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OLIGOPOLY EQUILIBRIUM WITH CONSISTENT CONJECTURES

Kunio KAWAMATA

Abstract: We study a conjectural variations oligopoly model and establish the existence of Bresnahan-Perry consistent equilibrium in a general setting. We also consider an extended game in which firms may strategically change their conjectures on the slopes of rivals' reaction functions in equilibrium and show that Hahn's reasonableness condition in this game implies the consistency of conjectures in the original model. To prepare for these, we prove the existence of oligopoly equilibrium in which all firms make positive profits and study the directions of changes in equilibrium outputs and profits with respect to changes in conjectural variations parameters.

1. INTRODUCTION

The paper is concerned with the existence and characterization of oligopoly equilibrium with "reasonable" conjectures. But before beginning our discussion we remark that there exist apparently opposing opinions about the effectiveness of using a consistency or rationality condition to narrow down the number of "plausible" oligopoly equilibria. On the one hand, Bresnahan (1981) and Perry (1982) have presented concrete examples in which either the consistent equilibrium is unique or there are at most a very small number of consistent equilibria. On the other hand, general existence results by Laitner (1980) and Hart (1985) state that there are infinitely many conjecture functions of the price and outputs, which are rational.

We may explain this as follows. For one thing, whereas the definition of consistency used by Bresnahan and by Perry attaches special importance to the correctness, in equilibrium, of the estimate of the slopes of the reaction functions (relating their rivals' aggregate outputs to their own outputs), there are possibilities that many different conjecture functions have the same slopes at points corresponding to the same output combinations.

Secondly, some general definitions of rationality do not require the correctness of the estimate of the slopes of the above reaction functions but they are justified

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in some other ways.[1] Ulph (1984) describes how the consistency in the sense of Bresnahan and Perry emerges and give examples of other rational conjectural equilibria, including the case where some firms are not making positive profits, and so may remain only as potential entrants.

In this paper we shall follow Bresnahan and Perry insofar as we require the correct estimate of the slopes of reaction functions of the above type. We shall try to justify this on the basis of firms' profit maximizing behavior. The chief end of this paper is to demonstrate the existence of consistent equilibria in the general case where there are n firms, the total revenue functions are (slightly stronger than) concave and the cost functions of all firms are convex (Theorem 6). To this end, we treat the slopes of the conjectured reaction functions $\beta=(\beta_1, \cdots, \beta_n)$ as unknown parameters (referred to as conjectural variations) and show that their equilibrium values are determined by a set of n equations.

To prepare for this, in an early stage of the analysis, we will arbitrarily fix the conjectural variations and give conditions for the existence of a Nash equilibrium in the oligopoly market with positive outputs (and positive profits) for all firms. From there we proceed to study the effects, on output levels, market shares and price-cost margins, of perturbing a particular β_i. Among other things, we show that if all β_i lie between -1 and 0 then a decrease in β_i increases i's output but decreases the outputs of the other firms. We also study the effects on welfare of a change in β_i.

We shall show that, so long as the conjectural variations of the other firms are fixed between -1 and 0, the profit of firm i is maximized at a β_i lying strictly between -1 and 0. This leads us to consider an extended game in which each firm may change its β strategically so as to increase its profit (pay-off), a procedure suggested by Hart (1985). The equilibrium of this extended game corresponds to what Hahn (1977) called the reasonable conjectural equilibrium when the conjectures are made on the slopes of the response functions. It is shown that any reasonable conjectural equilibrium of the present model is consistent in the sense of Bresnahan (1981) and Perry (1982) (Theorem 5).

At any non-consistent equilibrium, some firms can increase their profits simply by modifying their conjectures on their rivals' reaction. In this sense we may say that, although there may be infinitely many (general) rational conjectural functions, the equilibrium slopes of "plausible" conjectural functions (hence those of reaction functions) and the corresponding output levels are very limited in number.

2. BASIC ASSUMPTIONS

We shall consider an oligopolistic market in the traditional partial equilibrium framework in which all firms in the market produce a homogeneous good. The

[1] According to a general definition, the ordinary Cournot-Nash equilibrium is rational under the standard (Cournot-Nash) conjecture, but it is not a consistent equilibrium in the sense of Bresnahan and Perry.

buyer's side of the market is characterized by price taking behavior.

Let x be the market demand for the product and let p be its price. We shall assume that the inverse demand function can be expressed as

$$p=f(x)\,, \tag{1}$$

where f[2] is twice continuously differentiable and satisfies

A1 (i) $f(0)>0$,
(ii) $f'(x)<0$ and $f'(x)+xf''(x)<0$ for each $x>0$ such that $f(x)>0$.

The second condition of assumption A1(ii) is clearly implied by the first condition if, as is often assumed in the literature, f is a linear function of price. Admittedly this is a very stringent condition for some of the discussions below,[3] but it will help to unify and clarify the main argument of the present paper.

The total cost of the i-th firm corresponding to output level x_i will be denoted $C_i(x_i)$ $(i=1, 2, \cdots, n)$. We will assume that each cost function $C_i(\cdot)$ is twice continuously differentiable and satisfies the conditions

A2 (i) $C_i(0)\geq 0$
(ii) $C_i'(x_i)>0$ and $C_i''(x_i)>0$ for each $x_i>0$.

These conditions are standard, and we note only that they rule out the case of increasing returns to scale.

The condition for the balance of demand and supply of the product can be written as

$$x=x_1+\cdots+x_n\,. \tag{2}$$

We shall assume that this condition is always satisfied and refer sometimes to x as the total supply of the product. We shall also introduce a new variable x_{-i} to denote the aggregate output level of firms other than i. By definition we have

$$x=x_i+x_{-i}\,. \tag{2)'$$

The profit of the i-th firm can then be expressed as

$$\Pi_i(x_i, x)=f(x)x_i-C_i(x_i) \tag{3}$$

for each i. In view of (2) we may also write this function as $\Pi_i(x-x_{-i}, x)$, or as $\Pi_i(x_i, x_i+x_{-i})$, using any two of the three variables x_i, x_{-i} and x.

