < 0)$, and is a special case of our model, in which the demands are non-substitutable.

Since the demand vector $\mathbf{D}(\mathbf{p})$ makes sense only if $\mathbf{D}(\mathbf{p}) \geqq \mathbf{0}$, we can restrict the regions $\mathscr{P}$, $\mathscr{D}$ of meaningful price and demand vectors as follows:

$$\mathscr{P} \equiv \{\mathbf{p} \in \mathbf{R}^m_+ \mid -\mathbf{A}\mathbf{p} \leqq \mathbf{d}\},$$

$$\mathscr{D} \equiv \{\mathbf{x} \in \mathbf{R}^m_+ \mid \mathbf{x} \in \mathbf{D}(\mathscr{P})\}.$$

It directly follows from Conditions 1 and 2 on $\mathbf{A}$ that both $\mathscr{P}$ and $\mathscr{D}$ are compact convex polyhedra in $\mathbf{R}^m_+$ including $\mathbf{0}$. We shall further notice that $\mathbf{D}(\mathbf{p})$ gives a one-to-one bi-continuous correspondence beteween $\mathscr{P}$ and $\mathscr{D}$. We denote the inverse mapping as $\mathbf{D}^{-1}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x} - \mathbf{d})$, $\mathbf{x} \in \mathscr{D}$. It is easy to see that if an $\mathbf{x}$ belongs to the Parato frontier $\mathbf{PF} \equiv \{\mathbf{x} \in \mathscr{D} \mid \mathbf{y} \notin \mathscr{D} \text{ if } \mathbf{y} \geq \mathbf{x}\}$ of $\mathscr{D}$, then there corresponds a $\mathbf{p} \in \mathscr{P}$ such that $p^i = 0$ for some $i \in \mathbf{M}$, where $\mathbf{y} \geq \mathbf{x}$ implies $\mathbf{y} \geqq \mathbf{x}$ and $\mathbf{y} \neq \mathbf{x}$. Next the profit function of each firm is specified.

Definition 2. Profit Function

$$\pi_j(\mathbf{p}_j, \mathbf{q}_j) \equiv \mathbf{p}_j^T \mathbf{q}_j - F_j(\mathbf{q}_j). \tag{2}$$

The profit function consists of the expected revenue $\mathbf{p}_j^T\mathbf{q}_j$ and the determined cost $F_j(\mathbf{q}_j)$. We assume that the cost function F_j is made up of production cost and transportation cost which are both increasing to scale, and is strictly convex with respect to $\mathbf{q}_j$.

According to a usual rule adopted in non-cooperative game, we assume that each firm decides independently its production schedule and pricing strategy which is announced to each market at the same time. This requires that an initial market price vector $\hat{\mathbf{p}}$ is given in advance. Now we can formulate the profit maximization problem of firm j.

«**Problem** j» For any given initial market price vector $\hat{\mathbf{p}} \in \mathscr{P}$, maximize $\pi_j(\mathbf{p}_j, \mathbf{q}_j)$ subject to

$$\mathbf{0} \leqq \mathbf{p}_j \leqq \hat{\mathbf{p}}, \qquad \mathbf{0} \leqq \mathbf{q}_j \leqq \mathbf{D}(\mathbf{p}_j). \tag{3}$$

Note that if (a) $\hat{p}^i < p_j^i$ and $0 < q_j^i$ for some $i \in \mathbf{M}$, or (b) $D^i(\mathbf{p}_j) < q_j^i$ for some $i \in \mathbf{M}$, then we have (a) a situation that the firm can not sell the product q_j^i with the strategic price p_j^i, because the pre-described market rule makes consumers expect that they will be able to buy the commodity with a price not exceeding p^i, or (b) an excess supply $q_j^i - D^i(\mathbf{p}_j) > 0$, we shall further emphasize that $\mathbf{p}_j$ in the inequality $\mathbf{0} \leqq \mathbf{q}_j \leqq \mathbf{D}(\mathbf{p}_j)$ can not be replaced by $\hat{\mathbf{p}}$ since each firm optimistically estimate its strategic pricing is successful and, as a result, no one expects $\hat{\mathbf{p}}$ will always be realized. These will at least partly explain the reason why the constraint (3) must be imposed.

It follows from the definition of demand function (1) and the constraint (3) that the feasible region $\mathrm{T}(\hat{\mathbf{p}})$ of «Problem j» is a compact convex polyhedron in $\mathbf{R}_+^{2m}$ represented by

$$\mathrm{T}(\hat{\mathbf{p}}) = \{(\mathbf{p}_j, \mathbf{q}_j) \in \mathbf{R}_+^{2m} \mid \mathbf{p}_j \leqq \hat{\mathbf{p}},\ \mathbf{q}_j - \mathbf{A}\mathbf{p}_j \leqq \mathbf{d}\}.$$

Since $(\mathbf{0}, \mathbf{0}) \in \mathrm{T}(\hat{\mathbf{p}})$, $\mathrm{T}(\hat{\mathbf{p}})$ is nonempty for any $\hat{\mathbf{p}} \in \mathscr{P}$.

THEOREM 1. *For any $\hat{\mathbf{p}} \in \mathscr{P}$, there exists an unique optimal solution $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}), \bar{\mathbf{q}}_j(\hat{\mathbf{p}}))$ of* «**Problem** j»*, where we put $\bar{p}_j^i(\hat{\mathbf{p}}) = \hat{p}^i$ whenever $\bar{q}_j^i(\hat{\mathbf{p}}) = 0$.*

Proof. See Appendix 1.

THEOREM 2. *The optimal solution $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}), \bar{\mathbf{q}}_j(\hat{\mathbf{p}}))$ is continuous with respect to $\hat{\mathbf{p}} \in \mathscr{P}$.*

Proof. See Appendix 2.

