Title	A GENERAL MODEL OF TRADE AND OPTIMAL ACCUMULATION IN A SMALL OPEN ECONOMY: EXISTENCE AND TURNPIKE RESULTS
Sub Title	
Author	DATTA, Manjira
Publisher	Keio Economic Society, Keio University
Publication year	1992
Jtitle	Keio economic studies Vol.29, No.2 (1992.) ,p.37- 55
JaLC DOI	
Abstract	This paper analyzes optimal accumulation pattern of a small dynamic economy engaged in consumption, production and trade in a many commodity framework. This is modelled in the tradition of neo-classical growth theory (without endogenous growth). Since international prices are given, the many-sector model can be transformed into a one-sector model, where the single good is income or the market value of the stock of commodities owned. It is shown that the path of optimal income monotonically converges to a stationary level in the long-run (not entirely independent of the initial income) while the time-path of optimally chosen commodity bundles need not.
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19920002-0037

A GENERAL MODEL OF TRADE AND OPTIMAL ACCUMULATION IN A SMALL OPEN ECONOMY: EXISTENCE AND TURNPIKE RESULTS

Manjira DATTA*

Abstract: This paper analyzes optimal accumulation pattern of a small dynamic economy engaged in consumption, production and trade in a many commodity framework. This is modelled in the tradition of neo-classical growth theory (without endogenous growth). Since international prices are given, the many-sector model can be transformed into a one-sector model, where the single good is income or the market value of the stock of commodities owned. It is shown that the path of optimal income monotonically converges to a stationary level in the long-run (not entirely independent of the initial income) while the time-path of optimally chosen commodity bundles need not.

1. INTRODUCTION

The aim of the paper is to synthesize and extend the literature on optimal accumulation in an open economy. We focus on the nature of the sequence of nominal wealth resulting from the optimal behavior over time. Most of the earlier attempts to analyze this phenomenon have been in the framework of two-sector models, where the country has one consumption good and one capital good sector. The former enters the felicity function, while the latter determines output through a constant returns to scale technology [see, among others, Bardhan (1965), Findlay (1980), Srinivasan (1964), Takayama (1964), Uzawa (1961, 1963, 1964)]. The dynamic steady states of the relevant variables are then examined [Srinivasan and Bhagwati (1980)]. It has long been felt that the appropriate framework for discussing the properties of a path in which a country engaged in trade grows ought to be one that is general enough to allow for multiplicity of traded and non-traded commodities and joint production. The model should allow for existence of "pure" capital goods, "pure" consumption goods, non-traded primary factors of pro-

* I am indebted to Professors Mukul Majumdar and Tapan Mitra for their advice, encouragement, analytical guidance and help. I am also grateful for financial support which came partly from a National Science Foundation grant awarded to them and partly from the Warshow Endowment of Cornell University. This paper relies heavily on chapter 1 of my Ph. D. dissertation entitled "Essays on Intertemporal Allocations in Open Economies" submitted to Cornell University. I would like to thank Professors Leonard J. Mirman, Kazuo Nishimura, T. N. Srinivasan, Henry Y. Wan Jr. and Makoto Yano for their comments on an earlier draft. The shortcomings that remain are entirely mine.

duction and goods that enter into both production and consumption. This would make the theory closer to the literature on intertemporal general equilibrium. We allow for "pure" capital goods, "pure" consumption goods and goods that enter into both production and consumption. Non-traded goods and primary factors of production are not explicitly modelled. Yano (1990) considers a model having produced goods which can be either used as input or consumed, produced goods which are "pure" consumption goods, some produced goods are traded some are not and having primary goods which are not traded. He demonstrates the stability of a perfect foresight equilibrium path in dynamic trade models.

Jones (1961) considered a multi-country multi-commodity linear economic model. Jones and Scheinkman (1977) also analyzed a similar multi-sector model but the consumption basket chosen in these papers did not emerge from an optimization exercise. Wan and Majumdar (1980) developed a model of a competitive world in which several countries were engaged in production, consumption and exchange of many commodities sequentially over time. Each country produces a single good, and trades it for the required inputs and consumption goods chosen according to an intertemporal utility maximization. Here all the relevant functional relationships are assumed to be log-linear.

This paper considers the dynamic optimization problem of a "small" country which takes part in consumption, production and trade in a multi-commodity setting. By "small" it is meant that the country acts as a price-taker in the world market. Additionally, it also assumes the current set of prices to be an exact forecast of the future prices. All the commodities can, in principle, enter into the country's utility function and technology. In the initial period it is endowed with a vector of commodities. In each time period, the current stock of commodities (its endowment in the initial period and output thereater) evaluated at given international prices, determines its income. This income is used to buy a consumption bundle and a vector of inputs for use in production from the world market. The former gives it an immediate return in the form of utility and the latter results in a basket of output in the next period through its technology. The objective is to maximize the discounted sum of one-period utilities over an infinite horizon. The main focus of this paper is in proving the existence and the qualitative properties of the income path resulting from the optimal accumulation, consumption and trading decisions. Under certain regularity conditions on the utility function and technology, the existence of a set of stationary optimal income is established (similar to the "modified golden rule" stock in the one sector model). Further, the optimal income path monotonically converges to a stationary income.

Nishimura and Yano (1993) consider a two-good model of "large" country and show that the accumulation pattern of free trade equilibrium paths need not be monotone. Note that the models of multisector growth in a closed economy have pointed out the possibility of periodic and sometimes chaotic behavior of the optimal path [see Boldrin and Montrucchio (1984, 1985)]. In particular, as the discount factor moves from one to zero, the dynamical systems tend to lose their stability, i.e., the turnpike property [see Sutherland (1970), Samuelson (1973), Benhabib and Nishimura (1985)]. In the present analysis, we assume trade is balanced every period. The crucial difference between the multisector model of an open economy and a closed one is in the feasibility conditions. In the closed model feasibility requires that the consumption and input demand for a good cannot exceed its production. Whereas in the model of an open economy it requires the value of aggregate consumption and input demand not exceeding the value of its output. Since prices are assumed to remain unchanged this constraint also helps us to reduce the multisector model to a one-sector model, the single good being income. For this model, as was pointed out by Srinivasan and Bhagwati (1980), the turnpike property is to be "expected".

The paper is arranged in the following way. Section 2 outlines the model formally. In section 3 the original problem is transformed to a reduced form and this method enables us to draw upon some well-known mathematical arguments of intertemporal allocation theory. Section 4 establishes the existence of a stationary optimal income and indicates the shape of the optimal policy function. The appendix contains the proofs of the propositions of Section 4. Other straightforward proofs are omitted and can be found in Datta (1992).

Notation

Let $\mathscr{R}^n$ be the n-dimensional Euclidean space. For any two vectors $x=(x_i)$ and $y=(y_i)$ in $\mathscr{R}^n$ we write $x \gg y$ if $x_i > y_i$ for all $i=1, 2, \cdots, n$; $x \geq y$ if $x_i \geq y_i$ for all $i=1, 2, \cdots n$; $x > y$ if $x \geq y$ and $x_i > y_i$ for some i. The set $\{x \in \mathscr{R}^n : x \geq 0\}$ is denoted by $\mathscr{R}^n_+$ and the set $\{x \in \mathscr{R}^n : x \gg 0\}$ by $\mathscr{R}^n_{++}$.

A sequence a_t, for $t=0, 1, 2 \cdots$, is denoted by $\boldsymbol{a}=\langle a_t \rangle$. The consumption, input and output vectors are written as $c_t=(c_t^j)$, $x_t=(x_t^j)$ and $z_t=(z_t^j)$, respectively, where superscript j refers to the good and the subscript t refers to the time-period.