To rule out trivial cases we shall assume

[2] We do not rule out the possibility that $f(x)=0$ for some $x>0$. In this case the domain of f is not all of the non-negative reals R_+ but an interval $[0, \bar{x}]$, where $\bar{x}$ is the minimum of the x's satisfying the above condition. It does not matter if $f(0)$ is infinitely large.

[3] The second condition A1 (ii) is used only to ensure that $f'(x)+x_if''(x)$ is negative. And so if we have a good bound for x_i/x (as in the case of symmetric oligopoly), the condition can be weakened considerably. Also the condition needs to be satisfied only at equilibrium points, which we know lie in a closed interval $[\underline{X}, \overline{X}]$, where $\underline{X}$ and $\overline{X}$ are defined in (17).

A3 For each $i=1, 2, \cdots, n$

(i) $f(0)>C_1'(0)$ and $f(x_i)<C_i'(x_i)$ for some x_i and

(ii) there is an $x_i>0$ such that $\Pi_i(x_i, x_i)>0$.

Condition A3(ii) means that each firm could earn a profit if there were no other firms in the economy.

Let β_i denote the *conjectural variation* of the i-th firm ($i=1, 2, \cdots, n$). It shows how the firm estimates the change in the aggregate output of the other firms, x_{-i}, per unit change in x_i. For the following analysis it is convenient to define the *combined conjectural variation* α_i which is related to β_i by the formula

$$\alpha_i=\beta_i+1 . \tag{4}$$

This parameter shows how firm i estimates the change in aggregate output, x, per unit change in x_i. Behavioristically, α_i's may depend on x_i's, but, if nothing is said to the contrary, we shall only be concerned with the equilibrium values of α_i's. Of course if α_i's are constant as in the case of competitive conjecture ($\alpha_i=0$) or Cournot conjecture ($\alpha_i=1$) we need no such qualification.

Assume that firm i maximizes profit supposing that combined conjectural variation is α_i ($i=1, 2, \cdots, n$). The condition for the interior maximum of profit is then

$$R_i(x_i, x)\equiv f(x)+\alpha_i x_i f'(x)-C_i'(x_i)=0 \qquad (i=1, 2, \cdots, n) . \tag{5}$$

In view of (2)′ we may write each of the equations in (5) as

$$R_i(x-x_{-i}, x)=0 , \tag{5$'$}$$

or as

$$R_i(x_i, x_i+x_{-i})=0 . \tag{5$''$}$$

3. EXISTENCE OF EQUILIBRIUM UNDER FIXED CONJECTURAL VARIATIONS

In this section we will arbitrarily fix the parameters α_i ($i=1, 2, \cdots, n$) and investigate conditions for the existence of equilibrium with non-negative profits and positive outputs for all firms. We will also study the differentiability of the equilibrium allocation with respect to the α_i's.

If

$$x_i=x_i(x_{-i}) \tag{6}$$

maximizes $\Pi_i(x_i, x_i+x_{-i})$ given $\beta_i=dx_{-i}/dx_i$, then the function $x_i(\cdot)$ will be referred to as the *reaction function* of the i-th firm. The function $x=X_i(x_{-i})$ defined as

$$X_i(x_{-i})=x_i(x_{-i})+x_{-i} \tag{7}$$

(or as $x=X_i(x_{-i})$ that maximizes $\Pi_i(x-x_{-i}, x)$ given $\alpha_i=dx/dx_i$) will be called the *combined reaction function* (see, McManus (1964)).

Given $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$, we say that the vector of output levels $(x_1^*, x_2^*, \cdots, x_n^*, x^*)$ in R_+^{n+1} is an *oligopoly equilibrium* if it satisfies (2) and (x_i^*, x_{-i}^*) is on the (graph of) reaction function of the i-th firm, for $i=1, 2, \cdots, n$. We note that the equilibrium reduces to the competitive equilibrium if all α_i's are zero and it reduces to the Cournot-Nash equilibrium if all α_i's are 1. We will also be interested in the equilibrium where the profits are non-negative and outputs are positive for all firms. Notice that the non-negativity of the profit of the i-th firm implies $x_i^*>0$ if the fixed cost $C_i(0)$ is positive.

Now given x_{-i}, the reaction function is devied from (5)″ unless a corner maximum (the case where $x_i=0$ for some i) occurs. From (5)″ it follows that

$$\frac{dx_i}{dx_{-i}}=-\frac{f'+\alpha_i x_i f''}{(1+\alpha_i)f'+\alpha_i x_i f''-C_i''}\,, \tag{8}$$

which takes on a value between -1 and 0 if $0\leq\alpha_i\leq 1$. Hence for each x_{-i} in a suitable domain there is a unique x_i that solves (5)″. This shows that the reaction function is downward sloping and that the combined reaction function, derived from (5)′, is upward sloping.

To guarantee that all firms have non-negative profits or positive outputs in equilibrium, we need to impose an additional condition.[4] We will give two theorems which guarantee the existence of equilibrium with positive outputs and satisfy the differentiability condition with respect to α. Theorem 1A applies to the general case where fixed costs are non-zero. Theorem 1B applies only to the zero-fixed-cost case but the required additional condition is weaker and the conclusion is stronger. The rest of this section, except, perhaps, the statement of parts (b), (c) of Lemma 2, may be skipped at the first reading.

