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NON-COOPERATIVE N-PERSON BARGAINING GAMES AND SYMMETRIC RULES

Tetsuro OKAZAKI*

Abstract: We extend Rubinstein's (1982) two-person bargaining game with a fixed common discount factor to an N-person bargaining game. In this paper, game rules are required to be symmetric. Then we can prove that virtually any partition can be a subgame perfect equilibrium outcome, under 'any' symmetric rule, if the discount factor is more than or equal to a critical value given. So in any symmetric bargaining game, a subgame perfect equilibrium is infinite. In other words, it is impossible to construct a symmetric rule under which a subgame perfect equilibrium is unique. On the other hand, if the discount factor is less than this critical value, a symmetric rule can be constructed under which a subgame perfect equilibrium outcome is unique.

1. INTRODUCTION

As proved in Rubinstein (1982), a non-cooperative bargaining game with a fixed common discount factor has a unique subgame perfect equilibrium outcome in the case of two players. In the case of N players ($N \geq 3$), this result dose not generally hold. Any outcome can be a subgame perfect equilibrium outcome, if agreements require the approval of all players. This result is shown in Shaked's example (see Sutton (1986)) and in Haller (1986). In their models, one player offers a partition of a pie and the other players respond to the offer sequentially (in Sutton (1986)) or simultaneously (in Haller (1986)). In the case of sequential response, any outcome is a subgame perfect equilibrium outcome if the discount factor is large, while in the case of simultaneous response, any outcome is a subgame perfect equilibrium irrespective of the rate of the discount factor.

In contrast, Chae and Yang (1988) and Yang (1992) argue that, in the case of N players, there exists a unique subgame perfect equilibrium outcome using different rules. Under their rules, one player offers another player his share and this player responds to the offer. This sequence is then repeated between all of the players in turn. In their models, preceding players' accepted shares are honored by the succeeding players, i.e. once some shares are accepted, then these hold good forever. This means that agreements require only one player's approval. However it would be more appealing that agreements require unanimous approval.

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Moreover, under the game rules of Chae and Yang (1988) and Yang (1992), the first player starts the original game, but another player can not always start a subgame which is identical to the original game. The researcher believes, however, that it is more desirable that each player can start an identical subgame after his rejection, i.e. each player has the same opportunity as the first player. In Rubinstein's bargaining game, "the game form is 'almost' symmetric. The only asymmetry arises because of the need to specify who is the first player to make an offer" (see Rubinstein (1987) p. 200). In Rubinstein's game, each player has the same opportunity as the first player, while there exists strong asymmetry in the games of Chae and Yang (1988) and Yang (1992).

Because the game rules of Chae and Yang (1988) and Yang (1992) have strong asymmetry, the following question arises: can we construct a symmetric rule under which a subgame perfect equilibrium is unique? In this paper, we consider a bargaining game from this viewpoint.

At first, symmetric rules are characterized by representing some conditions which symmetric rules should satisfy. Under symmetric rules, agreements require unanimous approval and each player has the same opportunity as the first player. Therefore, asymmetry appears only in specifying the order in which players make offers.

Consider the case where the discount factor is more than or equal to the value $\underline{\delta}$ satisfying $1=\underline{\delta}^1+\cdots+\underline{\delta}^{N-1}$. Then, it is proved that, under 'any' symmetric game rule, a subgame perfect equilibrium outcome is not unique. In addition, if the discount factor is sufficiently close to 1, virtually any partition is a possible subgame perfect equilibrium outcome. In other words, it is impossible to construct a symmetric game rule under which a subgame perfect equilibrium outcome is unique. Unanimity and symmetry are natural requirements, but not compatible with uniqueness of a subgame perfect equilibrium outcome.

Sutton (1986) and Haller (1986) constructed a special game rule under which a subgame perfect equilibrium outcome is not unique. In this paper, we shall demonstrate that a subgame perfect equilibrium is not unique for a class of symmetric game rules.

Yang (1992) admits that his "bargaining game in extensive form is not 'natural', i.e. too restrictive". He states that "it is not certain yet whether there is a less restrictive model which has a unique perfect equilibrium" and that "we hope to find out such a model soon". The result here suggests that finding such a model may be difficult.

In addition, an assumption about the discount factor is necessary to this theorem. In fact, if the discount factor is less than $\underline{\delta}$, a symmetric rule can be constructed under which a subgame perfect equilibrium is unique.

2. MODEL

There are N players ($N \geq 2$). They bargain over partitions of a pie of size 1. A partition is denoted by $s=(s^1, \cdots, s^N)$ where $s^1 \geq 0, \cdots, s^N \geq 0$ and $\sum_i s_i = 1$. The set of partitions is denoted by S, i.e. $S=\{s^1, \cdots, s^N | s^1 \geq 0, \cdots, s^N \geq 0$ and $\sum_i s_i = 1\}$. We consider the case of simple preferences. Each player i has the same discount factor δ $(0<\delta<1)$. If the bargaining ends at time t with the partition $(s^1, \cdots, s^N)$, player i's utility is $\delta^{t-1}s^i$.