Each firm decides its production schedule $\bar{\mathbf{q}}_j(\hat{\mathbf{p}})$ and pricing strategy $\bar{\mathbf{p}}_j(\hat{\mathbf{p}})$ under an expectation that its announced price will be the least price of each market, and its product will be sold out at every market. First of all, however, there is no assurance that its price vector $\bar{\mathbf{p}}_j(\hat{\mathbf{p}})$ constitutes the winning market price vector $\bar{\mathbf{p}}(\hat{\mathbf{p}}) \equiv (\min_{j \in \mathbf{N}} \bar{p}_j^1(\hat{\mathbf{p}}), \cdots, \min_{j \in \mathbf{N}} \bar{p}_j^m(\hat{\mathbf{p}}))$. Furthermore, if there are several firms which offer the same winning market price in some market, there remains the possibility that the total amount of their products will exceed the demand in the market. That is optimal solutions of «**Problem** j» ($j \in \mathrm{N}$) under $\hat{\mathbf{p}} \in \mathscr{P}$ do not always satisfy the following feasibility conditions:

$$\sum_{j\in\bar{\mathbf{N}}^i} \bar{q}^i_j(\hat{\mathbf{p}}) \leqq D^i(\bar{\mathbf{p}}(\hat{\mathbf{p}}))\ (i\in\mathbf{M}) \quad \text{and} \quad \bar{q}^i_j(\hat{\mathbf{p}})=0 \quad (i\in\mathbf{M}, j\in\mathbf{N}\backslash\bar{\mathbf{N}}^i)\,, \tag{4}$$

where

$$\bar{\mathbf{N}}^i=\{j\in\mathbf{N}\,|\,\bar{p}^i_j(\hat{\mathbf{p}})=\bar{p}^i(\hat{\mathbf{p}})\ i\in\mathbf{M}\}.$$

4. EXISTENCE OF AN EQUILIBRIUM

Previous discussion makes it clear that the optimal solutions of firms do not always satisfy the feasibility condition (4) as a total. In this section, we will show the existence of an initial price vector $\hat{\mathbf{p}}^*\in\mathscr{P}$ under which the optimal solutions $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*),\ \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*))$ $(j\in\mathbf{N})$ satisfy the following two equalities:

$$\hat{\mathbf{p}}^*=\bar{\mathbf{p}}(\hat{\mathbf{p}}^*)\,, \tag{5}$$

$$\sum_{j\in\mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*)=\mathbf{D}(\bar{\mathbf{p}}(\hat{\mathbf{p}}^*))\,. \tag{6}$$

Note that (5) implies $\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)=\hat{\mathbf{p}}^*$ and we have $\bar{\mathbf{N}}^i=\mathbf{N}$ $(i\in\mathbf{M})$. Since (5) and (6) state that if the initial price vector is appropriately given then the same market price vector is realized under which balancing equality of demand and supply holds at every market, there is no insentive to alternate the initial price, and we say the set of $\hat{\mathbf{p}}^*$ and $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*),\ \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*))=(\hat{\mathbf{p}}^*,\ \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*))$ $(j\in\mathbf{N})$ the equilibrium.
Recalling that $\mathscr{D}\subset\mathbf{R}^m_+$ is compact convex, we have

$$\text{for } {}^\forall\hat{\mathbf{p}}\in\mathscr{P}\,,\quad \exists\lambda(\hat{\mathbf{p}})\equiv\max\left\{\lambda\in[0,\,1]\,\Big|\,\lambda\sum_{j\in\mathbf{N}}\bar{\mathbf{q}}_j(\hat{\mathbf{p}})\in\mathscr{D}\right\}. \tag{7}$$

By THEOREM 2, $\sum_{j\in\mathbf{N}}\bar{\mathbf{q}}_j(\mathbf{p})$ is continuous on $\mathscr{P}$, and we have the following lemma.

LEMMA. *$\lambda(\hat{\mathbf{p}})$ is continuous on $\mathscr{P}$.*

Proof. Since $\bar{\mathbf{q}}_j(\hat{\mathbf{p}})$ is the solution of «**Problem** j», $\bar{\mathbf{q}}_j(\hat{\mathbf{p}})\geqq\mathbf{0}$.

- Case 1: If $\sum_{j\in\mathbf{N}}\bar{\mathbf{q}}_j(\hat{\mathbf{p}})\in\mathscr{D}$,
  then from the above definition (7), $\lambda(\hat{\mathbf{p}})=1$.
- Case 2: If $\sum_{j\in\mathbf{N}}\bar{\mathbf{q}}_j(\hat{\mathbf{p}})\notin\mathscr{D}$,
  then $\mathbf{p}'$ satisfying $\mathbf{0}\leqq\sum_{j\in\mathbf{N}}\bar{\mathbf{q}}_j(\hat{\mathbf{p}})=\mathbf{A}\mathbf{p}'+\mathbf{d}$ does not belong to $\mathscr{P}$, which implies $p'^i<0$ for some $i\in\mathbf{M}$. Here we define the solution $\mathbf{p}^\circ$ of $\mathbf{A}\mathbf{p}+\mathbf{d}=0\cdot\sum_{j\in\mathbf{N}}\bar{\mathbf{q}}_j(\hat{\mathbf{p}})=\mathbf{0}$. Since $\mathbf{0}\in\mathscr{D}$ and $\mathbf{d}>\mathbf{0}$, $\mathbf{p}^\circ>\mathbf{0}$. We easily have $\lambda\sum_{j\in\mathbf{N}}\bar{\mathbf{q}}_j(\hat{\mathbf{p}})=\mathbf{A}\{(1-\lambda)\mathbf{p}^\circ+\lambda\mathbf{p}'\}+\mathbf{d}$, for ${}^\forall\lambda\in[0,\,1]$. There uniquely exists $\lambda^*\in[0,\,1]$ for which $\min_{i\in M}\{(1-\lambda)p^{\circ i}+\lambda p'^i\}=0$ holds. Since $\lambda\leqq\lambda^*$ implies $(1-\lambda)\mathbf{p}^\circ+\lambda\mathbf{p}'\geqq\mathbf{0}$, we see that $\lambda(\hat{\mathbf{p}})=\lambda^*$. It follows from THEOREM 1 that the solution $\mathbf{p}'$ of $\mathbf{A}\mathbf{p}+\mathbf{d}=\sum_{j\in\mathbf{N}}\bar{\mathbf{q}}_j(\hat{\mathbf{p}})$ is continuous with respect to $\hat{\mathbf{p}}$, and so is λ^*. If the distance between $\sum_{j\in\mathbf{N}}\ \bar{\mathbf{q}}_j(\hat{\mathbf{p}})$ and $\mathscr{D}$ tends to 0, we see that $\min_{i\in M}p'^i$ keeps negative but tends to 0, which implies the convergence of $\lambda(\hat{\mathbf{p}})$ to 1. □