Since

$$\frac{dx_i}{dx}=\frac{f'+\alpha_i x_i f''}{C_i''-\alpha_i f'} \tag{9}$$

is negative under our basic assumptions, (5) maybe solved for x_i, which we will write as

$$x_i=\tilde{x}_i(x)\,, \tag{10}$$

In view of (2) the function (10) carries the same information as the two reaction functions introduced before, and was used by Yakowitz and Szidarovsky (1977)) in their proof of existence of Cournot-Nash equilibrium. We will refer to (10) as the Yakowitz-Szidarovsky reaction function.

To investigate the existence problem we need to be more explicit about the domain of the Yakowitz-Szidarovsky reaction function. Thus let D_i be the set of

[4] Notice, for example, if all firms have constant marginal costs, an interior competitive equilibrium of the economy exists only when these values are the same.

all $x \geq x_i$ such taht $\Pi_i(x_i, x)$ can be made non-negative by a suitable choice of $x_i \geq 0$. Thus by definition

$$D_i = \{x \mid \Pi_i(x_i, x) \geq 0 \text{ for some } x \geq x_i \geq 0\} . \quad (11)$$

If in addition to $x \geq x_i$ we require only that $x_i \geq 0$ (but not the non-negativity of profits), then the corresponding domain will be denoted $\overline{D}_i$. In the sequel we will mainly be concerned with D_i and refer to $\overline{D}_i$ only parenthetically.

Since $f' < 0$, it follows that, for every x_i, $\Pi_i(x_i, x)$ is a decreasing function of x. Hence if we let

$$\overline{X}_i = \sup\{x \mid x \text{ is in } D_i\} \quad (12a)$$

and

$$\underline{X}_i = \inf\{x \mid x \text{ is in } D_i\} , \quad (12b)$$

we know that $D_i = [\underline{X}_i, \overline{X}_i]$ (when $\overline{X}_i = \infty$, the domain should be understood as $[\underline{X}_i, \infty)$). Notice that D_i is non-degenerate by A3 (ii). By the monotonicity of the combined reaction function, $\underline{X}_i$ is the solution of (5)′ when $x_{-i} = 0$, i.e., the solution of

$$f(x) + \alpha_i x f'(x) = C_i'(x) . \quad (13)$$

This $\underline{X}_i$ will be referred to as the *α_i-monopolistic solution* for firm i. When $\alpha_i = 0$ this solution is the *competitive solution* for firm i,[5] and when $\alpha_i = 1$ we have the standard monopoly solution in a partial equilibrium framework.

It is clear from the monotonicity of $\Pi_i(x_i, x)$ with respect to x, that the maximum in (12a) is attained when the profit is zero. Hence we will say that $\overline{X}_i$ is *the zero profit total output* for firm i. The corresponding price $\overline{p}_i = f(\overline{X}_i)$ is the "limit price" for firm i in the sense that any lower price discourages the firm from entering the market.

The domain $\overline{D}_i$ is also an interval of the form $[\underline{X}_i, \overline{X}_i']$ where $\overline{X}_i'$ is defined as the solution of (5) when $x_i = 0$ i.e., as $\overline{X}_i' = \min f^{-1} C_i'(0)$. To summarize, we have proved

LEMMA 1. *Under assumptions* A1–A3, *if* $0 \leq \alpha_i \leq 1$, *then the Yakowitz-Szidarovsky reaction function* (10) *is obtained by solving* (5) *for each x in $\overline{D}_i = [\underline{X}_i, \overline{X}_i']$* ($i = 1, 2, \cdots, n$). *These functions satisfy $\Pi_i \geq 0$ on $D_i = [\underline{X}_i, \overline{X}_i]$ and their slopes are negative.*

Another important result we will use below is that

$$\underline{X}_i = \tilde{x}_i(\underline{X}_i) \quad (14)$$

for each i ($i = 1, 2, \cdots, n$). This follows from the fact that X_i is smallest when the corresponding x_{-i} defined by (5)′ is zero.

[5] More precisely, it is the competitive solution in the case where only firm i exists in the market.

By construction, the vector $(x_1^*, x_2^*, \cdots, x_n^*, x^*)$ is a solution of equations (2) and (5) with non-negative profits for all firms if

$$x^* = \sum_{j=1}^{n} \tilde{x}_j(x^*) \tag{15}$$

and

$$\underline{X}_i \leq x^* \leq \overline{X}_i \qquad (i=1, 2, \cdots, n)\,. \tag{16}$$

To simplify notation, let us denote

$$\underline{X} = \max_i \underline{X}_i \tag{17a}$$

$$\overline{X} = \min_i \overline{X}_i\,. \tag{17b}$$

Then (16) is equivalent to

$$\underline{X} \leq x^* \leq \overline{X} \tag{16$'$}$$

and so in looking for the above equilibrium we may confine our attention to the interval $D = [\underline{X}, \overline{X}]$.

Since each $\tilde{x}_i(\cdot)$ is a monotone decreasing function, it is necessary for the existence of the above equilibrium (namely for (15) to hold) that

$$\underline{X} \leq \sum_{j=1}^{n} \tilde{x}_j(\underline{X}) \tag{18a}$$

and

$$\overline{X} \geq \sum_{j=1}^{n} \tilde{x}_j(\overline{X})\,. \tag{18b}$$

But since $\tilde{x}_i(\underline{X}_i) = \underline{X}_i$ by (14), the condition (18a) is automatically satisfied.

We are now ready to state our basic existence and differentiability results:

LEMMA 2. *Let assumptions* A1–A3 *be satisfied and* $0 \leq \alpha_i \leq 1$ $(i=1, 2, \cdots, n)$. *Then* (a) *a necessary and sufficient condition for the existence of an oligopoly equilibrium* $(x_1^*, x_2^*, \cdots, x_n^*, x^*)$, *with non-negative profits for all firms, is that*

(i) $\underline{X} \leq \overline{X}$ *and*

(ii) $\overline{X} \geq \sum_{j=1}^{n} \tilde{x}_j(\overline{X})$.

