Notes

VIABILITY AND OTHER RESULTS FOR AN EXTENDED INPUT-OUTPUT MODEL

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Abstract: The traditional viability theorem is proved and its connection with the nonsubstitution result studied for an input-output model extended to cover a limited degree of joint production.

1. INTRODUCTION

A well-known result in input-output theory states that an $n \times n$ input-output matrix has a nonnegative inverse if and only if it can realize a net output vector involving strictly positive quantities of *all* the goods. One of the best known proofs of this property is to be found in Gale (1960).[2] An objective of the present note is to show that Gale's proof is equally applicable to a class of production models which is somewhat more general than the ones to be found in the literature. The note then goes on to argue [by appealing to Chander (1974)] that it is this property of an (open-ended single primary factor) input-output system which plays a crucial role in the proof of the so-called nonsubstitution theorem also. As a result, the latter too is generalizable to a limited extent.[3]

In particular, it will be assumed in the sequel that *in order to realize a net output vector involving positive quantities of* all *the goods in the model, the number of activities to be operated at strictly positive levels must be at least as large as the number of goods.* Traditional theory ensures this property of the input-output system by restricting the analysis to single product activities alone. However, the absence of joint production, while sufficient, is not *necessary* for the property to be satisfied.

The next section spells out the model. The section following proves the major result of this note. The final section discusses the interconnection of the result with the nonsubstitution property of an input-output system.

[1] The author is indebted to two referees for helpful comments.

[2] See Theorem 9.1 in Gale (1960).

[3] Although the method of analysis employed here is due to the present author, the idea of a possible generalization of the input-output system appeared for the first time in Sinha (1979). An alternative to Sinha's approach was presented in Dasgupta and Sinha (1979). The present note, though greatly influenced by the latter work, offers a different as well as a simplified proof of the main result of that paper. Needless to say, Sinha cannot be held responsible for the opinions expressed here. A dynamic extension of the nonsubstitution result under joint production was attempted by Shiozawa (1977). However, the direction of Shiozawa's research was different from the one being reported here. The author is grateful to one of the referees for having drawn his attention to Shiozawa's work.

II. THE MODEL

There are n produced goods and m $(\geq n)$ activities in the system. A production activity is an element[4] of $\boldsymbol{R}_+^{2n}$ and characterized by the row vector

$$(\boldsymbol{b}_i, \boldsymbol{a}_i)=(b_{i1}, b_{i2}, \cdots, b_{in}; a_{i1}, a_{i2}, \cdots, a_{in}) \qquad i=1, \cdots, m$$

where, $b_{ij}=$ the gross output of the j-th good at unit level of operation of the i-th activity; and

$a_{ij}=$ the requirement of the j-th good as input for unit level of operation of the i-th activity.

Apart from the above description of the activities, the system is supposed to satisfy the usual assumptions imposed on an input-output system, including in particular the assumption of linearity. The model reduces to the standard Leontief system if $m=n$, $b_{ii}=1$ and $b_{ij}=0$, $i\neq j$.

At unit level of operation, activity i yields the net output vector $\boldsymbol{z}_i=\boldsymbol{b}_i-\boldsymbol{a}_i\in \boldsymbol{R}^n$ which forms the i-th row of the $m\times n$ matrix $\boldsymbol{Z}$. The net result of operating the activities at levels given by the row vector $\boldsymbol{x}=(x_1, x_2, \cdots, x_m)\in \boldsymbol{R}_+^m$ is represented by the product $\boldsymbol{xZ}$. The crucial assumption for the paper may be stated as follows:

(P) Let $\boldsymbol{xZ}\gg 0$ for some $\boldsymbol{x}\in \boldsymbol{R}_+^m$. Then, $\boldsymbol{x}$ has at least n strictly posivive components. That is, at least n activities must be operated to achieve postive net outputs of all the n goods.[5]

In particular, $\boldsymbol{Z}$ will be said to satisfy (P) whenever the above assumption holds.