Definition 3. $\mathbf{S}: \mathscr{P} \to \mathscr{D}$

For $^{\forall}\hat{\mathbf{p}} \in \mathscr{P}$, $\mathbf{S}(\hat{\mathbf{p}}) = \lambda(\hat{\mathbf{p}}) \sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}})$.

Notice that LEMMA ensures continuity of $\mathbf{S}$.

Definition 4. $\mathbf{G}: \mathscr{D} \to \mathscr{D}$

For $^{\forall}\mathbf{x} \in \mathscr{D}$, $\mathbf{G}(\mathbf{x}) = \mathbf{S}(\mathbf{D}^{-1}(\mathbf{x}))$.

Since $\mathbf{S}$ and $\mathbf{D}^{-1}$ are continuous on $\mathscr{P}$ and on $\mathscr{D}$ respectively, we easily see that the composite mapping $\mathbf{G}$ is also continuous on $\mathscr{D}$.

THEOREM 3. *There exists* $\hat{\mathbf{p}}^* \in \mathscr{P}$ *such that* $\mathbf{D}(\hat{\mathbf{p}}^*) = \mathbf{S}(\hat{\mathbf{p}}^*)$.

Proof. Since $\mathscr{D}$ is compact convex and $\mathbf{G}$ is continuous, by Brouwer Fixed Point Theorem, there exists $\mathbf{x}^* \in \mathscr{D}$ such that $\mathbf{G}(\mathbf{x}^*) = \mathbf{x}^*$, so we have $\mathbf{S}(\mathbf{D}^{-1}(\mathbf{x}^*)) = \mathbf{x}^*$. Then there corresponds to $\mathbf{x}^*$ a $\hat{\mathbf{p}}^* \in \mathscr{P}$ such that $\hat{\mathbf{p}}^* = \mathbf{D}^{-1}(\mathbf{x}^*)$, for which we have $\mathbf{S}(\hat{\mathbf{p}}^*) = \mathbf{x}^* = \mathbf{D}(\hat{\mathbf{p}}^*)$. □

THEOREM 4. *For* $\hat{\mathbf{p}}^* \in \mathscr{P}$ *satisfying* $\mathbf{D}(\hat{\mathbf{p}}^*) = \mathbf{S}(\hat{\mathbf{p}}^*)$, $\mathbf{D}(\hat{\mathbf{p}}^*) = \sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*)$.

Proof. Assume that $\sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*) \notin \mathscr{D}$. Because of the nonsingularity of $\mathbf{A}$, there exists $\mathbf{p}' \notin \mathscr{P}$ such that $\sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*) = \mathbf{A}\mathbf{p}' + \mathbf{d}$. Since the latter half of (3) implies $\mathbf{0} \leqq \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*)$ $(j \in \mathbf{N})$, the above assumption, $\sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*) \notin \mathscr{D}$, means $\mathbf{S}(\hat{\mathbf{p}}^*) \in \mathbf{PF}$, namely $\mathbf{D}(\hat{\mathbf{p}}^*) \in \mathbf{PF}$. Then there exists $i \in \mathbf{M}$ such that $\hat{p}^{*i} = 0$, for which $0 < d^i \leqq D^i(\hat{\mathbf{p}}^*)$ and $\sum_{j \in \mathbf{N}} \bar{q}_j^i(\hat{\mathbf{p}}^*) = 0$[2] hold. So we have

$$\sum_{j \in \mathbf{N}} \bar{q}_j^i(\hat{\mathbf{p}}^*) - D^i(\hat{\mathbf{p}}^*) < 0 . \tag{8}$$

On the other hand, we have

$$\sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*) - \mathbf{D}(\hat{\mathbf{p}}^*) = \sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*) - \mathbf{S}(\hat{\mathbf{p}}^*) = (1 - \lambda(\hat{\mathbf{p}}^*)) \sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*) \geqq \mathbf{0} . \tag{9}$$

These two inequalities (8), (9) are incompatible with each other. Hence $\sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*) \in \mathscr{D}$. Therefore by (7), we have $\lambda(\hat{\mathbf{p}}^*) = 1$, which gives $\mathbf{S}(\hat{\mathbf{p}}^*) = \sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*)$. Thus we have $\mathbf{D}(\hat{\mathbf{p}}^*) = \sum_{j \in \mathbf{N}} \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*)$. □

THEOREM 4 states that there exists an initial market price vector $\hat{\mathbf{p}}^* \in \mathscr{P}$ under which the optimal production schedules $\bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*) \in \mathscr{D}$ $(j \in \mathbf{N})$ satisfy (6).

THEOREM 5. *For* $\hat{\mathbf{p}}^* \in \mathscr{P}$ *satisfying* $\mathbf{D}(\hat{\mathbf{p}}^*) = \mathbf{S}(\hat{\mathbf{p}}^*)$, $\bar{\mathbf{p}}(\hat{\mathbf{p}}^*) = \hat{\mathbf{p}}^*$.