If the inequality in (ii) *is strict, the equilibrium profits are all positive. If only positivity of outputs, but not of profits, is required,* $\overline{X}$ *may be replaced by* $\overline{X}' = \min f^{-1}C_i'(0)$. *Moreover* (b) *the equilibrium, if it exists, is unique and* (c) *the equilibrium is a continuously differentiable function of* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$.

Proof. (a) Necessity has been shown above.

To show sufficiency, consider

$$\phi(x)=x-\sum_{i=1}^{n}\tilde{x}_i(x) \tag{19}$$

defined on $[\underline{X},\overline{X}]$. This function is clearly continuous and, by Lemma 1, it is strictly increasing.

Since $\phi(\underline{X})\leq 0$ by (18a) and $\phi(\overline{X})\geq 0$ by (ii), the intermediate value theorem guarantees that there is x^* such that $\underline{X}\leq x^*\leq\overline{X}$ and $\phi(x^*)=0$. This implies the equilibrium conditions (15) and (16). If the inequality in (ii) is strict, x^* is different from $\overline{X}_i$ for any i. Hence the equilibrium profit is positive since profit increases as x decreases.

In the case where only positivity of outputs is required, a similar analysis as above applies with the extended domain $\overline{D}_i=[\underline{X}_i,\overline{X}'_i]$, where $\overline{X}'_i=f^{-1}C'_i(0)$.

(b) The uniqueness of the equilibrium is clear since $\phi(\cdot)$ is strictly increasing.

(c) To show differentiability, we note that the system of equations (2) and (5) may be written as

$$\alpha_i=\frac{f(x_1+\cdots+x_n)-C'_i(x_i)}{-x_if'(x_1+\cdots+x_n)} \qquad (i=1,2,\cdots,n)\,. \tag{20}$$

Since, for each i and $j\neq i$,

$$\frac{\partial\alpha_i}{\partial x_j}=\frac{-(f'+\alpha_ix_if'')}{x_if'} \tag{21}$$

and

$$\frac{\partial\alpha_i}{\partial x_i}=\frac{C''_i-(1+\alpha_i)f'-\alpha_ix_if''}{x_if'}\,, \tag{22}$$

the Jacobian matrix of the transformation $\alpha\colon R^n_+\to R^n_+$, defined by (20), is of the form

$$J=-\begin{pmatrix} a_1+b_1\,, & a_1\,, & \cdots & a_1\\ a_2\,, & a_2+b_2\,, & \cdots & a_2\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ a_n\,, & a_n\,, & \cdots & a_n+b_n \end{pmatrix},$$

where all a_j and b_j are positive numbers. It is easy to verify that the principal minor of order n of J has the sign $(-1)^n$. One way to show this is to express J as the sum of two matrices one of which is a diagonal matrix whose i-th diagonal element is $-b_i$ for $i=1,2,\cdots,n$. Hence differentiability follows from the implicit function theorem.

REMARK. In fact we know that the transformation $\alpha\colon R^n\to R^n$ defined by (20) is globally invertible in an open convex domain of R^n by a Gale-Nikaido theorem (see, Nikaido (1968) Corollary to Theorem 20.8). What Lemma 2 asserts is that the range of the transformation contains the unit cube in R^n under the conditions given.

Let x_i^c be the competitive solution of firm i, i.e., the solution of $f(x_i)=C_i'(x_i)$. Also let $\bar{X}_i$ be the zero profit total output for firm i (see, (12a)). Using Lemma 2 we can establish

THEOREM 1A. *Under assumptions* A1–A3, *if* $0 \leq \alpha_i \leq 1$ $(i=1, 2, \cdots, n)$ *and*

$$\text{(C)}' \qquad \min_i \bar{X}_i \geq \sum_{j=1}^{n} x_j^c ,$$

then there exists a unique equilibrium $(x_1^*, x_2^*, \cdots, x_n^*, x^*)$ *of the system* (2) *and* (5) *with non-negative profits for all firms. Moreover, the equilibrium is a continuously differentiable function of* $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$. *If the inequality in* (C) *is strict, the equilibrium profits are all positive. In the case where only positivity of outputs, but not non-negativity of profits, is required.* (C) *may be replaced by*

$$\text{(C)} \qquad \min_i f^{-1}C_i'(0) > \sum_{j=1}^{n} x_j^c$$

and the other conclusions hold.

Proof. Since $\underline{X}_i$ is defined by (13) and x_i^c is obtained from the same equation by setting $\alpha_i=0$, we have $x_i^c \geq \underline{X}_i$ for each i.[6] Hence, using (C), we obtain

$$\bar{X} \geq \sum x_j^c \geq \sum \underline{X}_j , \tag{23}$$

where the summations are over $j=1, 2, \cdots, n$. Thus condition (i) of Lemma 2 is satisfied.

Moreover, since the Yakowitz-Szidarovsky reaction function is decreasing in x, (14) implies

$$\underline{X}_j > \tilde{x}_j(\bar{X}_j) \tag{24}$$

for each $i=1, 2, \cdots, n$. Hence summing over j and using (23), we derive the condition (ii) of Lemma 2. In the case where profits need not to be non-negative, a similar proof applies using the fact that $\bar{X}_i'=f^{-1}C_i'(0)$.