Let g be any real valued function defined on $\mathscr{R}^n$. g is said to be *weakly* (*strictly*) *increasing* if, for x, y in $\mathscr{R}^n$, $x \gg y$ ($x > y$) *implies* $g(x) > g(y)$. g is said to be *non-decreasing* if for x, y in $\mathscr{R}^n$, $x > y$ *implies* $g(x) \geq g(y)$. The norm on $\mathscr{R}^n$, denoted by $\|\cdot\|$, is the sum-norm. That is, for x in $\mathscr{R}^n$, $\|x\| = \sum_{i=1}^n |x_i|$.

2. THE MODEL

Suppose there are N producible and internationally traded commodities. The country is characterized by its vector of initial endowment $z_0 > 0 \in \mathscr{R}^N_+$, the available technology, Ω, its utility function u and the discount factor δ. The country aims to maximize the discounted sum of one-period utilities from consumption; and chooses the input, output or consumption basket accordingly. It is assumed that when the country formulates its plans in period zero, it treats the current prices (i.e., the prices quoted in period zero), $p \in \mathscr{R}^N_{++}$, as an exact forecast of all the future prices. The stock of commodities every period evaluated at these prices will be called its income at that period.

The technology of the country is described by a *production-possibility set* $\Omega \subset \mathscr{R}^N_+ \times \mathscr{R}^N_+$. A pair (x, z) belongs to Ω, if it is technically feasible to produce the output-vector z in the next period, by using the input-vector x in this period, *given the supply of primary resources*. Ω is assumed to satisfy:

(Ω1) (0, 0) belongs to Ω; and $(0, z)$ in Ω implies $z=0$.
(Ω2) Ω is closed and convex.
(Ω3) Given any (x, z) in Ω, there is (x', z') in Ω with $z' \gg z$.
(Ω4) For any (x, z) in Ω, $x' \geq x$ and $z \geq z' \geq 0$ implies (x', z') is in Ω.
(Ω5) If $(x^n, z^n) \in \Omega$ for $n=1, 2, \cdots$, and $\|x^n\| \to \infty$ as $n \to \infty$ then $\|z^n\|/\|x^n\| \to 0$ as $n \to \infty$.

Assumption (Ω1) states the possibility of inaction and impossibility of free production; (Ω2) ensures continuity in production and rules out increasing returns; (Ω3) is the condition of non-tightness (due to Malinvaud); (Ω4) allows free disposal of inputs and outputs; and (Ω5) is a kind of Inada condition which states that as inputs become very large, "average products" become very small. Note that (Ω5) implies that there is a number $\alpha > 0$, such that if $\|x\| > \alpha$ and (x, z) in Ω then $\|z\| \leq \|x\|$. That is, too high levels of inputs cannot be sustained.

The intertemporal preferences are summarized in terms of a one-period utility function and a discount factor $0 < \delta < 1$. The following assumptions are imposed on the function $u(\cdot)$:

(u1) $u: \mathscr{R}^N_+ \to \mathscr{R}$ and u is continuous on $\mathscr{R}^N_+$.
(u2) u is weakly increasing on $\mathscr{R}^N_+$.
(u3) If c and c' in $\mathscr{R}^N_+$ is such that $u(c) \neq u(c')$ then $u(\lambda c + (1-\lambda)c') > \lambda u(c) + (1-\lambda)u(c')$ for all $0 < \lambda < 1$.

Assumptions (u1) and (u2) are standard. (u3) is weaker than assuming strict concavity of u everywhere. Note, some goods could be "pure" capital good in our model, which do not affect utility if consumed.

We introduce one more assumption (Ω6) as a joint condition on technology and the discount factor, which can be interpreted as an index of productivity compared to the time-preference:

(Ω6) Ω is δ-productive, or, there exists some (x, z) in Ω such that $\delta z \gg x$.

A feasible program is a complete specification of the decisions on consumption, inputs, outputs etc., period after period, that satisfies the technological and balance of payments constraints. Formally, a *feasible program*, from the initial endowment z_0, can be specified by a triplet of non-negative sequences $\langle \boldsymbol{c}, \boldsymbol{x}, \boldsymbol{z} \rangle$ satisfying the following conditions, for all $t \geq 0$:

(1) $p(x_t + c_t) = pz_t$
(2) $(x_t, z_{t+1}) \in \Omega$

The objective of the country is to

$$\text{maximize} \quad \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{2.1}$$

over the set of all feasible programs $\langle \boldsymbol{c}, \boldsymbol{x}, \boldsymbol{z} \rangle$ from z_0.

The allocation decisions of the country can be described somewhat informally as follows. Given the initial price p and the stock of goods $z_t > 0$, the *income* of the country y_t is determined as:

$$y_t = pz_t \tag{2.2}$$

Its choice of an *expenditure allocation* $(y_t^I, y_t^C) \in \mathscr{R}_+^2$ satisfying

$$y_t^I + y_t^C = y_t \tag{2.3}$$

determines income in the next period according to the following exercise:

$$\begin{array}{ll} \text{maximize} & pz \\ \text{subject to} & px \leq y_t^I \\ \text{and} & (x, z) \in \Omega \end{array} \tag{2.4}$$

Next, we note that, the choice of expenditure allocation, y_t^C, also determines the immediate return from consumption in the following way:

$$\begin{array}{ll} \text{maximize} & u(c) \\ \text{subject to} & pc \leq y_t^C \text{ and } c \in \mathscr{R}_+^N \end{array} \tag{2.5}$$

A feasible program that solves the problem outlined in (2.1) must solve two static optimization problems (2.4) and (2.5). Thus, the expenditure allocation in one period is equivalent to choosing the income transition from one period to the next. Or, every sequence of income $\langle y_t \rangle$ has an equivalent sequence of expenditure allocations $\langle y_t^I, y_t^C \rangle$ and vice versa. This aspect will be utilized in the next section when the problem is set up in terms of income in period t and $(t+1)$.

Given a sequence of income, $\boldsymbol{y} = \langle y_t \rangle$, or, equivalently, a sequence of expenditure allocations, $(\boldsymbol{y}^I, \boldsymbol{y}^C) = \langle y_t^I, y_t^C \rangle$, the welfare attained by the country is

$$w(z_0, p) = \sum_{t=0}^{\infty} \delta^t u(\bar{c}_t) \tag{2.6}$$

where $\langle \bar{c}, \bar{x}, \bar{z} \rangle$ is a feasible program, $\bar{c}_t$, is a solution of (2.5) and $(\bar{x}_t, \bar{z}_{t+1})$ that of (2.4). Thus, the objective can be restated in terms of maximizing (2.6) over the set of feasible sequences of income. This is formally done in section 3.

Observe that $\bar{c}_t$, $\bar{x}_t$ determines the exports and imports of the country in period t, if for some commodity i,

$$\bar{c}_t^i + x_t^i > \bar{z}_t^i$$

then the commodity i is *imported* in period t. If for some commodity j,

$$\bar{c}_0^j + \bar{x}_t^j < \bar{z}_t^j$$

then the commodity j is *exported* in period t.

We present two examples to illustrate that some of the widely used models in the literature can be treated as special cases of the model developed here.