Proof. It suffices to prove $\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*) = \hat{\mathbf{p}}^*$ for all $j \in \mathbf{N}$. Suppose that there exists $j \in \mathbf{N}$ for which $\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*) \leq \hat{\mathbf{p}}^*$, that is, $\mathbf{M}' \equiv \{i \in \mathbf{M} \mid \bar{p}_j^i(\hat{\mathbf{p}}^*) < \hat{p}^{*i}\}$ is nonempty. Since we

[2] If $0 < \bar{q}_j^i(\hat{\mathbf{p}}^*)$ then $F_j(\bar{\mathbf{q}}'_j(\hat{\mathbf{p}}^*)) < F_j(\bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*))$ for $\bar{\mathbf{q}}'_j \equiv (\bar{q}_j^1, \cdots, \bar{q}_j^{(i-1)}, 0, \bar{q}_j^{(i+1)}, \cdots, \bar{q}_j^m)^T$. This leads to $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*), \bar{\mathbf{q}}'_j(\hat{\mathbf{p}}^*)) \in \mathrm{T}(\hat{\mathbf{p}}^*)$ and $\pi_j(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*), \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*)) = \bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)^T \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*) - F_j(\bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*)) = \bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)^T \bar{\mathbf{q}}'_j(\hat{\mathbf{p}}^*) - F_j(\bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*)) < \bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)^T \bar{\mathbf{q}}'_j(\hat{\mathbf{p}}^*) - F_j(\bar{\mathbf{q}}'_j(\hat{\mathbf{p}}^*)) = \pi_j(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*), \bar{\mathbf{q}}'_j(\hat{\mathbf{p}}^*))$ which contradicts to the optimality of $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*), \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*))$.

have $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)-\hat{\mathbf{p}}^*)^T\,\mathbf{A}(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)-\hat{\mathbf{p}}^*)<0$[3] from Condition 2, we see that there exists $i\in\mathbf{M}'$ such that $D^i(\hat{\mathbf{p}}^*)<D^i(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*))$. Now we suppose that $\bar{q}^i_j(\hat{\mathbf{p}}^*)<D^i(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*))$. If firm j changes the strategy from $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*),\ \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*))$ to $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)+\Delta p^i\mathbf{e}^i,\ \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*))$ by a small amount of $\Delta p^i>0$, then we see that $\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)\lneqq\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)+\Delta p^i\mathbf{e}^i\leqq\hat{\mathbf{p}}^*$, $\bar{q}^i_j(\hat{\mathbf{p}}^*)\leqq D^i(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)+\Delta p^i\mathbf{e}^i)<D^i(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*))$ and $\bar{q}^k_j(\hat{\mathbf{p}}^*)\leqq D^k(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*))\leqq D^k(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)+\Delta p^i\mathbf{e}^i)$ for all $k\neq i$, where $\mathbf{e}^i$ denotes the i-th unit vector in $\mathbf{R}^m$. Therefore the strategy $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)+\Delta p^i\mathbf{e}^i,\ \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*))$ is feasible for firm j. By noting that $\bar{q}^i_j=0$ implies $\bar{p}^i_j=\hat{p}^{*i}$, we directly see $\bar{q}^i_j>0$. This leads to $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)+\Delta p^i\mathbf{e}^i)^T\bar{\mathbf{q}}_j>\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)^T\bar{\mathbf{q}}_j$, a contradiction to the optimality of $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*),\ \bar{\mathbf{q}}_j(\hat{\mathbf{p}}^*))$. Thus we have $\bar{q}^i_j(\hat{\mathbf{p}}^*)=D^i(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*))>D^i(\hat{\mathbf{p}}^*)$. On the other hand by THEOREM 4, $\bar{q}^i_j(\hat{\mathbf{p}}^*)\leqq\sum_{j\in\mathbf{N}}\bar{q}^i_j(\hat{\mathbf{p}}^*)=D^i(\bar{\mathbf{p}}(\hat{\mathbf{p}}^*))$, we have a contradiction. □

5. CONCLUDING REMARKS

In this paper, an oligopolistic market model is considered where n firms supply a single homogeneous commodity to m different markets. In contrast to perfect competition, the model is oligopolistic in the sense that each firm decides not only the total production quantity of the commodity and its allocation to different markets, but also offering prices for individual markets.

Since a single homogeneous commodity is supplied by all firms, consumers in one market would buy the commodity from the firm with the least price, called the winning price of the market. All markets are substitutable to each other, i.e. consumers may move from one market to another when the price discrepancy between the two markets is sufficiently large. Each market has a potential demand for the commodity. By recognizing this demand, all firms realize upper bounds for setting their offering prices in the market. While different upper bounds may be perceived by different firms, it will be natural to expect that such different upper bounds will converge to a common value in pursuit of individual optimal pricing strategies by the firms, since only the winning price is accepted in each market. Hence, without loss of generality, we can assume that all firms recognize a common upper bound, called the guide price of the market. Given a set of guide prices for all markets, the firms set offering prices over m different markets with optimistic expectation that they would be a winner as long as their offering prices do not exceed the guide prices.

The main theorem of this paper reveals that there exists a unique set of guide prices over m different markets, which achieves the optimal pricing strategy for all firms and balances the demand and supply in all individual markets. Therefore the common guide price in this paper can be regarded as a variation of the rational expectation price, and as a result, all the firms will assign the same price with each

[3] Since matrix $\mathbf{A}$ satisfies Conditions 1 and 2, and that $\mathbf{A}+\mathbf{A}^T$ is symmetric, strictly diagonal dominant and has negative diagonal element. Then by Gerschgorin Circle Theorem, all the eigen values of $\mathbf{A}+\mathbf{A}^T$ are negative, and it is negative definite: for $^{\forall}\mathbf{x}\in\mathbf{R}^m$, $\mathbf{x}^T(\mathbf{A}+\mathbf{A}^T)\mathbf{x}=2\mathbf{x}^T\mathbf{A}\mathbf{x}<0$. Thus for $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)-\hat{\mathbf{p}}^*)\in\mathbf{R}^m$, $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)-\hat{\mathbf{p}}^*)^T\mathbf{A}(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}^*)-\hat{\mathbf{p}}^*)<0$.

other. This existing theorem for an oligopolistic equilibrium leads to the following open questions:

1: Would it be possible to implement a practical process of attaining the oligopolistic equilibrum ?