REMARKS. Condition (C) (and hence (C)′) of Theorem 1A is likely to be satisfied if (1) the inverse demand curve approaches zero very slowly and (2), for each firm, average cost remains low for low output levels and marginal cost becomes high before the output level becomes very large. The second condition of (2), would make x_i^c moderately small and (1), together with the first condition of (2), would make $f(x) > C_i(x_i)/x_i$ when x_i is small and x is very large, thus ensuring a very

[6] It is easy to show that the solution x of (10) is a decreasing function of α_i.

large $\overline{X}_j$. It is clear that Theorem 1 holds if the left hand side of (C) is replaced by $\min \bar{x}_{-i}$ where, for each i, $\bar{x}_{-i}$ is the cut-off value of the rival's output defined (independently of the firm i's output) by

$$\bar{x}_{-i} = \sup\{x_{-i} \mid \Pi_i(x_i, x_i + x_{-i}) \geq 0 \text{ for some } x_i \geq 0\} .$$

Another consequence of Lemma 2 is

THEOREM 1B. *Under assumptions* A1–A3, *if* $0 \leq \alpha_i \leq 1$ $(i = 1, 2, \cdots, n)$ *and the fixed cost of each firms is zero, then there exists an equilibrium* $(x_1^*, x_2^*, \cdots, x_n^*, x^*)$ *of the system* (2) *and* (5) *with all outputs positive if and only if*

$$\text{(C)} \qquad \min_i \overline{X}_i \geq \max_i \underline{X}_i ,$$

i.e., if and only if the minimum of the zero profit total ouput (*see* (12a)) *is not less than the maximum of the* α_i*-monopolistic solutions* (*the solution of* (13)). *Moreover, if the above conditions are satisfied, then the equilibrium is unique and it is a continuously differentiable function of* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. *Condition* (C) *is satisfied if all firms have the same technology and the same conjectural variation parameter or if*

$$\text{(C)}' \qquad \min_i \overline{X}_i \geq \max_i x_i^c$$

holds, where x_i^c *is the competitive solution for firm i.*

Proof. If fixed costs are zero for each firm, *then* $\overline{X}_i$ is $f^{-1}C_i'(0)$ and the corresponding x_i is zero. Hence condition (ii) of Lemma 2 is satified with strict inequality. Condition (i) of Lemma 2 is identical with condition (C) of Theorem 1B. The statement about the identical firm case follows since $\overline{X}_i \geq \underline{X}_i$ for each i. Finally, since $x_i^c \geq \underline{X}_i$ for each i, we know that (C)′ implies (C).

4. EFFECTS ON PRODUCTION, PRICE-COST MARGINS AND WELFARE

Let us suppose that the system of equations (2) and (5) has a solution in which the profits of all firms are positive. Lemma 2 gives a sufficient condition for this, which we shall assume often without mentioning it. In this section and in following sections we shall analyze the effects of a small change in a particular conjectural variation α_i on some important economic variables, including output levels, price-cost margins and profits. We shall also study the impact on economic welfare of a change in α_i.

Differentiating the system (2) and (5) with respect to α_i we have

$$(f' + \alpha_i x_i f'') \frac{\partial x}{\partial \alpha_i} + (\alpha_i f' - C_i'') \frac{\partial x_i}{\partial \alpha_i} = -x_i f' \tag{25a}$$

and

$$(f' + \alpha_j x_j f'') \frac{\partial x}{\partial \alpha_i} + (\alpha_j f' - C_j'') \frac{\partial x_j}{\partial \alpha_i} = 0 \tag{25b}$$

for each $i = 1, 2, \cdots, n$ and $j \neq i$.

Now to simplify notation, let us define functions $\gamma_i(x_i, x)$ and $\delta_i(x_i, x)$ by

$$\gamma_i = \frac{f' + \alpha_i x_i f''}{\alpha_i f' - C_i''} \qquad (i = 1, 2, \cdots, n) \tag{26}$$

and

$$\delta_i = \frac{x_i f'}{\alpha_i f' - C_i''} \qquad (i = 1, 2, \cdots, n)\,. \tag{27}$$

We notice that γ_i and δ_i are positive under assumptions A1 and A2 if $0 \leq \alpha_i \leq 1$.

Equations (25) may then be written as

$$\gamma_i \frac{\partial x}{\partial \alpha_i} + \frac{\partial x_i}{\partial \alpha_i} + \delta_i = 0 \tag{25a)'$$

and

$$\gamma_j \frac{\partial x}{\partial \alpha_i} + \frac{\partial x_j}{\partial \alpha_i} = 0 \tag{26b)'$$

for each $i = 1, 2, \cdots, n$ and $j \neq i$.

Hence, noticing that

$$\frac{\partial x}{\partial \alpha_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial \alpha_i}\,,$$

we have[7]

$$\frac{\partial x}{\partial \alpha_i} = -\delta_i / \upsilon\,, \tag{28}$$

$$\frac{\partial x_i}{\partial \alpha_i} = -\delta_i (1 - \gamma_i / \upsilon)\,, \tag{29}$$

and

$$\frac{\partial x_j}{\partial \alpha_i} = \delta_i \gamma_j / \upsilon \tag{30}$$

for each $i = 1, 2, \cdots, n$ and $j \neq i$, where $\upsilon = \upsilon(x_i, x)$ is defined by

[7] Formula (28) may be obtained by adding all equations in (25).

$$v = 1 + \sum_{j=1}^{n} \gamma_j . \tag{31}$$

These relations immediately imply the following:

THEOREM 2. *Under assumptions* A1–A3 *and* $0 \leq \alpha_i \leq 1$ $(i = 1, 2, \cdots, n)$, *it follows that* $\partial x_i / \partial \alpha_i < 0$, $\partial x_j / \partial \alpha_i > 0$, for each $j \neq i$, *and* $\partial x / \partial \alpha_i < 0$.

Next define the market share of firm i by

$$s_i = x_i / x \tag{32}$$

for each i. The next proposition is a direct consequence of Theorem 2.

COROLLARY 1. *Under assumptions* A1–A3 *and* $0 \leq \alpha_i \leq 1$ $(i = 1, 2, \cdots, n)$, *it follows that* $\partial s_i / \partial \alpha_i < 0$ *and* $\partial s_j / \partial \alpha_i > 0$ *for* $j \neq i$.