3. SYMMETRIC RULES

In this section, we present some conditions which symmetric rules should satisfy and make what is intended by symmetry clear.

We describe the structure of the games as follows: The set of players is denoted by I. At each time t, some player makes an offer. The index of the player who makes an offer at time t is denoted by o_t. For simplicity, we assume that, at any time t, o_t is independent of histories until time $t-1$ which is denoted by h_{t-1}. We assume that there is only one player who makes an offer at any time t and that each player's action at time t is one of making his offer, responding to an offer, or doing nothing. In other words, there is no outside option. Moreover we consider the case where $o_i = i$ for any $i \in I$. In other words, the player who makes an offer at time i is called player i.

In Sutton (1986) and Haller (1986), player o_t offers some partition and the other players respond to the offer sequentially (in Sutton (1986)) or simultaneously (in Haller (1986)). In Chae and Yang (1988) and Yang (1992), player o_t sequentially offers another player's share and this player responds. So there are some variations with respect to structures of offers and responses.

In order to represent the structure of offers and responses in general form, notations are introduced. The set of offers which are available for player o_t at time t, give h_{t-1}, is denoted by $O_t(h_{t-1})$. In general, $O_t(h_{t-1})$ needs not to be S. So an offer at time t may not be some s. The profile of the shares which are offered at time t is denoted by π_t and the set of indices of players whose shares are in π_t by $I(\pi_t)$. For example, if player o_t offers player i's, j's, and k's shares, $\pi_t = \{s_i, s_j, s_k\}$ and $I(\pi_t) = \{i, j, k\}$. The set of indices of the players who respond to an offer π_t at time t is denoted by $R_t(\pi_t)$ and the response of player i in $R_t(\pi_t)$ by r^i. In brief, at time t, player o_t offers some π_t in $O_t(h_{t-1})$ and player i in $R_t(\pi_t)$ responds to the offer π_t. We denote by h_t^i the information of player i at time t. That is, h_t^i is a set of actions at time t which player i knows when he takes his action at time t.

In Sutton and Haller, $O_t(h_{t-1}) = S$ for any h_{t-1} and $R_t(\pi_t) = I \backslash \{o_t\}$ for any π_t. On the contrary, in Chae and Yang and Yang, $O_t(h_{t-1})$ directly depends on h_{t-1} and $R_t(\pi_t)$ indirectly depends on h_{t-1}. In Sutton, at time 1, $\pi_1 \in S$, $R_1(\pi_1) = I \backslash \{1\}$, and $h_1^i = \{\pi_1, r^2, \cdots, r^{i-1}\}$; in Haller, $\pi_1 \in S$, $R_1(\pi_1) = I \backslash \{1\}$, and $h_1^i = \{\pi_1\}$; in Chae and Yang, if there is no rejection at time 1, $\pi_1 = \{s^2, \cdots, s^N\}$, $R_1(\pi_1) = I \backslash \{1\}$, and $h_1^i = \{s^2, r^2, \cdots, s^{i-1}, r^{i-1}, s^i\}$; in Yang, if there is no rejection at time 1, $\pi_1 = \{s^2, \cdots, s^N\}$, $R_1(\pi_1) = I \backslash \{1\}$, and $h_1^i = \{s^N, r^N, \cdots, s^{i+1}, r^{i+1}, s^i\}$.

We present some conditions which the structure of offers and responses should satisfy.

CONDITION 1. *If π_t is accepted by all players in $R_t(\pi_t)$, then, for any $\pi_{t+1} \in O_{t+1}(h_t)$ and any $j \in I(\pi_t)$, j belongs to $I(\pi_{t+1})$ and player j's share in π_t is equal to player j's share in π_{t+1}.*

CONDITION 2. (1) *Let π_t be accepted by all players in $R_t(\pi_t)$. For any $\pi_{t+1} \in O_{t+1}(h_t)$, if i is in $I(\pi_{t+1}) \backslash I(\pi_t)$ and i is not equal to o_{t+1}, then i belongs to $R_t(\pi_{t+1})$.* (2) *Let π_t be rejected by some player in $R_t(\pi_t)$ or $t=0$. For any $\pi_{t+1} \in O_{t+1}(h_t)$, if i is in $I(\pi_{t+1})$ and i is not equal to o_{t+1}, then i belongs to $R_t(\pi_{t+1})$.*

CONDITION 3. *For any t and any $\pi_t \in O_t(h_{i-t})$, o_t belongs to $I(\pi_t)$.*

Condition 1 means that if π_t is accepted by all players in $R_t(\pi_t)$, then this offer is not changed until a rejection occurs. Assuming Conditions 1 and 2, if player i's share is offered, he can respond to this offer. From Condition 3, if player i makes his offer, his share is included in his offer. In other words, assuming Condition 1, player i's share is offered by some player until time i.