2: Would a similar existence theorem be present when a demand function is nonlinear but has a Jacobian matrix satisfying Conditions 1 and 2 regarding $\mathscr{P}$?

3: Would it be possible to extend the model where multiple firms supply a common set of multiple commodities to different markets which are still substitutable to each other ?

Appendix 1 (Proof of THEOREM 1).

1: the existence of the optimal solution

By the definition of profit function π_j (2), at any $(\mathbf{p}, \mathbf{q}) \in \mathrm{T}(\hat{\mathbf{p}})$, $\mathbf{p}_j^T \mathbf{q}_j$ and $F_j(\mathbf{q}_j)$ are continuous, therefore π_j is also continuous. And further, the feasible region $\mathrm{T}(\hat{\mathbf{p}})$ of «**Problem** j» is compact, convex and nonempty, hence π_j has at least one maximum, namely the optimal solution $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}), \bar{\mathbf{q}}_j(\hat{\mathbf{p}}))$.

2: the uniqueness of the optimal solution

For $\exists \hat{\mathbf{p}} \in \mathscr{P}$, we supppose that «**Problem** j» has two different optimal solutions $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}), \bar{\mathbf{q}}_j(\hat{\mathbf{p}}))$ and $(\bar{\bar{\mathbf{p}}}_j(\hat{\mathbf{p}}), \bar{\bar{\mathbf{q}}}_j(\hat{\mathbf{p}}))$. In the rest of proof we shall abbreviate them $(\bar{\mathbf{p}}_j, \bar{\mathbf{q}}_j)$ and $(\bar{\bar{\mathbf{p}}}_j\ \bar{\bar{\mathbf{q}}}_j)$ respectively.

(a) $\bar{\mathbf{p}}_j = \bar{\bar{\mathbf{p}}}_j (\equiv \mathbf{p}_j^0)$:

We have $\pi_j(\bar{\mathbf{p}}_j, \bar{\mathbf{q}}_j) = \mathbf{p}_j^{0T} \bar{\mathbf{q}}_j - F_j(\bar{\mathbf{q}}_j)$, $\pi_j(\bar{\bar{\mathbf{p}}}_j, \bar{\bar{\mathbf{q}}}_j) = \mathbf{p}_j^{0T} \bar{\bar{\mathbf{q}}}_j - F_j(\bar{\bar{\mathbf{q}}}_j)$. Under the fixed $\mathbf{p}_j^0$, inner product $\mathbf{p}_j^{0T} \mathbf{q}_j$ is linear for $\mathbf{q}_j$ and by the definition, $F_j(\mathbf{q}_j)$ is strictly convex. Thus for $\bar{\mathbf{q}}_j$, $\bar{\bar{\mathbf{q}}}_j$ satisfying $\bar{\mathbf{q}}_j \neq \bar{\bar{\mathbf{q}}}_j$, we have $\pi_j(\mathbf{p}_j^0, \bar{\mathbf{q}}_j) = \pi_j(\mathbf{p}_j^0, \bar{\bar{\mathbf{q}}}_j) < \pi_j(\mathbf{p}_j^0, \lambda \bar{\mathbf{q}}_j + (1-\lambda) \bar{\bar{\mathbf{q}}}_j)$ and $\mathbf{0} \leqq \lambda \bar{\mathbf{q}}_j + (1-\lambda) \bar{\bar{\mathbf{q}}}_j \leqq \mathbf{D}(\mathbf{p}_j^0)$ for $^\forall \lambda \in (0, 1)$. This contradicts that both $(\bar{\mathbf{p}}_j, \bar{\mathbf{q}}_j)$ and $(\bar{\bar{\mathbf{p}}}_j, \bar{\bar{\mathbf{q}}}_j)$ are the optimal solutions of «**Problem** j».

(b) $\bar{p}_j^i \neq \bar{\bar{p}}_j^i$:

Without loss of generality, we can suppose $\bar{\bar{p}}_j^i < \bar{p}_j^i$. By the constraints (3) of «**Problem** j» $\bar{\bar{p}}_j^i < \bar{p}_j^i$ implies $\bar{\bar{p}}_j^i < \hat{p}^i$. If $\bar{\bar{q}}_j^i < D^i(\bar{\bar{\mathbf{p}}}_j)$, then we can increase only the i-th component of $\bar{\bar{\mathbf{p}}}_j$ from $\bar{\bar{\mathbf{p}}}_j$ to $\bar{\bar{\mathbf{p}}}_j + \varDelta p^i \mathbf{e}^i$ ($\leqq \hat{\mathbf{p}}$) by a small amount of $\varDelta p^i > 0$. Since $\bar{\bar{p}}_j^i < \hat{p}^i$ implies $0 < \bar{\bar{q}}_j^i$, we have $0 < \bar{\bar{q}}_j^i \leqq D^i(\bar{\bar{\mathbf{p}}}_j + \varDelta p^i \mathbf{e}^i) < D^i(\bar{\bar{\mathbf{p}}}_j)$. On the other markets, by the Conditions 1 and 2 of matrix $\mathbf{A}$, $\bar{\bar{q}}_j^k \leqq D^k(\bar{\bar{\mathbf{p}}}_j) \leqq D^k(\bar{\bar{\mathbf{p}}}_j + \varDelta p^i \mathbf{e}^i)$ for $^\forall k \neq i$. Therefore $(\bar{\bar{\mathbf{p}}}_j + \varDelta p^i \mathbf{e}^i, \bar{\bar{\mathbf{q}}}_j)$ satisfies (3) and is feasible. However, $\pi_j(\bar{\bar{\mathbf{p}}}_j, \bar{\bar{\mathbf{q}}}_j) < \pi_j(\bar{\bar{\mathbf{p}}}_j + \varDelta p^i \mathbf{e}^i, \bar{\bar{\mathbf{q}}}_j)$ contradicts that $(\bar{\bar{\mathbf{p}}}_j, \bar{\bar{\mathbf{q}}}_j)$ is the optimal solution. Consequently, $\bar{\bar{p}}_j^i < \bar{p}_j^i$ means $0 < \bar{\bar{q}}_j^i = D^i(\bar{\bar{\mathbf{p}}}_j)$. Since $\bar{q}_j^i \leqq D^i(\bar{\mathbf{p}}_j)$ and $\bar{\bar{q}}_j^i = D^i(\bar{\bar{\mathbf{p}}}_j)$ imply $\bar{q}_j^i - \bar{\bar{q}}_j^i \leqq D^i(\bar{\mathbf{p}}_j) - D^i(\bar{\bar{\mathbf{p}}}_j)$,