Proof. The second part is clear from Theorem 1, and the first part follows from it since market shares add up to 1.

The price-cost margin of firm i is defined by

$$m_i = (p - C_i')/p \tag{33}$$

for each i. The effects of a change in α_i on m_j $(j = 1, 2, \cdots, n)$ are given by

COROLLARY 2. *Under assumptions* A1–A3 *and* $0 \leq \alpha_i \leq 1$ $(i = 1, 2, \cdots, n)$, *it follows that* (i) $\partial m_i / \partial \alpha_i > 0$ *and* (ii) $\partial m_j / \partial \alpha_i > 0$ *if and only if* $1 + ff''/(f')^2 < C_j'/x_j C_j''$.

Proof. The first part of the statement easily follows from

$$\frac{\partial m_i}{\partial \alpha_i} = -\frac{1}{f^2}\left(f C_i'' \frac{\partial x_i}{\partial \alpha_i} - f' C_i' \frac{\partial x}{\partial \alpha_i} \right), \tag{34}$$

and Theorem 2.

For the second part, we compute $\partial m_j / \partial \alpha_i$ just as above and then obtain

$$\frac{\partial m_j}{\partial \alpha_i} = -\frac{\delta_i}{f^2 v} (f C_j'' \gamma_j + C_j' f') \tag{35}$$

using (28) and (30). We then infer, from (26) and (5), that the expression in parenthesis is negative if and only if

$$1 + \frac{ff''}{(f')^2} - \frac{C_j'}{x_j C_j''} < 0 . \tag{36}$$

Hence (ii) follows.

REMARK. If we denote the demand elasticity by ε and the supply elasticity by η_j for each j, we know that the condition (36) is satisfied if either $f'' < 0$ and $\eta_j > 1$ or $\eta_j > \varepsilon + 1$.

One of the most interesting and important consequences of a change in α_i is the effect which it has on the profits of firms in the economy. In the next section, we analyze this in detail, together with some related problems.

To conclude this section we study the effects of a perturbation of α_i on economic welfare. We measure welfare by

$$W = \int_0^x f(y)dy - \sum_{k=1}^{n} C_k(x_k), \tag{37}$$

which is equivalent to the sum of consumer's and producers' surpluses.

The next theorem gives conditions under which a decrease in α_i increases welfare.

THEOREM 3. *Under assumptions* A1–A3 *and* $0 \leq \alpha_i \leq 1$ $(i = 1, 2, \cdots, n)$, *if* (C) $\alpha_i x_i \geq \alpha_k x_k$ *for any* k (*i.e., the elasticity of firm i's conjectured change in demand* $\alpha_i x_i / x$ *is the largest of all*) *then* $\partial W / \partial \alpha_i < 0$.

Proof. In view of (5) we have

$$\begin{aligned} \frac{\partial W}{\partial \alpha_i} &= f \frac{\partial x}{\partial \alpha_i} - \sum C_k' \frac{\partial x_k}{\partial \alpha_i} \\ &= -\left(\sum \alpha_k x_k \frac{\partial x_k}{\partial \alpha_i} \right) f', \end{aligned} \tag{38}$$

where the summation is over $k = 1, 2, \cdots, n$. Hence making use of (C) and Theorem 2 we have

$$\begin{aligned} \frac{\partial W}{\partial \alpha_i} &\leq -\alpha_i x_i f' \sum \frac{\partial x_k}{\partial \alpha_i} \\ &= -\alpha_i x_i f' \frac{\partial x}{\partial \alpha_i} \\ &< 0. \end{aligned} \tag{39}$$

5. THE EFFECTS OF A CHANGE IN α_i ON PROFITS

In this section we study the effects of a change in α_i on the profits of the firms in the economy. We will continue to assume the conditions of Lemma 2. Our basic result is the following:

THEOREM 4. *Under assumptions* A1–A3 *and* $0 < \alpha_i < 1$ $(i = 1, 2, \cdots, n)$,

(i) $\partial \Pi_i / \partial \alpha_i$ *is positive if* α_i *is sufficiently close to zero and negative if* α_i *is sufficiently close to* 1. *Given* $0 < \alpha_j < 1$ $(j \neq i)$, Π_i *attains a maximum at a point where* $\partial x / \partial \alpha_i = \alpha_i \, \partial x_i / \partial \alpha_i$.

(ii) $\partial \Pi_j / \partial \alpha_i > 0$ *for* $j \neq i$.

(iii) $\partial (\sum_k \Pi_k) / \partial \alpha_i > 0$ *if* $\alpha_i x_i \leq \alpha_k x_k$ *for all* k.

Proof. Using (5) and then (28) and (29) we have

$$\frac{\partial \Pi_i}{\partial \alpha_i}=(f-C_i')\frac{\partial x_i}{\partial \alpha_i}+f'x_i\frac{\partial x}{\partial \alpha_i}$$

$$=\left(\frac{\partial x}{\partial \alpha_i}-\alpha_i\frac{\partial x_i}{\partial \alpha_i}\right)x_if'$$

$$=\left(-x_if'\delta_i\left[(1-\alpha_i)-\alpha_i\sum_{k\neq i}\gamma_k\right]\right)\Big/v\,. \tag{40}$$

The final expression of (40) may be regarded as a continuous function of $(\alpha_1, \cdots, \alpha_n)$, by Lemma 2. We find that it is positive if $\alpha_i=0$ and negative if $\alpha_i=1$. It is now easy to complete the proof of part (i).

To show (ii), we use (5) to obtain

$$\frac{\partial \Pi_j}{\partial \alpha_i}=(f-C_j')\frac{\partial x_j}{\partial \alpha_i}+f'x_j\frac{\partial x}{\partial \alpha_i}$$

$$=\left(\frac{\partial x}{\partial \alpha_i}-\alpha_j\frac{\partial x_j}{\partial \alpha_i}\right)x_jf'\,. \tag{41}$$

Hence the result follows from Theorem 2.