The set of game rules which satisfy Conditions $1 \sim 3$ includes many rules. In fact, Sutton's, Haller's, Chae and Yang's and Yang's rules satisfy these conditions. (In Chae and Yang's and Yang's rules, π_1 gives the partition $(1-\sum s^i, s^2, \cdots, s^N)$.) As an example of the rule satisfying these conditions which is different from their rules, see Section 5.

We define $T = \min\{t \mid I(\pi_t) = I$ for some history until time $t-1\}$. Then T is the minimum time length for the bargaining to end. From Conditions 1 and 3, we have $T \leq N$. We assume that if there is no rejection until time T, the bargaining always ends at time T. In other words, we assume that there is no waste offer.

Next, we consider the following two conditions.

CONDITION 4. *For any t and π_t, if some player $i \in R_t(\pi_t)$ rejects the offer, the subgame which starts at time $t+1$ has the same structure as the original game except the discounted utilities and the order of players to move.*

This condition means that any player's share is not determined if there is a player who does not agree. In this sense, under the game rules satisfying Condition 4, agreements require the indirectly unanimous approval.

CONDITION 5. *For any $i \in I$, any t such that $o_t = i$, any h_{t-2}, and any $\pi_{t-1} \in O_{t-1}(h_{t-2})$, i belongs to $R_{t-1}(\pi_{t-1})$.*

Assuming Conditions 4 and 5, player i can make the first offer at time t by rejecting an offer at time $t-1$. So, for any player, if the bargaining does not end before his opportunity to make his offer comes, he can make the first offer in the subgame which is the same as the original game, *i.e.* he has the same opportunity as the first player. Therefore, the game rules satisfying Conditions 4 and 5 are symmetric.

4. MULTIPLICITY OF EQUILIBRIUM OUTCOMES

In this section, we prove that a subgame perfect equilibrium outcome is not unique under any symmetric rule if the discount factor is more than or equal to a certain value. Therefore, the result achieved here would indicate that Sutton and Haller's multiplicity is a special example of the result here and that the results in Chae and Yang and Yang do not hold if the conditions of symmetry are imposed.

We define a subset of S, P, as $P=\{s\in S \mid 0\leq s^i\leq \delta^{i-1}$ for every $i\}$.[1] It should be noted that P includes almost every partition if the discount factor is sufficiently close to 1. In the following theorem it is proved that under any symmetric game rule, virtually any partition is a possible subgame perfect equilibrium for the discount factor close to 1.

Denote by $\underline{\delta}$ the discount factor δ satisfying $1=\delta^1+\cdots+\delta^{N-1}$. Then we can derive the following result.

THEOREM 1. *Suppose that $\delta\geq\underline{\delta}$. In any bargaining game, if the rule of this bargaining game satisfies Conditions 1 ~ 5, for every partition $s\in P$, there is a subgame perfect equilibrium whose outcome is s.*

Proof. For simplicity, we assume that $o_{i+kN}=i$ for any $i\in I$ and $k=0, 1, 2, \cdots$. Consider the subgame after a rejection at time t and some offer in this subgame π_y where $y>t$. It can be said that the original game is the subgame after a rejection at time 0. We define the set $A(\pi_y;\, t)$ as follows.

$$A(\pi_y;\, t)=\{o_{y+1}, \cdots, o_{t+N}\}\setminus I(\pi_y) \tag{1}$$

$A(\pi_y;\, t)$ is the set of players who do not make their first offers in this subgame and whose shares are not offered at time y. For any $i\in R_y(\pi_y)$, we define the set $B^i(\pi_y;\, t)$ and the function $n(i;\, t)$ as follows.

$$B^i(\pi_y;\, t)=\{k \mid k\in\{o_{y+1}, \cdots, o_{t+N}\}\cap I(\pi_y) \text{ and } r^k\notin h^i_y\} \tag{2}$$

$$n(i;\, t)=\begin{cases} i-o_t & \text{when } i-o_t>0 \\ N-(i-o_t) & \text{when } i-o_t\leq 0 \end{cases} \tag{3}$$

$B^i(\pi_y;\, t)$ is the set of the players who do not make their first offers in this subgame and do not make their responses before player i makes his response. $n(i;\, t)$ means that player i is the $n(i;\, t)$-th player in this subgame. We define $\alpha(\pi_y;\, t)$ as follows.

$$\alpha(\pi_y;\, t)=\sum_{i\in A(\pi_y;\, t)} \delta^{n(i;\, t)-1} \tag{4}$$

We denote by $S(\pi_y)$ the summation of the shares in π_y.