$$(\bar{p}_j^i - \bar{\bar{p}}_j^i)(\bar{q}_j^i - \bar{\bar{q}}_j^i) \leqq (\bar{p}_j^i - \bar{\bar{p}}_j^i)(D^i(\bar{\mathbf{p}}_j) - D^i(\bar{\bar{\mathbf{p}}}_j)) . \tag{10}$$

or

$$(\bar{\bar{p}}_j^i - \bar{p}_j^i)((\bar{\bar{q}}_j^i - \bar{q}_j^i) \leqq (\bar{\bar{p}}_j^i - \bar{p}_j^i)(D^i(\bar{\bar{\mathbf{p}}}_j) - D^i(\bar{\mathbf{p}}_j)) . \tag{11}$$

For any component where $\bar{p}_j \neq \bar{\bar{p}}_j$ (10) or (11) holds. Summing all components, we have $(\bar{\mathbf{p}}_j-\bar{\bar{\mathbf{p}}}_j)^T(\bar{\mathbf{q}}_j-\bar{\bar{\mathbf{q}}}_j)\leqq(\bar{\mathbf{p}}_j-\bar{\bar{\mathbf{p}}}_j)^T(\mathbf{D}(\bar{\mathbf{p}}_j)-\mathbf{D}(\bar{\bar{\mathbf{p}}}_j))<0$[4]. If we consider K, L, defined for $^\forall\lambda\in(0,1)$ as follows

$$K\equiv\lambda\bar{\mathbf{p}}_j^T\bar{\mathbf{q}}_j+(1-\lambda)\bar{\bar{\mathbf{p}}}_j^T\bar{\bar{\mathbf{q}}}_j\,,$$
$$L\equiv(\lambda\bar{\mathbf{p}}_j+(1-\lambda)\bar{\bar{\mathbf{p}}}_j)^T(\lambda\bar{\mathbf{q}}_j+(1-\lambda)\bar{\bar{\mathbf{q}}}_j)\,,$$

we have $K-L=\lambda(1-\lambda)(\bar{\mathbf{p}}_j-\bar{\bar{\mathbf{p}}}_j)^T(\bar{\mathbf{q}}_j-\bar{\bar{\mathbf{q}}}_j)<0$, which means that $\mathbf{p}_j^T\mathbf{q}_j$ is strictly concave on the interval between $(\bar{\mathbf{p}}_j,\bar{\mathbf{q}}_j)$ and $(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)$. As in the case (a), this contradicts that both $(\bar{\mathbf{p}}_j,\bar{\mathbf{q}}_j)$ and $(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)$ are the optimal solutions of « **Problem** j ». □

Appendix 2 (Proof of THEOREM 2).

Consider a convergent sequence $\{\mathbf{p}^\nu\}$ in $\mathscr{P}$ to $\hat{\mathbf{p}}$. Since $\mathscr{P}$ is compact, we easily see that $\hat{\mathbf{p}}\in\mathscr{P}$. Our aim is to show that the corresponding sequence $\{(\bar{\mathbf{p}}_j(\mathbf{p}^\nu),\bar{\mathbf{q}}_j(\mathbf{p}^\nu))\}$ of the optimal solutions converges to $\{(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\hat{\mathbf{p}}))\}$.

If we deny the conclusion, we have $\varepsilon>0$ and a subsequence $\{\mathbf{p}^{\nu'}\}$ of $\{\mathbf{p}^\nu\}$ such that $(\bar{\mathbf{p}}_j(\mathbf{p}^{\nu'}),\bar{\mathbf{q}}_j(\mathbf{p}^{\nu'}))\notin N((\bar{\mathbf{p}}_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\hat{\mathbf{p}})),\varepsilon)$ as $\nu'\to\infty$, where by N we mean the usual Euclidian neighborhood. Since $\bigcup_{\mathbf{p}\in\mathscr{P}}\mathrm{T}(\mathbf{p})$ is compact[†1] there exists a convergent subsequence $\{(\bar{\mathbf{p}}_j(\mathbf{p}^\mu),\bar{\mathbf{q}}_j(\mathbf{p}^\mu))\}\subset\bigcup_{\mathbf{p}\in\mathscr{P}}\mathrm{T}(\mathbf{p})$ of $\{(\bar{\mathbf{p}}_j(\mathbf{p}^{\nu'}),\bar{\mathbf{q}}_j(\mathbf{p}^{\nu'}))\}$ to, say, $(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)$. It directly follows from the facts $\bar{\mathbf{p}}_j\to\hat{\mathbf{p}}$, $(\bar{\mathbf{p}}_j(\mathbf{p}^\mu),\bar{\mathbf{q}}_j(\mathbf{p}^\mu))\to(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)$, $\bar{\mathbf{p}}_j(\mathbf{p}^\mu)\leqq\mathbf{p}^\mu$ and $\bar{\mathbf{q}}_j(\mathbf{p}^\mu)-\mathbf{A}\bar{\mathbf{p}}_j(\mathbf{p}^\mu)\leqq\mathbf{d}$, that $\bar{\bar{\mathbf{p}}}_j\leqq\hat{\mathbf{p}}$ and $\bar{\bar{\mathbf{q}}}_j-\mathbf{A}\bar{\bar{\mathbf{p}}}_j\leqq\mathbf{d}$.