Finally, if the condition in (iii) holds, in view of (40), (41) and Theorem 2, we have

$$\frac{\partial(\sum \Pi_k)}{\partial \alpha_i}=f'\left(\frac{\partial x}{\partial \alpha_i}x-\sum_k \alpha_k x_k\frac{\partial x_k}{\partial \alpha_i}\right)$$

$$=f'\left(\frac{\partial x_i}{\partial \alpha_i}(x-\alpha_i x_i)+\sum_{k\neq i}\frac{\partial x_k}{\partial \alpha_i}(x-\alpha_k x_k)\right)$$

$$\geq f'(x-\alpha_i x_i)\frac{\partial x}{\partial \alpha_i}$$

$$\geq 0\,. \tag{42}$$

6. CONSISTENT AND REASONABLE EQUILIBRIUM

So far we have analyzed the behavior of a particular firm assuming that other firms' conjectural variations are fixed. We will now proceed to analyze the situation where firms may change α_i's depending on the other firms' behavior. The equilibrium values of them will be our major concern in the following discussion.

Following Bresnahan [1981] and Perry [1982] we will say that firm j's conjecture is *consistent* given $\alpha=(\alpha_1, \cdots, \alpha_n)$ if its conjectural variation α_j equals the slope of the actual response function of the other firms. Formally, let $i\neq j$ and differentiate (5) with respect to x_j. We may write the resulting equation as

$$\gamma_i \frac{dx}{dx_j} + \frac{dx_i}{dx_j} = 0 \qquad (i \neq j),$$

using (26). In view of (2) we may then obtain the slope of the aggregate response function with respect to the change in x_j as

$$\frac{dx}{dx_j} = \frac{1}{1 + \sum_{k \neq j} \gamma_k}. \tag{43}$$

Consistency requires that $\alpha_j = dx/dx_j$, where the slope is evaluated at the equilibrium point given $\alpha = (\alpha_1, \cdots, \alpha_n)$. The oligopoly equilibrium with combined conjectural variations $\alpha = (\alpha_1, \cdots, \alpha_n)$ is *consistent* if each firm's conjecture is consistent in the above sense.

We next proceed to define another related concept of equilibrium. To each vector of combined conjectural variations $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, let us associate the profit vector $\Pi[\alpha] = (\Pi_1[\alpha], \Pi_2[\alpha], \cdots, \Pi_n[\alpha])$ which is determined uniquely under the hypothesis of Lemma 2 or Theorem 1 (which includes the condition that $0 < \alpha_i < 1$ for all i).

We shall refer to $(x_1^*, \cdots, x_n^*, x^*, \alpha_1^*, \cdots, \alpha_n^*)$, or simply $\alpha^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)$, as an *oligopolistic equilibrium of the extended game* if $(x_1^*, \cdots, x_n^*, x^*)$ is the equilibrium of the original game given $\alpha^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)$ and no single firm i can increase its profit by any small perturbation of α_i^*.[8]

The last condition can be stated formally as

$$\Pi_i[\alpha^*] \geq \Pi_i[\alpha^*/\alpha_i] \qquad (i = 1, 2, \cdots, n), \tag{44}$$

for any α_i in some neighborhood of α_i^* in the real line, where (α^*/α_i) denotes the vector obtained from α^* by replacing its i-th component by α_i. This notion of equilibrium corresponds to what Hahn (1978) called the *reasonable conjectural equilibrium* when the conjecture is made on the slopes of response functions. Henceforward we will use the same terminology for the above equilibrium.

Since, by Theorem 4(i), the maximum of Π_i with respect to α_i is attained in the interior or the unit interval, in looking for the extended equilibrium such that $0 \leq \alpha_i \leq 1$ $(i = 1, 2, \cdots, n)$, we may concentrate our attention on the condition that $\partial \Pi_i / \partial \alpha_i = 0$ for all i. In view of (40) this may be expressed as

[8] A stronger requirement is that any perturbation of firm i's α_i will (strictly) decrease its profit. This definition reduces to the original one in the regular case where each local maximum of profit is attained at an isolated point. If the demand function and all cost functions are sufficiently smooth, Sard's theorem guarantees that the set of critical values has measure zero. We may also require that profit is maximized globally rather than locally.

Implicit in this definition is the supposition that the adjustment in conjectures is made relatively slowly and in full anticipation of the adjustment in outputs. The equilibrium concept here corresponds to the subgame perfectness in the game theory.

$$\alpha_i = \frac{1}{1+\sum_{k \neq i} \gamma_k(\alpha)} \qquad (i=1, 2, \cdots, n), \tag{45}$$

where γ_k is defined by (26).

Since (45) is the condition defining the consistency of firms' conjectures we have established

THEOREM 5. *Under conditions* A1–A3, *if* $E=(x_1^*, \cdots, x_n^*, x^*, \alpha_1^*, \cdots, \alpha_n^*)$ *is a reasonable conjectural equilibrium with* $0 \leq \alpha_i^* \leq 1$ *for all* i, *then* E *is a consistent equilibrium.*

We will now state and prove the main existence theorem for consistent equilibrium, which generalizes a result of Bresnahan [1981] to a much broader situation:

THEOREM 6. *Under assumptions* A1–A3 *and condition* (C) *of Theorem* 1A (*or condition* (C)′ *of Theorem* 1B *if fixed costs are zero*) *there exists a consistent equilibrium such that* $0 < \alpha_i < 1$ *for all* $i = 1, 2, \cdots, n$.