Fix a partition $s^*=(s^1, \cdots, s^N)$ which is in P. Consider the following strategies.

At first, we describe player i's strategy at time t in the case where there is no rejection before player i takes his action at time t. We say that π_t conincides with

[1] Since $\delta\geq\underline{\delta}$, there are infinite partitions in P.

s^* if $s^k = s^{*k}$ for any $k \in I(\pi_t)$ where s^k is player k's share in π_t.

states s^*

player i, at time t such that $i \in R_t(\pi_t)$ (and $i = o_{t+1}$)

1. if π_t coincides with s^* he accepts this offer π_t (and if π_t is accepted by all $j \in R_t(\pi_t)$ he offers π_{t+1} which coincides with s^*)
2. if π_t does not coincide with s^*, $s^j \geq \delta^{j-1}$ for any $j \in B^i(\pi_t; 0)$ and $1 - S(\pi_t) \geq \alpha(\pi_t; 0)$, he accepts this offer π_t (and if π_t is accepted by all $j \in R_t(\pi_t)$ he offers π_{t+1} such that $s^j = \delta^{j-1}$ for any $j \in I(\pi_{t+1}) \backslash I(\pi_t)$ and $j \neq i$)
3. if π_t does not coincide with s^* and $s^j < \delta^{j-1}$ for some $j \in B^i(\pi_t; 0)$ or $1 - S(\pi_t) < \alpha(\pi_t; 0)$, he rejects this offer π_t

(at time 1, player 1 offers π_1 which coincides with s^*)

Next we describe strategies after a rejection occurs. Suppose that some player rejects the offer π_y at time y. In this case, each player's strategy in the subgame after this rejection is as follows. Denote by C the set $\{k \in \{o_{y+1}, \cdots, o_N\} \mid \text{player } k \text{ rejects } \pi_y\}$. If $C = \varnothing$ and some player rejects π_y, each player's strategy is one in state e^j where $j = o_{y+1}$. Let $C \neq \varnothing$ and $j = \min C$. If player j's rejection violates the above rule, *i.e.* if player j rejects π_y which he, following the strategy in state s^*, would accept, each player's strategy is one in state e^{j+1} (if $j = N$, $e^{j+1} = e^1$). Otherwise, each player's strategy is one in state e^j.

state e^j (e^j is the j-th unit vector)

player i, at time t such that $i \in R_t(\pi_t)$ (and $i = o_{t+1}$)

1. if π_t coincides with e^j he accepts this offer π_t (and if π_t is accepted by all $k \in R_t(\pi_t)$ he offers π_{t+1} which coincides with e^j)
2. if π_t does not coincide with e^j, $s^k \geq \delta^{n(k;y)-1}$ for any $k \in B^i(\pi_t; y)$ and $1 - S(\pi_t) \geq \alpha(\pi_t; y)$, he accepts this offer π_t (and if π_t is accepted by all $k \in R_t(\pi_t)$ he offers π_{t+1} such that $s^k = \delta^{n(k;y)-1}$ for any $k \in I(\pi_{t+1}) \backslash I(\pi_t)$ and $k \neq i$)
3. if π_t does not coincide with e^j and $s^k < \delta^{n(k;y)-1}$ for some $k \in B^i(\pi_t; y)$ or $1 - S(\pi_t) < \alpha(\pi_t; y)$, he rejects this offer π_t

(at time $y+1$, player o_{y+1} offers π_{y+1} which coincides with e^j)

In state e^j mentioned above, we describe player i's strategy in the case where there is no rejection before player i takes his action. If some player rejects the offer π_z at time z in state e^j, the following transition occurs. Denote by C' the set $\{k \in \{o_{z+1}, \cdots, o_{y+N}\} \mid \text{player } k \text{ rejects } \pi_z\}$. If $C' = \varnothing$ and some player rejects π_z, each player's strategy is one in state e^j where $j = o_{z+1}$. Let $C' \neq \varnothing$ and $j = \min C'$. If player j's rejection violates the above rule, each player's strategy is one in state e^{j+1}. Otherwise, each player's strategy is one in state e^j.

We can prove that the combination of these strategies is a subgame perfect equilibrium (see Appendix A) and s^* is a partition in this equilibrium outcome. Therefore, for every partition $s \in P$, there is a subgame perfect equilibrium where the outcome is s. ■

5. EXISTENCE OF A SYMMETRIC GAME RULE WITH A UNIQUE OUTCOME

In section 4, we proved that it is impossible to construct a symmetric game rule under which a subgame perfect equilibrium outcome is unique. An assumption about the discount factor is necessary to this theorem. In this section, we consider the case where $1>\delta^1+\cdots+\delta^{N-1}$. Then, it is shown that there is a symmetric game rule under which a subgame perfect equilibrium outcome is unique. We denote by G the game considered in this section and describe the rule of G below.