Example (1). The Srinivasan and Bhagwati (1980) economy can be specified in terms of the immediate return function U, the discount factor $0<\delta<1$, and the production functions F_1 and F_2 for the consumption and investment goods respectively. For simplicity, consider no population growth. $U: \mathscr{R}_+ \to \mathscr{R}_+$ is assumed to be continuous, increasing and strictly concave. $F_i: \mathscr{R}_+^2 \to \mathscr{R}_+$, for $i=1$ and 2, is twice differentiable, concave and satisfies the Inada conditions. We show that this economy can be represented as a special case ($N=2$) of the model presented above. Let us denote the first commodity as the consumption good and the second as investment good. Say, without loss of generality, one unit of labor is available. Define λ as the fraction of labor going into the production of consumption good. According to the terminology of our model, the technology of the Srinivasan and Bhagwati economy can be described by the production-possibility set, Ω, as follows:

$$\Omega = \{(x, z) \in \mathscr{R}_+^2 \times \mathscr{R}_+^2 : z^1 \leq F_1(x^{21}, \lambda), \quad z^2 \leq F_2(x^{22}, 1-\lambda), \quad x^{21} + x^{22} \leq x^2 \quad \text{and} \quad 0 \leq \lambda \leq 1\}$$

where x^{2i} is the amount of investment good devoted to the production of the ith good for $i=1$ and 2. The utility function relevant for the economy can be defined as $u(c^1, c^2) = U(c^1)$. It is easy to check that the assumptions made on Ω and u, in the model, are satisfied.

Example (2). A slightly modified version of the Wan and Majumdar (1980) model can also be interpreted as a special case of the model outlined above. Here we use the Cobb-Douglas type structure on both preference and technology. The utility function can be defined as $u(c) = (c^1)^{a_1} \cdot (c^2)^{a_2} \cdots (c^N)^{a_N}$, where $\sum_{j=1}^N a_j < 1$. And the technology can be summarized in terms of the production-possibility set, as follows:

$$\Omega = \left\{(x, z) \in \mathscr{R}_+^N \times \mathscr{R}_+^N : z \leq (x^{1i})^{b_{1i}} (x^{2i})^{b_{2i}} \cdots (x^{Ni})^{b_{Ni}}, \quad \sum_{j=1}^N b_{ji} < 1 \quad \text{and} \quad \sum_{j=1}^N x^{ij} \leq x^i \quad \text{for all } i = 1, 2, \cdots, N\right\}$$

It can be easily verified that the assumptions made on u and Ω, are satisfied by this particular specification of preference and technology. ■

3. TRANSFORMATION OF THE ORIGINAL PROBLEM TO REDUCED FORM

In this section, we reduce the multi-sectoral allocation problem to an equivalent single sector in which the country "produces" income, "consumes" a part of it, and "saves" the rest to accumulate capital [Bhagwati and Srinivasan (1983)]. In other words, we reduce the country's multi-sectoral problem into a one-dimensional problem, which is defined in terms of its income path. This transformation enables us to use the framework of one-good model of optimal growth in the tradition of Koopmans (1963) and Cass (1965). Nishimura and Yano (1993) also reduce their two-sector model in a similar fashion. A nice summary of the literature on one-sector economic (convex) growth can be found in Mirman (1980).

3.1. Technology and the Definition of Transition Possibility Set

Define an *income generating function* $h\colon \mathscr{R}_+ \to \mathscr{R}_+$ as follows,

$$h(y) = \max\{pz : (x, z) \in \Omega \text{ and } px \leq y\} \tag{3.1}$$

Thus, $h(y)$ is the level of maximum income the country can attain in the next period, if its current income is y.

In view of (Ω5) and $p \gg 0$, $\| z \|$ is bounded and $h(y)$ is well-defined by Weierstrass's Theorem. The function h has the following properties:

(h1) $h(0) = 0$
(h2) h is increasing, concave and continuous on $\mathscr{R}_+$.
(h3) There exists a y^0 such that $h(y^0) = y^0$ and $h(y) < y$ for all $y > y^0$.

That is, y^0 is the "highest level of sustainable income" and eventually, h lies below the 45° line.

Next we define the *transition-possibility set* γ,

$$\gamma = \{(y, y') \in \mathscr{R}_+^2 : y' \leq h(y)\} \tag{3.2}$$

$(y, y') \in \gamma$ means that attaining an income level y' in the next period is technically feasible if the current period's income is y. We list some of the properties of γ, which follow from those of h:

(γ1) $(0, 0) \in \gamma$; and $(0, y') \in \gamma$ implies $y' = 0$.
(γ2) γ is closed and convex.
(γ3) There is a y^0 such that $y > y^0$ and $(y, y') \in \gamma$ imply that $y' < y$.

Figure 1 captures the properties of the technological aspects of the transformed model.

3.2. Preference Structure and Definition of a "Reduced" Utility Function

In this sub-section we consider the aspects of consumption side by defining a "reduced" utility function v on the transition-possibility set γ. An indirect utility function, $g(\cdot)$,

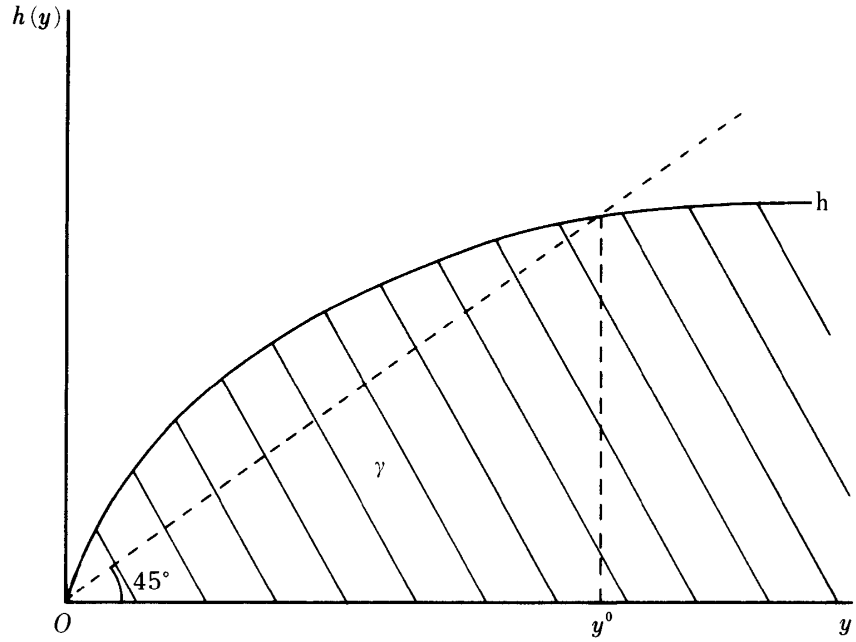

Fig. 1. Income Generating Function h and Transition Possibility Set γ.

$$g(y)=\max\{u(c)\colon c\in\mathscr{R}_{+}^{N} \text{ and } pc\leq y\} \tag{3.3}$$

is well-defined by Weierstrass's Theorem and it is continuous, increasing on $\mathscr{R}_{+}$ and strictly concave on $\mathscr{R}_{++}$.

Now, using g, we can formally define the "reduced" utility function v, on the transition-possibility set, as

$$v(y, y')=g[y-H(y')] \qquad \text{where} \quad H=h^{-1} \tag{3.4}$$

H is well-defined since h is increasing and $H(y')$ is the amount of income to be allocated to production today in order to generate y' tomorrow. Thus $v(y, y')$ is the indirect utility or the immediate return to the country, if it generates an income y' in the next period, from an income level y in the current period.

In terms of the notation introduced in this section the optimization problem can be written as:

$$\begin{aligned} &\text{maximize} \quad \sum_{t=0}^{\infty} \delta^{t} v(y_{t}, y_{t+1}) \\ &\text{subject to} \quad (y_{t}, y_{t+1})\in\gamma \quad \text{for all } t\geq 0 \text{ and given } y_{0}>0 \end{aligned} \tag{3.5}$$

We note some of the properties of v:

($v1$) If $(y, y')\in\gamma$, $\xi\geq y$ and $0\leq\xi'\leq y'$ then $(\xi, \xi')\in\gamma$ and $v(\xi, \xi')\geq v(y, y')$.
($v2$) There is β in $\mathscr{R}$ such that (y, y') in γ implies $v(y, y')\geq\beta$.
($v3$) v is continuous and strictly concave on γ.
($v4$) v is supermodular, that is, for (y_i, y_i') in γ for $i=a$ and b, $y_a>y_b$ and $y_a'>y_b'$

imply $v(y_a, y'_a)+v(y_b, y'_b)>v(y_a, y'_b)+v(y_b, y'_a)$.