Thus we have $(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)\in\mathrm{T}(\hat{\mathbf{p}})\setminus N((\bar{\mathbf{p}}_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\hat{\mathbf{p}})),\varepsilon)$. By the uniqueness of the optimal solution in $\mathrm{T}(\hat{\mathbf{p}})$ we can take ε' as follows:

$$\pi_j(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\hat{\mathbf{p}}))-\pi_j(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)\equiv\varepsilon'>0\,. \tag{12}$$

Further for any $\delta>0$, there is $\mu^*\in\{\mu\}$ such that

$$\mathrm{T}(\mathbf{p}^{\mu_k})\cap N((\bar{\mathbf{p}}_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\hat{\mathbf{p}})),\delta)\,, \tag{13}$$

includes a point, say, $(\tilde{\mathbf{p}}^{\mu_k},\tilde{\mathbf{q}}^{\mu_k})$[†2], for any $\mu_k\geqq\mu^*$. Again by the uniqueness of the optimal solution in $\mathrm{T}(\mathbf{p}^{\mu_k})$,

$$\pi_j(\tilde{\mathbf{p}}_j^{\mu_k},\tilde{\mathbf{q}}_j^{\mu_k})\leqq\pi_j(\bar{\mathbf{p}}_j(\mathbf{p}^{\mu_k}),\bar{\mathbf{q}}_j(\mathbf{p}^{\mu_k}))\,. \tag{14}$$

From (13), the fact that $\{(\bar{\mathbf{p}}_j(\mathbf{p}^{\nu'}),\bar{\mathbf{q}}_j(\mathbf{p}^{\nu'}))\}$ converges to $(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)$ and π_j is continuous, we have $\max\{|\pi_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}))-\pi_j(\tilde{\mathbf{p}}_j^{\mu_k},\tilde{\mathbf{q}}_j^{\mu_k})|,\ |\pi_j(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\hat{\mathbf{p}}))-\pi_j(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)|\}<\varepsilon'/3$, for sufficiently large μ_k. And (12), (14) imply

$$\begin{aligned}
\varepsilon'&=\pi_j(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\hat{\mathbf{p}}))-\pi_j(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)\\
&=\{\pi_j(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\hat{\mathbf{p}}))-\pi_j(\tilde{\mathbf{p}}_j^{\mu_k},\tilde{\mathbf{q}}_j^{\mu_k})\}+\{\pi_j(\tilde{\mathbf{p}}_j^{\mu_k},\tilde{\mathbf{q}}_j^{\mu_k})-\pi_j(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)\}\\
&\leqq\{\pi_j(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}),\bar{\mathbf{q}}_j(\hat{\mathbf{p}}))-\pi_j(\tilde{\mathbf{p}}_j^{\mu_k},\tilde{\mathbf{q}}_j^{\mu_k})\}+\{\pi_j(\bar{\mathbf{p}}_j(\mathbf{p}^{\mu_k}),\bar{\mathbf{p}}_j(\mathbf{p}^{\mu_k}))-\pi_j(\bar{\bar{\mathbf{p}}}_j,\bar{\bar{\mathbf{q}}}_j)\}\\
&<2\varepsilon'/3\,.
\end{aligned}$$

This is a contradiction. □

[4] Since matrix A satisfies Conditions 1 and 2, same discussion as the footnote 3 holds.

Remarks

†1

$$\bigcup_{\mathbf{p}\in\mathscr{P}} \mathrm{T}(\mathbf{p}) = \bigcup_{\mathbf{p}\in\mathscr{P}} \{(\mathbf{p}_j(\mathbf{p}), \mathbf{q}_j(\mathbf{p}))\in\mathbf{R}_+^{2m} \mid \mathbf{q}_j \leqq \mathbf{A}\mathbf{p}_j+\mathbf{d},\ \mathbf{p}_j\leqq\mathbf{p}\in\mathscr{P}\}$$

$$= \bigcup_{\lambda\in\Lambda} \{(\mathbf{p}_j(\mathbf{p}^\lambda), \mathbf{q}_j(\mathbf{p}^\lambda))\in\mathbf{R}_+^{2m} \mid \mathbf{q}_j \leqq \mathbf{A}\mathbf{p}_j+\mathbf{d},\ \mathbf{p}_j\leqq\mathbf{p}^\lambda\in\mathscr{P}\}\,.$$

Since $\mathscr{P}$ is compact convex, for each $i\in\mathbf{M}$ there is $\max_{\lambda\in\Lambda} p^{\lambda i}$. Supposing that $\mathbf{p}_{\max}$ is the vector which is composed of these $\max_{\lambda\in\Lambda} p^{\lambda i}$, then $\bigcup_{\mathbf{p}\in\mathscr{P}}\mathrm{T}(\mathbf{p})\subset\mathrm{T}(\mathbf{p}_{\max})$ since $\mathbf{p}\leqq\mathbf{p}_{\max}$ holds for any $\mathbf{p}\in\mathscr{P}$. $\mathrm{T}(\mathbf{p}_{\max})$ is compact by definition. Thus we have the boundedness of $\bigcup_{\mathbf{p}\in\mathscr{P}}\mathrm{T}(\mathbf{p})$. Let $\{(\mathbf{p}_j^\lambda, \mathbf{q}_j^\lambda)\}\subset\bigcup_{\mathbf{p}\in\mathscr{P}}\mathrm{T}(\mathbf{p})$ be a convergent sequence to a point $(\bar{\mathbf{p}}_j, \bar{\mathbf{q}}_j)$, and $\{\mathbf{p}^\lambda\}\subset\mathscr{P}$ be the corresponding sequence such that $\{(\mathbf{p}_j^\lambda, \mathbf{q}_j^\lambda)\}\subset\mathrm{T}(\mathbf{p}^\lambda)$. Then by the compactness of $\mathscr{P}$, $\{\mathbf{p}^\lambda\}$ has a convergent subsequence $\{\mathbf{p}^\sigma\}$ to a point $\bar{\mathbf{p}}\in\mathscr{P}$. Therefore $\mathbf{q}_j^\sigma\leqq\mathbf{A}\mathbf{p}_j^\sigma+\mathbf{d}$, $\mathbf{p}_j^\sigma\leqq\mathbf{p}^\sigma$, $\mathbf{q}_j^\sigma\to\bar{\mathbf{q}}_j$ and $\mathbf{p}_j^\sigma\to\bar{\mathbf{p}}_j$ hold when $\mathbf{p}^\sigma$ tends to $\bar{\mathbf{p}}$, which imply that $(\bar{\mathbf{p}}_j, \bar{\mathbf{q}}_j)\in\mathrm{T}(\bar{\mathbf{p}})\subset\bigcup_{\mathbf{p}\in\mathscr{P}}\mathrm{T}(\mathbf{p})$. Consequently $\bigcup_{\mathbf{p}\in\mathscr{P}}\mathrm{T}(\mathbf{p})$ is closed.