At time 1, player 1 offers his own share s^1 and player 2 accepts or rejects the offer s^1. If player 2 accepts the offer, player 1's share s^1 'temporarily' holds good. If player 2 rejects, player 1 responds to player N's offer at time N, and his next opportunity to offer his share comes at time $N+1$.

At time 2, player 2 offers his own share s^2 and player 3 accepts or rejects the offer s^2. If player 3 rejects player 2's offer, both player 1 and player 2 have to go behind player N keeping the same order, *i.e.* player 1 (respectively player 2) responds to player N's (respectively player 1's) offer at time N (respectively $N+1$), and his next opportunity to offer his share comes at time $N+1$ (respectively $N+2$).

In general, at time $i+kN$ ($k=0, 1, 2, \cdots$), player i offers his own share s^i and player $i+1$ accepts or rejects this offer s^i. If player $i+1$ rejects the offer s^i, not only s^i but also other shares which have been accepted in advance become void, and the bargaining starts all over again from player $i+1$'s offer. Therefore each player has the opportunity to make the first offer in the same subgame as the original game, that is, each player has the same opportunity as the first player. Therefore, this game rule satisfies symmetry.

Let all the shares except one share, for example $(s^1, \cdots, s^{N-1})$, be accepted sequentially. Since player N knows that his share becomes $s^N=1-\sum_{i=2}^{N-1} s^i$ by accepting the offer s^{N-1}, this means that $s^N=1-\sum_{i=2}^{N-1} s^i$ is also accepted by player N and there is no rejection with respect to $(s^1, \cdots, s^N)$. If all the offers $(s^1, \cdots, s^N)$ are accepted sequentially in the above sense, the bargaining ends with the partition $s=(s^1, \cdots, s^N)$. Under this rule, agreements require the indirectly unanimous approval. So, this game rule satisfies unanimity.

It is easy to check that this rule satisfies the conditions mentioned in Section 3. (This rule does not satisfy Condition 1, but we can reconstruct the rule so as to satisfy Condition 1 without affecting the result.)

In the following theorem, we assert that, under this rule, there is a unique subgame perfect equilibrium outcome if $\delta<\underline{\delta}$, *i.e.* if $1>\delta^1+\cdots+\delta^{N-1}$. Therefore, there is a symmetric game rule under which a subgame perfect equilibrium outcome is unique.

THEOREM 2. *In the bargaining game G, if $\delta < \underline{\delta}$, there is a unique subgame perfect equilibrium outcome with the partition*

$$(s^{*1}, s^{*2}, \cdots, s^{*N}) = \left(\frac{1-\delta}{1-\delta^N}, \delta^1 \frac{1-\delta}{1-\delta^N}, \cdots, \delta^{N-1} \frac{1-\delta}{1-\delta^N}\right). \tag{5}$$

Proof. Denote by s_t the offer made at time t. If there is no rejection until time t, we say that the last rejection occurs at time 0.

Consider the following strategies. At time $i-1+kN$ $(k=0, 1, 2, \cdots)$, player i takes his action as follows: (1) if the last rejection occurs at time $i-1-t+kN$ and $1-\sum_{j=i-t+kN}^{i-1+kN} s_j \geq \sum_{j=t+1}^{N} s^{*j}$, he accepts the offer s_{i-1+kN} at time $i-1+kN$ and offers $1-\sum_{j=i-t+kN}^{i-1+kN} s_j - \sum_{j=t+2}^{N} s^{*j}$ at time $i+kN$; (2) otherwise, he rejects the offer at time $i-1+kN$ and offers s^{*1} at time $i+kN$.

The combination of these strategies is a subgame perfect equilibrium, and the partition is $(s^{*1}, \cdots, s^{*N})$ in the equilibrum outcome of these strategies. So there is at least one subgame perfect equilibrium outcome.

Next, we show the uniqueness of a subgame perfect equilibrium. We denote the supremum of player i's utilities in the set of subgame perfect equilibrium outcomes by M^i and the infimum by m^i. Write $M = \max\{M^1, \cdots, M^N\}$.

Now we can prove that

$$\delta^{N-2} - (\delta^1 M + \cdots + \delta^{N-1} M) \leq m^1 \tag{6}$$

holds (see Claim 1 in Appendix B).

Similarly, we can get

$$\delta^{N-2} - (\delta^1 m^1 + \cdots + \delta^{N-1} m^1) \geq M \tag{7}$$

(see Claim 2 in Appendix B).