These are used in the next section to derive the main propositions of the paper.

4. EXISTENCE OF A NON-TRIVIAL STATIONARY OPTIMAL INCOME

We begin this section with a list of definitions:

An *income program* from $\bar{y}>0$, is a sequence $\langle y_t \rangle$ such that $y_0=\bar{y}$ and $(y_t, y_{t+1})\in\gamma$ for all $t\geq 0$.

An income program $\langle y_t^* \rangle$ from $\bar{y}$, is an *optimal income program*, if for every program $\langle y_t \rangle$, from $\bar{y}$, we have, $\sum_{t=0}^{\infty}\delta^t v(y_t, y_{t+1})\leq\sum_{t=0}^{\infty}\delta^t v(y_t^*, y_{t+1}^*)$. Also, associated with an optimal income program there is an equivalent sequence of expenditure allocations which is called an *optimal expenditure allocation program*.

An income program $\langle y_t \rangle$ from $\bar{y}$, is *stationary*, if $y_t=\bar{y}$ for all $t\geq 0$.

An income program $\langle y_t \rangle$ from $\bar{y}$, is a *stationary optimal income program*, if it is both stationary and optimal. If a stationary optimal income program exists from y, then y is called a *stationary optimal income*.

The *optimal policy function*, $\phi: \mathscr{R}_+\to\mathscr{R}_+$, is defined as follows:

$$\phi(y_0)=\{y_1 : \langle y_t \rangle \text{ is an optimal program from } y_0>0\}$$

We first establish the existence of a unique optimal income program, it is then shown to be monotonic in t. The monotonicity and boundedness of the sequence $\langle y_t^* \rangle$ is used to assert certain turnpike properties. We prove the existence of a (non-trivial) stationary optimal income program. Though we cannot claim uniqueness of stationary optimal incomes, we can show that the set of stationary optimal incomes will at most be a closed interval $[y^*, y^{**}]$ of the strictly positive real line. In fact, every point in that interval is a stationary optimal income in the sense that if the initial income y_0 lies in $[y^*, y^{**}]$ then the optimal income program from y_0 is the stationary program $\langle y_t \rangle$ where $y_t=y_0$ for all $t\geq 0$. We indicate the shape of the optimal policy function and prove that the optimal income program eventually converges to a stationary optimal income. The results on the optimally chosen sequence of consumption, input and output vectors are, however, less definitive.

The propositions are stated below. Some of the proofs are contained in the appendix.

PROPOSITION 4.1. *There exists a unique optimal income program from any $y>0$.*

For establishing the monotonicity of the optimal program $\langle y_t^* \rangle$, the following lemma is useful.

LEMMA 4.1. *Let $\langle y_t \rangle$ and $\langle y'_t \rangle$ be optimal income programs starting from y_0 and y'_0 respectively. If $y_0>y'_0$, then $y_1\geq y'_1$.*

Using Lemma 4.1, we can now state and prove our result in monotonicity of the optimal program. This is similar to the properties of optimal stock derived in Dechert and Nishimura (1983) or Majumdar and Nermuth (1982); both these papers allow for non-convexity in the production structure.

PROPOSITION 4.2. *Let* $\langle y_t \rangle$ *be the optimal income program starting from* y_0. *Then either,* $y_t \leq y_{t+1}$ *for all* $t \geq 0$, *or* $y_t \geq y_{t+1}$ *for all* $t \geq 0$.

Proof. First, let us suppose $y_0 < y_1$. By the principle of optimality, $\langle y_{1+t} \rangle$ is the optimal program from y_1. From Lemma 4.1, we know that $y_1 \leq y_2$; and by an induction on t, we have $y_t \leq y_{t+1}$, for all $t \geq 0$. For $y_0 > y_1$, a similar argument shows that $y_t \geq y_{t+1}$, for all $t \geq 0$. If $y_0 = y_1$, then the principle of optimality and the uniqueness of optimal income program implies that $y_t = y_{t+1}$, for all $t \geq 0$. ■

Now, we extend the idea of δ-productivity of the production-possibility set to the transition-possibility set. The proof of existence of a stationary optimal income uses this in a crucial way.

LEMMA 4.2. γ *is* δ*-productive i.e., there is some* $\bar{y} > 0$ *such that* $\delta \bar{y} > H(\bar{y})$.

To see this, note from ($\Omega 6$) there is some $(\bar{x}, \bar{z})$ in Ω such that $\delta \bar{z} \gg \bar{x}$. Choose $\bar{y} = p\bar{z}$ since $p \gg 0$. Then $H(\bar{y}) < p\bar{x} < p\delta\bar{z} = \delta\bar{y}$.

THEOREM 4.1. *A stationary optimal income exists. And if it is not unique, then the set of stationary optimal incomes is a closed interval on* $\mathscr{R}_{++}$.

In the next theorem, we summarize the characteristics of the optimal policy function, which enable us to derive its shape, see Figure 2.

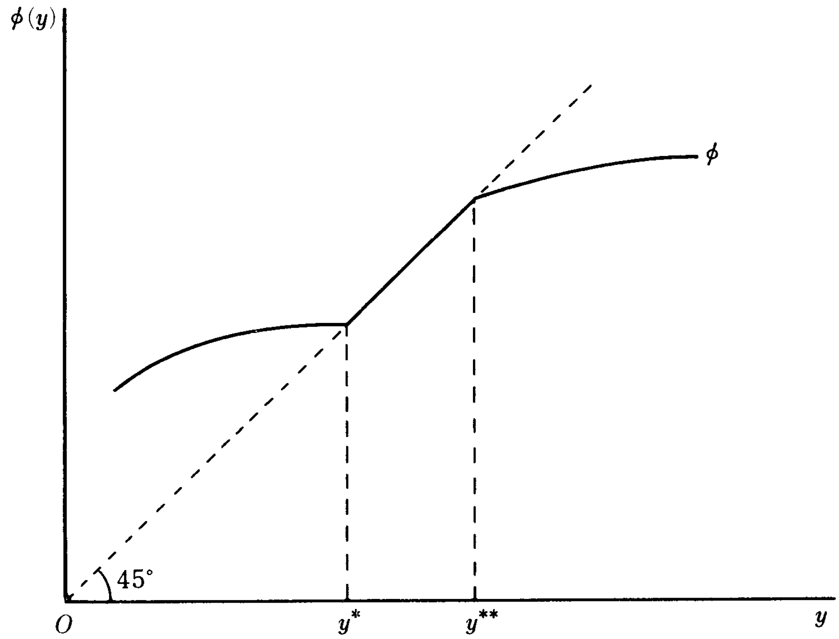

Fig. 2. Optimal Policy Function ϕ.

THEOREM 4.2. *The optimal policy function, ϕ, has the following properties:*
(1) *ϕ is continuous and increasing on $\mathscr{R}_{++}$.*
(2) *Let $[y^*, y^{**}]$ be the set of stationary optimal incomes. Then,*

(a) *$\phi(y) < y$ for all $y > y^{**}$; and*
(b) *$\phi(y) > y$ for all $0 < y < y^*$.*

Proof of (1) follows from Proposition 1 in Dutta and Mitra (1989) and from Proposition 4.2 above. In order to prove (2), the technique used by Mitra and Ray (1984) is adapted. We require two lemmas, which are stated and proved in the appendix. Also, we note that in some cases the interval $[y^*, y^{**}]$ may be degenerate, for example, when h is strictly concave.