Proof. Let us consider the function

$$\psi_i(\alpha) = \frac{1}{1+\sum_{k \neq i} \gamma_k(\alpha)} \qquad (i=1, 2, \cdots, n) \tag{46}$$

defined on the unit cube $I^n = \{\alpha \in R^n \; 0 \leq \alpha_i \leq 1, \; i=1, 2, \cdots, n\}$.

Since each γ_k is positive, $\psi = (\psi_1, \cdots, \psi_n)$ maps the unit cube into another unit cube. By Lemma 2 and continuity of the functions defining γ_i (see, (26)), ψ is a continuous function of α. Hence by Brower's fixed point theorem there exists $\alpha^* \in I^n$ such that $\psi(\alpha^*) = \alpha^*$. By construction this α^* satisfies all of the conditions in (45). Also it is clear that no α_i^* can be 0 or 1. (Note that $\underline{X}_i$ in (C) of Theorem B may depend on α_i and the competitive solution is the maximum of all $X_i(\alpha_i)$'s when α_i varies in the unit interval).

Since there are n equations in (45), consistency is usually enough to determine the admissible values of conjectural variations. We will show below that the consistency condition in Bresnahan [1981] can be derived directly from (45).

If, for each i, there exists a unique α_i which satisfies (45) given all the other conjectural variations, then by Theorem 4(i), it gives the maximum profit for the firm. Hence it defines the reasonable conjectural equilibrium. But in general (45) may also give other kind of stationary points. Since the second derivative of $\Pi_i[\alpha]$ with respect to α_i depends on the third order derivative of f and of C_i, it is usually

[9] Hart [1985, p. 130] states that at any reasonable equilibrium each firm's conjectured slope of the demand function must equal the actual slope. He also gives a brief proof of it assuming differentiability and interiority of solution, which are justified here under the stated condition.

hard to prove directly that a stationary point is a local maximum.

PROPOSITION. *If all n firms have the same cost function* $C(\cdot)$ *and the fixed cost is zero, then there exists an equilibrium of the extended game in which all* α_i^* $(i=1, 2, \cdots, n)$ *are the same and strictly between* 0 *and* 1.

Proof. We look for the solution where all x_i's are the same and drop subscripts of the cost functions. From (40) we know that $\partial \Pi_i/\partial \alpha_i \geq 0$ if and only if

$$g(\alpha_i) \equiv (f' + (n-1)x_i f'')\alpha_i^2 + ((n-2)f' - C'')\alpha_i + C'' \geq 0 . \tag{47}$$

Since $g(0)>0$, $g(1)<0$ and $g(\alpha_i)$ is continuous, there exists $0<\alpha_i^*<1$ such that $g(\alpha_i^*)=0$ with g decreasing there (the typical case is $g'(\alpha^*)<0$). Thus Π_i is locally maximized at α_i^*. By construction this α_i^* defines the symmetric solution of the extended game. The existence of consistent equilibrium in this case was established in Perry (1982).

Example. Consider a duopoly with a linear demand function and quadratic cost functions:

$$p = -ax + b$$

and

$$C_i(x_i) = c_i x_i^2 + d_i x_i + e_i \qquad (i=1, 2) .$$

In this case, (45) may be written as

$$\alpha_i = \frac{a\alpha_j + 2c_j}{(1+\alpha_j)a + 2c_j} \qquad (i=1, 2, i \neq j) . \tag{45$'$}$$

Hence for each α_j there exists a unique α_i $(i \neq j)$ satisfying (45); and so the equilibrium of the extended game is given by

$$\alpha_i^* = [-c_i + \sqrt{(a+c_i)[(ac_j/(a+c_j) + c_i)}]/a \qquad (i=1, 2, i \neq j)$$

together with the corresponding outputs determined by (5). That this defines a consistent equilibrium was established in Bresnahan (1981).

7. CONCLUSION

We have examined the nature of oligopolistic equilibrium in the traditional partial-equilibrium framework. In Section 3 we supposed that each firm chooses its optimal output level under a fixed (combined) conjectural variation parameter α_i. To show the existence of equilibrium in this set-up requires assumptions similar to those which guarantee the existence of Cournot-Nash equilibrium in the standard framework as in Okuguchi (1976), Friedman (1977) and Yakowitz and Szidarovsky (1977). We required somewhat stronger assumptions because in our framework some firms may behave as price takers while others may be somewhere between the Cournot-oligopolist and the price taker and because we needed to have the existence of equilibrium with non-negative profits and positive outputs for all firms. The result on the differentiability of equilibrium allocations with respect to α_i is also used in the later part of the paper.

Interesting elements of the existence problem which are related to the present paper but not discussed explicitly here (such as conjectured demand functions, entry of firms and dynamic plays) are found in the work of Negishi (1960–1961), Friedman (1977), Seade (1980), Novshek (1980) and Hart (1985). Other important work which dealt with the existence problem includes McManus (1964) and Roberts and Sonnenschein (1977).

In Sections 4 and 5 we have studied the effects of a change in α_i on such economic variables as output levels, profits, price-cost margins and economic welfare. Some of the results (e.g. Theorems 2, 3 and 4) may be of independent interest, although partial results (e.g. comparisons between the values of economic variables in the competitive equilibrium and those in the Cournot equilibrium) have been obtained under somewhat restrictive assumptions.

In Section 6 we have defined the concepts of consistent equilibrium and reasonable conjectural equilibrium which restrict the admissible values of conjectural variations. We have established that the consistent equilibrium exists under fairly normal conditions. Since consistent conjectural variations are characterized by a set of n equations, the consistency condition is usually enough to determine equations, the consistency condition is usually enough to determine the equilibrium values of conjectural variations. We have shown that a reasonable conjectural equilibrium must be consistent. Suggestive work which discussed the strategic behavior of oligopolists in somewhat different situations includes Frisch (1933) and Ono (1978).

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