From two inequalities (6) and (7) and an inequality $1 > \delta^1 + \cdots + \delta^{N-1}$, we have $M \leq \delta^{N-2}(1-\delta)/(1-\delta^N)$ and $m^1 \geq \delta^{N-2}(1-\delta)/(1-\delta^N)$. This implies that $m^1 = M^1 (= M) = \delta^{N-2}(1-\delta)/(1-\delta^N)$. So player 1's utility in the subgame perfect equilibrium is only $\delta^{N-2}(1-\delta)/(1-\delta^N)$. Using this result, we can show that player i's utility is $\delta^{N-2}\delta^{i-1}(1-\delta)/(1-\delta^N)$ for each i. Since the game ends at time $t-1$, player i's share is $\delta^{i-1}(1-\delta)/(1-\delta^N)$. ∎

Shaked's example show that, in the case of $N=3$, any partition can be a subgame perfect equilibrium outcome if $\delta \geq 1/2$. On the contrary, a subgame perfect equilibrium outcome is unique even if $(\sqrt{5}-1)/2 \approx 0.618 > \delta \geq 1/2$ under the rule in this section. Therefore, the range where a subgame perfect equilibrium outcome is unique is enlarged under this rule.

6. CONCLUDING REMARKS

It should be concluded, from what is shown in this paper, that $\underline{\delta}$ is a critical value. If $\delta < \underline{\delta}$, a symmetric game rule can be constructed under which a subgame perfect equilibrium outcome is unique. On the contrary, if $\delta \geq \underline{\delta}$, such a symmetric game rule can not be constructed.

The main conclusion is that it is impossible to construct a game rule under which a subgame perfect equilibrium is unique where $\delta \dot{\geq} \underline{\delta}$. Therefore, to construct such a game rule, symmetry must be abandoned like in Chae and Yang (1988) or Yang (1992).

APPENDIX: A

In this appendix, we prove that the combination of strategies which appears in the proof of Theorem 1 is a subgame perfect equilibrium. Fixing player j's strategy ($j \neq i$), we check that player i's strategy is his best action.

At first, consider the strategy which is in state s^*.

(Case 1.1) Consider the case where π_t coincides with s^* and $i \neq o_{t+1}$. If he accepts this offer, his share in equilibrium outcome is s^{*i}. If he rejects this offer, his share in equilibrium outcome is 0. Therefore his strategy is his best action.

(Case 1.2) Consider the case where π_t coincides with s^* and $i = o_{t+1}$. If he accepts this offer and offers π_{t+1} which coincides with s^*, his share in equilibrium outcome is s^{*i}. If he rejects this offer, his share in equilibrium outcome is 0. If he accepts this offer and offers π_{t+1} which does not coincide with s^* and in which $s^i > s^{*i}$, this offer π_{t+1} is rejected and his share in equilibrium outcome is 0. (If $i=1$, there is no s such that $s^i > 0$ and $s^j \geq \delta^{j-1}$ for any $j \in I \setminus \{1\}$ because $1 \leq \delta + \cdots + \delta^{N-1}$. If $i = 2, \cdots, N$ there is no π_{t+1} such that $s^i > s^{*i}$, $s^k = s^{*k}$ for any $k \in I(\pi_t)$, and $s^j \geq \delta^{j-1}$ for any $j \in I(\pi_{t+1}) \setminus I(\pi_t)$, because $s^* \in P$. From these, π_{t+1} is rejected.) Therefore, his strategy is his best action.

(Case 2.1) Consider the case where π_t does not coincide with s^*, $s^j \geq \delta^{j-1}$ for any $j \in B^i(\pi_t; 0)$ and $1 - S(\pi_t) \geq \alpha(\pi_t; 0)$, and $i \neq o_{t+1}$. If he accepts this offer, his share in equilibrium outcome is s^i in π_t or δ^{i-1}. If he rejects this offer, his share is 0. Therefore his strategy is his best action.

(Case 2.2) Consider the case where π_t does not coincide with s^*, $s^j \geq \delta^{j-1}$ for any $j \in B^i(\pi_t; 0)$, $1 - S(\pi_t) \geq \alpha(\pi_t; 0)$, and $i = o_{t+1}$. If he accepts this offer, and offers π_{t+1} such that $s^j = \delta^{j-1}$ for any $j \in I(\pi_{t+1}) \setminus I(\pi_t)$ and $j \neq i$, his share in equilibrium outcome is $1 - S(\pi_t) - \alpha(\pi_{t+1}) + \delta^{i-1}$. If he rejects this offer, his share in equilibrium outcome is 0. If he accepts this offer and offers π_{t+1} such that $s^j < \delta^{j-1}$ for some $j \in I(\pi_{t+1}) \setminus I(\pi_t)$, this offer is rejected and his share in equilibrium outcome is 0. If he accepts this offer and offers π_{t+1} such that $s^j > \delta^{j-1}$ for some $j \in I(\pi_{t+1}) \setminus I(\pi_t)$, his share in equilibrium outcome cannot be increased. (If some player rejects the offer, his share is 0. Even if the offer is accepted, his share is less than $1 - S(\pi_t) - \alpha(\pi_{t+1}) + \delta^{i-1}$.) Therefore his strategy is his best action.