Regardless of the initial income, the optimal income program converges to a stationary optimal income. If the initial income is less than y^*, the optimal income path monotonically increases and converges to y^*. And, if the initial income is greater than y^{**}, then the optimal path monotonically decreases and converges to y^{**}. Any other income in between is stationary optimal. So, unlike the traditional models of one-sector economic growth, the long-run income is not entirely independent of the initial level of income. It is interesting to note that the "flat" portion of the optimal policy function is similar to the region of incomplete specialization in the Srinivasan-Bhagwati framework. However, the analysis of complete and incomplete specializations are difficult to handle in the model we present. In a model of two traded goods, without joint production, Manning (1980) answers some of the questions relating to specialization and dynamics. Also, the monotonicity of the optimal program $\langle y_t \rangle$ is translated into the same property for the optimal sequence of the expenditure allocations $\langle y_t^I, y_t^C \rangle$.

COROLLARY 4.1. *Let $\langle y_t^I, y_t^C \rangle$ be the optimal sequence of expenditure allocations. Then, either we have $y_t^I \leq y_{t+1}^I$ and $y_t^C \leq y_{t+1}^C$ for all $t \geq 0$; or, $y_t^I \geq y_{t+1}^I$ and $y_t^C \geq y_{t+1}^C$ for all $t \geq 0$.*

We note that monotonicity of y_t^I over t can be derived even if we allow for non-convex technology but the same is not necessarily true for y_t^C. Next, Corollary 4.2 is stated which has some interesting implications on the sequence of optimally chosen consumption, input and output vectors.

COROLLARY 4.2. *Let the optimal income program $\langle y_t \rangle$, from y_0, converge to $y > 0$. Then the optimal sequence of expenditure allocations $\langle y_t^I, y_t^C \rangle$ converges to $\langle y^I, y^C \rangle$, where $y^I = H(y)$ and $y^C = y - H(y)$.*

Theorem 4.2 asserts that the optimal income program $\langle y_t \rangle$ is a convergent sequence. Now, $y_t^I = H(y_{t+1})$, $y_t^C = y_t - H(y_{t+1})$ and the continuity of H prove Corollary 4.2.

Define,

$$\Gamma(y_t^C) = \text{Argmax}\{u(c) : pc \leq y_t^C \text{ and } c \in \mathscr{R}_+^N\}$$

$$\rho(y_t^I) = \text{Argmax}\{pz : px \leq y_t^I \text{ and } (x, z) \in \Omega\}$$

Thus, every element in set $\Gamma(y_t^C)$ is a solution to the one-period utility maximization problem (2.5) and, similarly, every element of $\rho(y_t^I)$ is a solution to the profit maximization problem (2.4). Following Berge's maximum theorem [see, for example, Border (1985)], Γ and ρ are upper-semicontinuous as correspondences. If the solutions to (2.4) and (2.5) are unique then Γ and ρ are, in fact, continuous functions. Thus, Corollary 4.2 will then imply convergence of $\Gamma(y_t^C)$ and $\rho(y_t^I)$ in the long-run. In other words, the sequence of optimally chosen consumption, input and output vectors converge to a steady-state over time. In the case of Wan and Majumdar (1980) log-linear economy this is true. This turnpike result is, in spirit, similar to McKenzie (1986) and contrasts with those of Boldrin and Montrucchio (1984, 1985). In general, however, the model does not give a definitive answer regarding the long-run behavior of the optimally chosen commodity bundles. Consider an optimal income program $\langle y_t \rangle$, where $y_t = y^*$ for all t, and equivalently, the optimal expenditure allocations program $\langle y_t^I, y_t^C \rangle$ with $(y_t^I, y_t^C) = (y^{I*}, y^{C*})$. Now, if $\Gamma(y^{C*})$ or $\rho(y^{I*})$ has multiple solutions then it is possible that, even in the long-run, the optimally chosen consumption or the input-output vector moves in periodic, cyclical or chaotic fashion, although the income and the expenditure allocations have reached a steady-state. Thus, we note that the results derived here inherit some properties of both the one-sector and multi-sector closed economy models of optimal growth, while exactly mimicking none.

The main shortcoming of the present paper is, of course, the assumption of static price expectations. This has been done in the interest of simplicity and clarity. How the wealth dynamics outlined here would change with introduction of more interesting forms of expectation formation (such as adaptive expectation or rational expectation) is an open question. Another limitation of the analysis in this paper is the assumption of non-tightness on the technology which rules out the case of linear technology with primary factors of production.

APPENDIX

PROPOSITION 4.1. *There exists a unique optimal income program from any $y > 0$.*

Proof. Define for each $y > 0$, $B(y) = \max(y, y^0)$. Then, if $\langle y_t \rangle$ is a program from y then $y_t \leq B(y)$ for all $t \geq 0$. Thus by (v2), (v3) and (v4), $\sum_{t=0}^{\infty} \delta^t v(y_t, y_{t+1})$ is absolutely convergent.

Let $\sigma = \sup\{\sum_{t=0}^{\infty} \delta^t v(y_t, y_{t+1}) : \langle y_t \rangle$ is a program from $y\}$. By definition of σ, there is a sequence of program $\mathbf{y}^n = \langle y_t^n \rangle$ from y such that $\sum_{t=0}^{\infty} \delta^t v(y_t^n, y_{t+1}^n) > [\sigma - (1/n)]$ for all $n \geq 1$ and $y_t^n \leq B(y)$ for all $t \geq 0$ and $n \geq 1$. $\mathbf{y}^n$ is bounded for each n, therefore, by a standard "diagonalization" argument [see Rudin (1976), pp. 156–157], there exists a convergent subsequence $\mathbf{y}^{n'} = \langle y_t^{n'} \rangle$ such that for each $t \geq 0$, $y_t^{n'} \to y_t'$ as $n' \to \infty$. Thus, for all $t \geq 0$, $(y_t^{n'}, y_{t+1}^{n'}) \to (y_t', y_{t+1}')$ as $n' \to \infty$. γ is closed, therefore, $(y_t', y_{t+1}') \in \gamma$ for all $t \geq 0$. And $y_0^{n'} = y$ for all n' implies $y_0' = y$. Hence, $\langle y_t' \rangle$ is a program from y and it is routine to show that $\langle y_t' \rangle$ is optimal from y.

Next, to show that an optimal program from $y>0$ is unique, use convexity of γ and strict concavity of v. ■

LEMMA 4.1. *Let $\langle y_t \rangle$ and $\langle y'_t \rangle$ be optimal income programs starting from y_0 and y'_0 respectively. If $y_0 > y'_0$, then $y_1 \geq y'_1$.*

Proof. Let us suppose $y_1 < y'_0$; then supermodularity of v implies that

$$v(y_0, y_1) + v(y'_0, y'_1) < v(y_0, y'_1) + v(y'_0, y_1)$$

and this would contradict the optimality criterion. We define the *value function* $V: \mathscr{R}_+ \to \mathscr{R}_+$, as follows

$$V(y) = \max\left\{ \sum_{t=0}^{\infty} \delta^t v(y_t, y_{t+1}) : \langle y_t \rangle \text{ is a program from } y \right\}$$

So, by the optimality criterion, we have $V(y_0) = v(y_0, y_1) + \delta V(y_1) \geq v(y_0, y'_1) + \delta V(y'_1)$. And, similarly, $V(y'_0) = v(y'_0, y'_1) + \delta V(y'_1) \geq v(y'_0, y_1) + \delta V(y_1)$. Hence, by combining these two inequalities, we get,

$$\begin{aligned} v(y_0, y_1) + v(y'_0, y'_1) &= [V(y_0) - \delta V(y_1)] + [V(y'_0) - \delta V(y'_1)] \\ &= [V(y_0) - \delta V(y'_1)] + [V(y'_0) - \delta V(y_1)] \end{aligned}$$