†2

First we note that $\bar{\mathbf{p}}_j(\hat{\mathbf{p}})\leqq\hat{\mathbf{p}}$ and $\bar{\mathbf{p}}_j(\hat{\mathbf{p}})-\mathbf{A}\bar{\mathbf{p}}_j(\hat{\mathbf{p}})\leqq\mathbf{d}$. If $\bar{\mathbf{p}}_j(\hat{\mathbf{p}})\mathbf{p}^\mu$ is satisfied, we directly see $(\bar{\mathbf{p}}_j(\hat{\mathbf{p}}), \bar{\mathbf{q}}_j(\hat{\mathbf{p}}))\in\mathrm{T}(\mathbf{p}^\mu)$. Suppose there is an subsequence $\{\mu'\}$ of $\{\mu\}$ on which $\bar{\mathbf{p}}_j(\hat{\mathbf{p}})\leqq\mathbf{p}^{\mu'}$ does not hold. Then we can take a sequence $\{\lambda^{\mu'}\}\in(0, 1)$ defined as $\lambda^{\mu'}=\max\{\lambda\in(0,1)\mid\lambda\bar{\mathbf{p}}_j(\hat{\mathbf{p}})\leqq\mathbf{p}^{\mu'}\}$. For each μ', there corresponds an $i\in\mathbf{M}$ such that $\hat{p}^i\geqq\bar{p}_j^i(\hat{\mathbf{p}})>\lambda^{\mu'}\bar{p}_j^i(\hat{\mathbf{p}})=p^{i\mu'}\geqq 0$. Since $\mathbf{p}^{\mu'}\to\hat{\mathbf{p}}$, we must have $\lambda^{\mu'}\to 1$. If $\bar{\mathbf{q}}_j(\hat{\mathbf{p}})-\mathbf{A}(\lambda^{\mu'}\bar{\mathbf{p}}_j(\hat{\mathbf{p}}))\leqq\mathbf{d}$ is satisfied for any μ', $\lambda^{\mu'}\to 1$ implies that for ${}^\forall\delta>0$, $(\lambda^{\mu'}\bar{\mathbf{p}}_j(\hat{\mathbf{p}}), \bar{\mathbf{q}}_j(\hat{\mathbf{p}}))\in\mathrm{T}(\mathbf{p}^{\mu'})\cap N((\bar{\mathbf{p}}_j(\hat{\mathbf{p}}), \bar{\mathbf{q}}_j(\hat{\mathbf{p}}), \delta)$ holds whenever μ' is large. Suppose there is an subsequence $\{\mu''\}$ of $\{\mu'\}$ on which $\bar{\mathbf{q}}_j(\hat{\mathbf{p}})+\mathbf{A}(\lambda^{\mu''}\bar{\mathbf{p}}_j(\hat{\mathbf{p}}))\leqq\mathbf{d}$ is violated. Let $\bar{\mathbf{q}}_j^{\mu''}$ be defined as $\bar{q}_j^{i\mu''}=d^i+[\mathbf{A}(\lambda^{\mu''}\bar{\mathbf{p}}_j(\hat{\mathbf{p}}))]^i$if $\bar{q}_j^i(\hat{\mathbf{p}})-[\mathbf{A}(\lambda^{\mu''}\bar{\mathbf{p}}_j(\hat{\mathbf{p}}))]^i>d^i$ and $[\mathbf{A}(\lambda^{\mu''}\bar{\mathbf{p}}_j(\hat{\mathbf{p}}))]^i>0$, $\bar{q}_j^{i\mu''}=\bar{q}_j^i(\hat{\mathbf{p}})$ otherwise. It is easy to see $\bar{q}_j^{i\mu''}-\mathbf{A}(\lambda^{\mu''}\bar{\mathbf{p}}_j(\hat{\mathbf{p}}))\leqq\mathbf{d}$. Further, for any $i\in\mathbf{M}$ such that $\bar{q}_j^i(\hat{\mathbf{p}})-\bar{q}_j^{i\mu''}>0$, since $\bar{q}_j^i(\hat{\mathbf{p}})\leqq d^i+[\mathbf{A}\bar{\mathbf{p}}_j(\hat{\mathbf{p}})]^i$, we have $(1-\lambda^{\mu''})[\mathbf{A}\bar{\mathbf{p}}_j(\hat{\mathbf{p}})]^i\geqq\bar{q}_j^i(\hat{\mathbf{p}})-\bar{q}_j^{i\mu''}$. Again $\lambda^{\mu''}\to 1$ implies $\bar{\mathbf{q}}_j^{\mu''}\to\bar{\mathbf{q}}_j(\hat{\mathbf{p}})$.

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REFERENCES

G. Debreu, "Theory of Value", Yale Univ. Press, 1959.

F. Nikaido, "Convex Structures and Economic Theory", Academic Press, 1968.

M. Intriligator, "Mathematical Optimization and Economic Theory", Prentice-Hall Inc., 1971.

P. J. Lederer, "The Effect of Production and Transport Competitive Spatial Pricing and Firm Location", William E. Simon Graduate School, 1989.

J. M. Ortega, W. C. Rheinboldt, "Iterative Solution of Non-linear Equations in Several Variables", Academic Press, 1970.

T. M. Apostol, "Mathematical Analysis 2nd Edition", Addison Wesley, 1973.