(Case 3.1) Consider the case where (1) π_t does not coincide with s^*, (2) $s^j < \delta^{j-1}$ for some $j \in B^i(\pi_t; 0)$ or $1 - S(\pi_t) < \alpha(\pi_t; 0)$, (3) $i = o_{t+1}$, and (4) there is $j \in \{o_{t+1}, \cdots, o_N\} \cap R_t(\pi_t)$ such that $j \neq i$ and $r^j \notin h^i_t$. If he rejects this offer and offers π_{t+1} which coincides with e^i, his share in equilibrium outcome is 1. If he accepts this offer, this offer is rejected by player j, and his share is 0. Therefore his strategy is his best action.

(Case 3.2) Consider the case where (1) π_t does not coincide with s^*, (2) $s^j < \delta^{j-1}$ for some $j \in B^i(\pi_t; 0)$ or $1 - S(\pi_t) < \alpha(\pi_t; 0)$, (3) $i = o_{t+1}$, and (4) there is no $j \in \{o_{t+1}, \cdots, o_N\} \cap R_t(\pi_t)$ such that $j \neq i$ and $r^j \notin h^i_t$. If there is some j such that $r^j \in h^i_t$, player j rejects the offer. Since there is no rejection before player i takes his action at time t, there is no j such that $r^j \in h^i_t$. So there is no $j \in \{o_{t+1}, \cdots, o_N\} \cap R_t(\pi_t)$ such that $j \neq i$ from (4). Moreover, for any $j \in \{o_{t+1}, \cdots, o_N\} \cap I_{t-1}(\pi_{t-1})$, $s^j \geq \delta^{j-1}$ because π_{t-1} is accepted at time $t-1$. From these, there is no $j \in B^i(\pi_t; 0)$ such that $s^j < \delta^{j-1}$ and $j \neq i$. Therefore we have $s^i < \delta^{i-1}$ or $1 - S(\pi_t) < \alpha(\pi_t; 0)$ from (2). If he rejects this offer and offers π_{t+1} which coincides with e^i, his share in equilibrium outcome is 1 and his utility is $\delta^{i-1}\delta^{T-1}$. If he accepts this offer and some player k rejects this offer, his share is 1 and his utility is $\delta^{i-1}\delta^{T-1}$. (Note that $k \notin \{o_{t+1}, \cdots, o_N\}$ from (4).) If this offer is accepted by all players in $R_t(\pi_t)$, his share in equilibrium outcome is s^i $(<\delta^{i-1})$ or 0[2] and his utility is $\delta^{Y-1}s^i$ or 0 where Y is the time length for the bargaining to end if there is no rejection. We have $Y \geq T$. Therefore his strategy is his best action.

(Case 3.3) Consider the case where (1) π_t does not coincide with s^*, (2) $s^j < \delta^{j-1}$ for some $j \in B^i(\pi_t; 0)$ or $1 - S(\pi_t) < \alpha(\pi_t; 0)$, (3) $i \neq o_{t+1}$, and (4) there is $j \in \{o_{t+1}, \cdots, o_N\} \cap R_t(\pi_t)$ such that $j \neq i$ and $r^j \notin h^i_t$. If he rejects this offer, his share is 1 or 0. If he accepts this offer, his share is 0. Therefore his strategy is his best action.

(Case 3.4) Consider the case where (1) π_t does not coincide with s^*, (2) $s^j < \delta^{j-1}$ for some $j \in B^i(\pi_t; 0)$ or $1 - S(\pi_t) < \alpha(\pi_t; 0)$, (3) $i \neq o_{t+1}$, and (4) there is no $j \in \{o_{t+1}, \cdots, o_N\} \cap R_t(\pi_t)$ such that $j \neq i$ and $r^j \notin h^i_t$. In this case, we have $s^i < \delta^{i-1}$ or $1 - S(\pi_t) < \alpha(\pi_t; 0)$. If he rejects this offer, transition to state e^i occurs and his share is 1. If he accepts this offer and some player k rejects this offer, transition to state e^j occurs where $j = o_{t+1}$ and his share is 0. If this offer is accepted by all players in $R_t(\pi_t)$, his share in equilibrium outcome is s^i or 0. Therefore his startegy is his best action.

Using similar reasoning, we can prove that the strategy which is in state e^j is his best actions. ∎

APPENDIX: B

In this appendix, we complete the proof of Theorem 2.

CLAIM 1. *$\delta^{N-2} - (\delta^1 M + \cdots + \delta^{N-1} M) \leq m^1$ holds.*

Proof. Suppose that there exists player 1's utility u in some subgame perfect equilibrium outcome such that $\delta^{N-2} - (\delta^1 M + \cdots + \delta^{N-1} M) > u$.