Therefore, $v(y_0, y_1) + v(y'_0, y'_1) \geq v(y_0, y'_1) + v(y'_0, y_1)$, which is a contradiction. ■

THEOREM 4.1. *A stationary optimal income exists. And if it it not unique, then the set of stationary optimal incomes is a closed interval on $\mathscr{R}_{++}$.*

Proof. First, we show that a modified golden rule exists. Then, by a well-known result in the literature, we claim the existence of a stationary optimal income. We define a *modified golden rule* to be a pair (y^*, π^*) such that $y^* \geq 0$, $\pi^* \geq 0$ and

(1) $y^* - H(y^*) > 0$
(2) $\pi^*[\delta y^* - H(y^*)] \geq \pi^*[\delta y - H(y)]$ for all $y \geq 0$.
(3) $g[y^* - H(y^*)] - \pi^*[y^* - H(y^*)] > g(\xi) - \pi^* \xi$ for all $\xi \geq 0$

Let $\eta = \max(\bar{y}, y^0)$ and y^* maximize $[\delta y - H(y)]$ over $[0, \eta]$, where $\bar{y}$ is such that $\delta \bar{y} > H(\bar{y})$. Therefore, $\delta y^* - H(y^*) \geq \delta \bar{y} - H(\bar{y}) > 0$, which implies $y^* - H(y^*) > 0$. Now, from ($h3$), we have, $\delta y - H(y) < 0$ for all $y > \eta$ and for $0 \leq y \leq \eta$, $\delta y^* - H(y^*) \geq \delta y - H(y)$. We combine these to get, $\delta y^* - H(y^*) \geq \delta y - H(y)$ for all $y \geq 0$.

Now, define $\pi^* = g'_+[y^* - H(y^*)]$, where g'_+ is the right-hand derivative of the function g. For any $(y, y') \in \gamma$, from the concavity of g, we have,

$$\begin{aligned} g[y - H(y')] - g[y^* - H(y^*)] &\leq g'_+[y^* - H(y^*)][\{y - H(y')\} - \{y^* - H(y^*)\}] \\ &= \pi^*(y - y^*) - \pi^*[H(y') - H(y^*)] \\ &= \pi^*(y - y^*) + \pi^*[\delta y - H(y')] - \pi^*[\delta y^* - H(y^*)] + \pi^* \delta y^* - \pi^* \delta y' \end{aligned}$$

$$\leq \pi^* y - \pi^* y^* + \pi^* \delta y^* - \pi^* \delta y'$$

Or,

$$g[y - H(y')] + \pi^* \delta y' - \pi^* y \leq g[y^* - H(y^*)] + \pi^* \delta y^* - \pi^* y^*$$

Following the proof of Proposition 3 in DasGupta and Mitra (1990), (y^*, π^*) is a modified golden rule. Then by Lemma 4.9 of Mitra and Ray (1984), $\langle y_t \rangle$ given by $y_t = y^*$ for all $t \geq 0$, is a non-trivial stationary optimal program and y^* is a stationary optimal income.

Next, we show that if a stationary optimal income is not unique then the set of stationary optimal incomes is a closed interval in $\mathscr{R}_{++}$.

For this we first, prove that a non-trivial stationary optimal income maximizes $[\delta y - H(y)]$ among all $y \geq 0$. Thus, the set of non-trivial stationary optimal income and the set of y that maximizes $[\delta y - H(y)]$ over the positive reals are identical. Then, we show that, if both y^* and y^{**} are non-trivial stationary optimal incomes, then every y in the interval $[y^*, y^{**}]$ is a stationary optimal income. Suppose y^* is a stationary optimal income. Then we must have $y^* \geq H(y^*)$, but equality here would imply expenditure allocated to consumption is zero every period which is not optimal, therefore, we have $y^{C*} = [y^* - H(y^*)] > 0$. Theorem 8.6 in Peleg and Ryder (1972) asserts there is a sequence $\langle \pi_t^* \rangle$, $\pi_t^* > 0$ for all $t \geq 0$, such that

$$\delta^t g[y^* - H(y^*)] - \pi_t^*[y^* - H(y^*)] \geq \delta^t g(\xi) - \pi_t^* \xi \qquad \text{for all} \quad \xi \geq 0 \qquad \text{(A.1)}$$

and

$$\pi_{t+1}^* y^* - \pi_t^* H(y^*) \geq \pi_{t+1}^* y - \pi_t^* H(y) \qquad \text{for all} \quad y \geq 0 \qquad \text{(A.2)}$$

Now, if $[\delta y - H(y)]$ is not maximized at $y = y^*$, then there is a $\xi \geq 0$ such that

$$\delta\xi - H(\xi) > \delta y^* - H(y^*) \qquad \text{or}, \quad \delta(\xi - y^*) > H(\xi) - H(y^*) \qquad \text{(A.3)}$$

Define $q_t^* = (\pi_{t+1}^* / \pi_t^*)$ for all $t \geq 0$. From (A.2), we have

$$q_t^*[\xi - y^*] \leq H(\xi) - H(y^*) \qquad \text{(A.4)}$$

Clearly, either (i) $\xi > y^*$ or, (ii) $\xi < y^*$.

Case (i): $\xi > y^*$

Define $L_1(\xi, y^*) \equiv [H(\xi) - H(y^*)]/(\xi - y^*)$. From (A.3), we know that $\delta > L_1(\xi, y^*)$. Let $\varepsilon \equiv \delta - L_1(\xi, y^*)$. Using (A.4) $q_t^* \leq L_1(\xi, y^*) = \delta - \varepsilon$. Denoting (ε/δ) by θ, we get,

$$(q_t^*/\delta) \leq (1 - \theta) \qquad \text{(A.5)}$$

From (A.1) with $\xi = 2y^{C*}$, and from (A.5) we have,

$$\delta^t[g(2y^{C*}) - g(y^{C*})] \leq \pi_t^* y^{C*} = q_{t-1}^* q_{t-2}^* \cdots q_0^* \pi_0^* y^{C*}$$

Or,

$$[g(2y^{C*})-g(y^{C*})]\leq(1-\theta)^t\pi_0^* y^{C*}$$

The left-hand side is positive, since $\pi_0^*>0$ and $y^{C*}>0$, and independent of t. The right-hand side converges to 0 as $t\to\infty$, which means that the above inequality is contradicted for large t.

Case (ii) $(\xi>y^*)$

Define $L_2(y^*,\xi)\equiv[H(y^*)-H(\xi)]/(y^*-\xi)$. From (A.3), we have that $\delta<L_2(y^*,\xi)$. Let $\varepsilon\equiv L_2(y^*,\xi)-\delta$. Using (A.4), $q_t^*\geq L_2(y^*,\xi)=\delta+\varepsilon$. Denoting (δ/ε) by θ, we get,

$$(q_t^*/\delta)\geq(1+\theta) \tag{A.6}$$

From (A.1) with $\xi=(y^{C*}/2)$, and from (A.6) we have,

$$\delta^t[g(y^{C*})-g(y^{C*}/2)]\geq\left(\frac{1}{2}\right)\pi_t^* y^{C*}=\left(\frac{1}{2}\right)q_{t-1}^* q_{t-2}^*\cdots q_0^*\pi_0^* y^{C*}$$

Or,

$$[g(y^{C*})-g(y^{C*}/2)]\geq\left(\frac{1}{2}\right)(1+\theta)^t\pi_0^* y^{C*}$$

The left-hand side is positive number independent of t, while the right-hand side increases to ∞ as $t\to\infty$, which is a contradiction.

Therefore, $[\delta y-H(y)]$ is maximized at $y=y^*$.

Now, suppose y^* and y^{**} are two distinct non-trivial stationary optimal incomes. Consider $\bar{y}=\lambda y^*+(1-\lambda)y^{**}$ for any $\lambda\in[0,1]$. The above result and convexity of H implies,

$$\delta y^*-H(y^*)\geq\delta\bar{y}-H(\bar{y})\geq\lambda[\delta y^*-H(y^*)]+(1-\lambda)[\delta y^{**}-H(y^{**})$$
$$\geq\delta y^*-H(y^*)$$

Thus, $\bar{y}$ also maximizes $[\delta y-H(y)]$ over the positive reals. Hence it is a stationary optimal income for all $\lambda\in[0,1]$. ■

In order to prove part (2) of Theorem 4.2, we need the following Lemmas, using the techniques of Mitra and Ray (1984).

LEMMA A.1. *There is* $(\bar{y},\bar{\eta})$ *such that for all* $(y,\eta)\leq(\bar{y},\bar{\eta})$ *with* $y\geq0$ *and* $\eta>0$ *we have,* $1<[(\delta\eta)/\{H(y+\eta)-H(y)\}]$.

Proof. Pick $\bar{\eta}>0$ such that $\delta\bar{\eta}>H(\bar{\eta})$. This is possible given Lemma 4.2; $H(0)=0$, then by continuity of H, there is a $\bar{y}>0$ such that

$$1<[\delta\{(\bar{y}+\bar{\eta})-\bar{y}\}/\{H(\bar{y}+\bar{\eta})-H(y)\}]$$

Now choose $(y,\eta)\leq(\bar{y},\bar{\eta})$ with $y\geq0$ and $\eta>0$, by convexity of H we have,

$$[\{H(\bar{y}+\eta)-H(\bar{y})\}/\eta]\leq[\{H(\bar{y}+\bar{\eta})-H(\bar{y})\}/\bar{\eta}]$$

and

$$[\{H(y+\eta)-H(y)\}/\eta] \le [\{H(\bar{y}+\eta)-H(\bar{y})\}/\eta]$$

Combining the above two inequalities, we get

$$[\{H(y+\eta)-H(y)\}/\eta] \le [\{H(\bar{y}+\bar{\eta})-H(\bar{y})\}/\bar{\eta}] < \delta$$

which completes the proof of Lemma A.1. ■

This result is crucial for us to establish the following lemma.

LEMMA A.2. *If* $\langle y_t \rangle$ *be the optimal income program from* $y>0$, *then* $\inf_{t\ge 0}\langle y_t\rangle > 0$.

Proof. From Proposition 4.2, we know that, the optimal income program $\langle y_t \rangle$ is either monotonically increasing or decreasing. If it is a monotonic increasing sequence, then $\inf_{t\ge 0}\langle y_t\rangle = y > 0$. Now, the result has to be proved for the case of decreasing sequence. Suppose the hypothesis is not true; then the optimal income program $\langle y_t \rangle$ from some $y>0$, is such that y_t decreases to 0 as $t\to\infty$. Here we can have two distinct possibilities, either (i) $y_t=0$ for some t, or, (ii) y_t approaches 0 in the limit. If $y_t=0$ for some t, let T be the first period (≥ 1) when this happens. Then, clearly, $y_T^C > y_{T+1}^C$. Otherwise, $y_t^C=0$ for all t, which implies that $y_t^I>0$ for all $t\ge 0$. Hence, $y_t>0$ for all t. But this is contradictory to the supposition that $y_t=0$ for some t. In the other case, where $y_t\downarrow 0$ as $t\to\infty$ define T such that $y_{T-1}<\min(\bar{y},\bar{\eta})$, and $y_T^C>y_{T+1}^C$. Such a choice of T is possible because zero sequence of consumption expenditure is not optimal. Also, note that such choice of T takes care of both the cases (i) and (ii). Then $y_T<\bar{y}$; and from Lemma A.1, with $y=0$ and $\eta=y_{T-1}$, $\delta y_{T-1}>H(y_{T-1})$. Therefore,

$$H(y_{T-1}) > y_{T-1} \qquad \text{or}, \quad y_T \le y_{T-1} < h(y_{T-1})$$

Now pick $\eta'>0$, such that $\eta'<\min[\bar{\eta}, y_T-H(y_{T+1})]$ and $[y_{T-1}-H(y_T+\eta')] > [y_T-H(y_{T+1})]$. Noting that $[y_{T-1}-H(y_T+\eta')]>0$, we have,

$$V(y_{T-1}) \ge g[y_{T-1}-H(y_T+\eta')] + \delta g[y_T+\eta'-H(y_{T+1})] + \delta^2 V(y_{T+1}) \quad \text{(A.7)}$$

Also,

$$V(y_{T-1}) = g[y_{T-1}-H(y_T)] + \delta g[y_T-H(y_{T+1})] + \delta^2 V(y_{T+1}) \quad \text{(A.8)}$$

Combining (A.7) and (A.8),

$$g[y_{T-1}-H(y_T)] + \delta g[y_T-H(y_{T+1})]$$
$$\ge g[y_{T-1}-H(y_T+\eta')] + \delta g[y_T+\eta'-H(y_{T+1})]$$
$$g[y_{T-1}-H(y_T)] - g[y_{T-1}-H(y_T+\eta')]$$
$$\ge \delta g[y_T+\eta'-H(y_{T+1})] - \delta g[y_T-H(y_{T+1})]$$
$$\{g[y_{T-1}-H(y_T)] - g[y_{T-1}-H(y_T+\eta')]\}/\{H(y_T+\eta')-H(y_T)\}$$

$$\geq \{\delta\eta'/[H(y_T+\eta')-H(y_T)]\}(1/\eta')\{g[y_T+\eta'-H(y_{T-1})]-g[y_T-H(y_{T+1})]\}$$

Since $(y_T, \eta') \leq (\bar{y}, \bar{\eta})$, the result of Lemma A.2 contradicts with the concavity of g when $[y_{T-1}-H(y_T+\eta')]>[y_T+H(y_{T+1})]$. Hence $\inf_{t\geq 0} y_t>0$. ■

THEOREM 4.2(2). *Let $[y^*, y^{**}]$ be the set of stationary optimal incomes. Then, ϕ, has the property that:*

(a) $\phi(y)<y$ for all $y>y^{**}$; and
(b) $\phi(y)>y$ for all $0<y<y^*$.

Proof. We note that the optimal program $\langle y_t\rangle$ from $y_0>0$ is monotonic from Proposition 4.2. If $y_t \geq y_{t+1}$ for all $t\geq 0$, then $y_t \rightarrow (\inf_{t\geq 0} y_t)=y'>0$. If $y_t \leq y_{t+1}$ for all $t\geq 0$ then it is an increasing sequence which is bounded above by $B(y_0)$, hence it converges to some income $y''>0$, when $y_0>0$. By definition, $\phi(y_t)=y_{t+1}$ for all $t\geq 0$. Hence the continuity of ϕ implies that both y' and y'' are stationary optimal incomes so they will belong to the interval $[y^*, y^{**}]$. It can be shown that this result will be contradicted if (a) and (b) were not true. First, to show (a) holds:

For $y>y^0$, the maximum sustainable income, $h(y)<y$, and so $\phi(y)\leq h(y)<y$. Thus, if (a) were not true, then there would be some $\xi>y^{**}$ such that $\phi(\xi)\geq\xi$. By continuity of ϕ, there would then be some $\xi'>\xi$ for which we must have $\phi(\xi')=\xi'$. But this means $\xi'\in[y^*, y^{**}]$, which is a contradiction.