Consider the following player 2's action: player 2 accepts any offer s_1 such that

[2] If $1 - S(\pi_t) < \alpha(\pi_t; 0)$ holds, any offer at time $t+1$ is rejected.

$1-s_1>(\delta^1 M+\cdots+\delta^{N-1}M)/\delta^{N-2}$ and makes an offer after which the bargaining ends at time $N-1$. If this action is his best action, player 1 can get the utility which is strictly larger than u, and this is a contradiction. Therefore it is sufficient to prove that player 2 accepts any offer s_1 such that $1-s_1>(\delta^1 M+\cdots+\delta^{N-1}M)/\delta^{N-2}$ and makes an offer after which the bargaining ends at time $N-1$ as his best action.

First, we prove that, for any set of offers $(s_1,\cdots,s_{N-1})$, if $1-(s_1+\cdots+s_{N-1})>\delta^{N-1}M/\delta^{N-2}$ and there is no rejection until time $t-2$, player N accepts the offer s_{N-1} as his best action. If player N accepts the offer s_{N-1}, his share is $1-(s_1+\cdots+s_{N-1})$, and his utility is $\delta^{N-2}(1-(s_1+\cdots+s_{N-1}))$. If he rejects the offer, he makes the first offer in the subgame which starts at time N, and his utilities in the set of subgame perfect equilibrium outcomes of this subgame are not larger than M. The discounted value of this M is $\delta^{N-1}M$ and $\delta^{N-1}M<\delta^{N-2}(1-(s_1+\cdots+s_{N-1}))$ holds. Therefore player N accepts the offer s_{N-2} as his best action, if $1-(s_1+\cdots+s_{N-1})>\delta^{N-1}M/\delta^{N-2}$ and there is no rejection until time $t-2$.

Using this fact, we can demonstrate that, for any $(s_1,\cdots,s_{N-2})$, if $1-(s_1+\cdots+s_{N-2})>(\delta^{N-2}M+\delta^{N-1}M)/\delta^{N-2}$ and there is no rejection until time $t-3$ player $N-1$ accepts the offer s_{N-2} and makes an offer which is accepted at time $N-1$ as his best action. (If he accepts s_{N-2} and makes an offer which is rejected, he is the N-th player in the subgame which starts at time N and his utility is less than $\delta^{N-1}M$. If he rejects s_{N-2}, his utility is less than $\delta^{N-2}M$.)

Similarly, we can prove that, for any i $(i=2,\cdots,N-2)$ and any $(s_1,\cdots,s_{i-1})$, if $1-(s_1+\cdots+s_{i-1})>(\delta^{i-1}M+\cdots+\delta^{N-1}M)/\delta^{N-2}$ and there is no rejection in advance, player i accepts the offer s_{i-1} and makes an offer after which the bargaining ends at time $N-1$ as his best action. ■

CLAIM 2. *$\delta^{N-2}-(\delta^1 m^1+\cdots+\delta^{N-1}m^1)\geq M$ holds.*

Proof. Suppose that this does not hold. Since M is the supremum of some player j's utility, there exists u such that $\delta^{N-2}-(\delta^1 m^1+\cdots+\delta^{N-1}m^1)<u<M$ and u is player j's utility in some subgame perfect equilibrium outcome. Notice that $1<(\delta^1 m^1+\cdots+\delta^{N-1}m^1+u)/\delta^{N-2}$ holds. Let t be the time when the last rejection occurs on this equilibrium path ($t=0$ if a rejection does not occur on the equilibrium path) and s^i be player i's share in this subgame perfect equilibrium outcome. Let k be the index of the player who makes the last rejection at time t (if $t=0$, $k=N$). We define $\tau(i)$ by $\tau(i)=t+i-k$ if $i-k>0$ and $\tau(i)=N+t+i-k$ if $i\leq k$.

We can show that there exists $i\in\{1,\cdots,N\}\setminus\{j\}$ such that $\delta^{t+N-2}s^i<\delta^{\tau(i)-1}m^1$. In fact, if it is not true, we have $1<(\delta^{\tau(1)-1}m^1+\cdots+\delta^{\tau(j-1)-1}m^1+u+\delta^{\tau(j+1)-1}m^1+\cdots+\delta^{\tau(N)-1}m^1)/\delta^{t+N-2}\leq s^1+\cdots+s^{j-1}+(u/\delta^{t+N-2})+s^{j+1}+\cdots+s^N$ and this is a contradiction since $(s^1,\cdots,s^{j-1},u/\delta^{t+N-2},s^{j+1},\cdots,s^N)$ is a partition in this subgame perfect equilibrium outcome. But this contradicts the fact that player i can make the first offer in the subgame which starts at time $\tau(i)$

by rejecting the offer at time $\tau(i)-1$. Therefore we have $\delta^{N-2}-(\delta^1 m^1+\cdots+\delta^{N-1}m^1)\geq M$.

Keio University

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