Now, to show (b) holds:

Using the continuity of ϕ and the fact that the set of non-trivial stationary optimal incomes is the closed interval $[y^*, y^{**}]$, we can conclude that if (b) were violated, then we ought to have $\phi(y)<y$ for *all* $0<y<y^*$. $\langle y_t\rangle$ is then a monotone decreasing sequence and from Lemma A.2 it converges to $0<y'<y^*$, which is not possible. ■

COROLLARY 4.1. *Let $\langle y_t^I, y_t^C\rangle$ be the optimal sequence of expenditure allocations. Then, either we have $y_t^I \leq y_{t+1}^I$ and $y_t^C \leq y_{t+1}^C$ for all $t\geq 0$; or, $y_t^I \geq y_{t+1}^I$ and $y_t^C \geq y_{t+1}^C$ for all $t\geq 0$.*

Proof. From Proposition 4.2 we have either $y_t \leq y_{t+1}$ or $y_t \geq y_{t+1}$ for all $t\geq 0$. Consider the first case, i.e., $y_t \geq y_{t+1}$ for all $t\geq 0$. Now, $y_{t-1}^I=H(y_t)$ and H is increasing which imply $y_{t-1}^I \leq y_t^I$. Repeating this argument for each t, we get $y_t^I \leq y_{t+1}^I$ for all $t\geq 0$. The value function, V, is strictly concave which implies $y_t^C \leq y_{t+1}^C$ for $y_t \leq y_{t+1}$ [see Dechert and Nishimura (1983)]. $y_t^I \geq y_{t+1}^I$ and $y_t^C \geq y_{t+1}^C$ for all $t\geq 0$ can be proved in exactly the same way for the other case. ■

BIBLIOGRAPHY

Bardhan, P. K. (1965), "Optimum Accumulation and International Trade", *Review of Economic Studies*, vol. 32, no. 91, pp. 241–244.

Benhabib, J. and K. Nishimura (1985), "Competitive Equilibrium Cycles", *Journal of Economic Theory*, vol. 35, no. 1, pp. 284–306.

Bhagwati, J. N. and T. N. Srinivasan (1983), *Lectures on International Trade*, M.I.T. Press.

Boldrin, M. and L. Montrucchio (1984), "The Emergence of Dynamical Complexities in Models of Optimal Growth: The Role of Impatience", *Rochester Centre for Economic Research*, Working Paper no. 7, Rochester, New York.

Boldrin, M. and L. Montrucchio (1985), "Cyclic and Chaotic Behavior in Intertemporal Optimization Models", in X. J. R. Avula, G. Leithmann, C. D. Mote and E. Y. Rodin (eds.) *Mathematical Modelling in Science and Technology*, Pergamon Press.

Border, K. C. (1985), *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press.

Cass, D. (1965), "Optimum Growth in an Aggregative Model of Capital Accumulation", *Review of Economic Studies*, vol. 32, no. 3, pp. 233–240.

DasGupta, S. and T. Mitra (1990), "On Price Characterization of Optimal Plans in a Multi-Sector Economy", in B. Dutta, S. Gangopadhyay, D. Mookherjee, D. Ray (eds.) *Essays in Honor of D. Banerjee*, Oxford University Press.

Datta, M. (1992), *Essays on Intertemporal Allocations in Open Economies*, Ph. D. Dissertation, Cornell University, Ithaca, New York.

Dechert, W. D. and K. Nishimura (1983), "Complete Characterization of Optimal Growth Paths in an Aggregated Model with a Non-Concave Production Function", *Journal of Economic Theory*, vol. 31, no. 2, pp. 332–354.

Dutta, P. K. and T. Mitra (1989), "Maximum Theorems for Convex Structures with an Application to the Theory of Optimal Intertemporal Allocation", *Journal of Mathematical Economics*, vol. 18, no. 1, pp. 77–86.

Findlay, R. (1980), "The Terms of Trade and Equilibrium Growth in the World Economy", *American Economic Review*, vol. 70, no. 3, pp. 291–299.

Jones, R. W. (1961), "Comparative Advantage and the Theory of Tariffs: A Multi-Country, Multi-Commodity Model", *Review of Economic Studies*, vol. 28, no. 77, pp. 161–175.

Jones, R. W. and J. A. Scheinkman (1977), "The Relevance of the Two-Sector Production Model in Trade Theory", *Journal of Political Economy*, vol. 85, no. 5, pp. 909–935.

Koopmans, T. C. (1963), "On the Concept of Optimal Economic Growth", *The Econometric Approach to Development Planning*, North-Holland, pp. 225–287.

Majumdar, M. and M. Nermuth (1982), "Dynamic Optimization in Non-Convex Models with Irreversible Investment: Monotonicity and Turnpike Results", *Zeitschrift für Nationalökonomie*, vol. 42, no. 4, pp. 339–362.

Manning, R. (1980), "Specialization and Dynamics in a Trade Model", *The Economic Record*, pp. 251–260.

McKenzie, L. W. (1986), "Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics", in K. J. Arrow and M. D. Intrilligator (eds.) *Handbook of Mathematical Economics*, vol. III, North-Holland.

Mirman, L. J. (1980), "One Sector Economic Growth and Uncertainty: A Survey" in M. A. H. Dempster (ed.) *Stochastic Programming*, Academic Press.

Mitra, T. and D. Ray (1984), "Dynamic Optimization on a Non-Convex Feasible Set: Some General Results for Non-Smooth Technologies", *Zeitschrift für Nationalökonomie*, vol. 44, no. 2, pp. 151–175.

Nishimura, K. and M. Yano (1993), "Interlinkage in the Endogenous Real Business Cycles of International Economics", *Economic Theory*, vol. 3, no. 1, pp 151–168.

Peleg, B. and H. Ryder (1972), "On Optimal Consumption Plans in a Multi-Sector Economy," *Review of Economic Studies*, vol. 39, no. 118, pp. 159–169.

Rudin, W. (1976), *Principles of Mathematical Analysis*, Third Edition, McGraw Hill.

Samuelson, P. A. (1973), "Optimality of Profits-Including Prices Under Ideal Planning", *Proceedings of the National Academy of Sciences*, vol. 70, pp. 2109–2111.

Smith, M. A. M. (1977), "Capital Accumulation in the Open Two-Sector Economy", *The Economic Journal*, vol. 87, no 346, pp. 273–282.

Srinivasan, T. N. (1964), "Optimal Savings in a Two-Sector Model of Growth", *Econometrica*, vol. 32, no. 3, pp. 358–373.

Srinivasan, T. N. and J. N. Bhagwati (1980), "Trade and Welfare in a Steady State", in J. S. Chipman and C. P. Kindleberger (eds.) *Flexible Exchange Rates and the Balance of Payments*, North-Holland.

Sutherland, W. (1970), "On Optimal Developments in Multi-Sectoral Economy: The Discounted Case", *Review of Economic Studies*, vol. 37, no. 112, pp. 585–589.

Takayama, A. (1964), "Economic Growth and International Trade", *Review of Economic Studies*, vol. 31, no. 87, pp. 207–220.

Uzawa, H. (1962), "On a Two-Sector Model of Economic Growth", *Review of Economic Studies*, vol. 29, no. 78 pp. 40–47.

Uzawa, H. (1963), "On a Two-Sector Model of Economic Growth: II", *Review of Economic Studies*, vol. 30, no. 83, pp. 105–118.

Uzawa, H. (1964), "Optimal Growth in a Two-Sector Model of Capital Accumulation", *Review of Economic Studies*, vol. 31, no. 85, pp. 1–24.

Wan, H. Y. and M. Majumdar (1980), "Trade Under Temporary Equilibrium", *Journal of International Economics*, vol. 10, no. 1, pp. 37–62.

Yano, M. (1990), "Von Neumann Facets and the Dynamic Stability of Perfect Foresight Equilibrium Paths in Neo-Classical Trade Models", *Zeitschrift für Nationalökonomie*, vol. 51, no. 1, pp. 27–69.