Let $\boldsymbol{Z}^n$ denote an $n\times n$ real matrix with the property that $\sum_{i=1}^n x_i z_i\gg 0$ for some $(x_i, x_2, \cdots, x_n)\geq 0$. The matrix $\boldsymbol{Z}^n$ will be referred to as *productive*, though this involves a slight abuse of Gale's terminology [Gale (1960)].

3. VIABILITY OF THE SYSTEM

The central theorem may now be stated.

THEOREM. *Suppose* $\boldsymbol{Z}^n$ *satisfies* (P). Then $\boldsymbol{Z}^n$ *is productive iff for any* $y\geq 0$ *the equation*

$$\sum_{i=1}^{n} x_i z_i = y$$

has a solution $(x_1, x_2, \cdots, x_n)\geq 0$.

[4] Following standard practice, the n dimensional Euclidean space and its nonnegative orthant are represented by $\boldsymbol{R}^n$ and $\boldsymbol{R}_+^n$ respectively. Also, a vector $\boldsymbol{x}\in\boldsymbol{R}_+^n$ will often be indicated by $\boldsymbol{x}\geq 0$. Similarly, $\boldsymbol{x}\gg 0$ implies that $\boldsymbol{x}$ is contained in the interior of $\boldsymbol{R}_+^n$.

[5] It is easy to see that (P) rules out joint production when $n=2$. For $n=3$, however, the matrix formed out of $z_1=(-1, 1, 1)$, $z_2=(1, -1, 1)$ and $z_3=(1, 1, -1)$ satisfies (P). *In other words, the model admits joint production when* $n\geq 3$. The example was suggested by T. N. Sinha.

A major step in the proof of the theorem is the following lemma.

LEMMA. *Suppose* $\boldsymbol{Z}^n$ *satisfies* (P). *If* $\boldsymbol{Z}^n$ *is productive and* $\sum_{i=1}^n x_i z_i \geq 0$, *then* $(x_1, x_2, \cdots, x_n) \geq 0$.

The method of proof employed below follows Gale's proof of Lemma 9.1.

Proof of the Lemma. By definition, there is a vector $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \geq 0$ such that $\sum_{i=1}^n \bar{x}_i z_i \gg 0$. Then (P) implies $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \gg 0$. Suppose now that $(x_1, x_2, \cdots, x_n)$ satisfies $\sum_{i=1}^n x_i z_i \geq 0$ but $x_j < 0$ for some j. Let $\theta = \max_i[-x_i/\bar{x}_i] = -x_1/\bar{x}_1$ (say). Then $\theta > 0$ and $(x'_1, x'_2, \cdots, x'_n) = (x_1, x_2, \cdots, x_n) + \theta(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \geq 0$, with $x'_1 = 0$. On the other hand, $\sum_{i=1}^n x'_i z_i = \sum_{i=1}^n x_i z_i + \theta \sum_{i=1}^n \bar{x}_i z_i \gg 0$, which contradicts (P). Q.E.D.

To prove the theorem, it is now merely necessary to repeat the steps following Lemma 9.1 in Gale (1960). For the sake of completeness, these are reproduced below.

Proof of the Theorem. To start with, it may be noted that $\boldsymbol{Z}^n$ is a nonsingular matrix. For consider the equation $\boldsymbol{x}\boldsymbol{Z}^n = 0$ for $x \in \boldsymbol{R}^n$. The Lemma implies that $x \geq 0$. On the other hand, $\boldsymbol{x}\boldsymbol{Z}^n = 0$ must also mean that $(-\boldsymbol{x})\boldsymbol{Z}^n = 0$ and hence, by the Lemma once again, that $-x \geq 0$. Therefore, $x = 0$, so that $\boldsymbol{Z}^n$ is nonsingular.

Thus, for any $y \in \boldsymbol{R}^n$, $\exists a$ unique x satisfying $\boldsymbol{x}\boldsymbol{Z}^n = y$. Applying the Lemma once again, $x \geq 0$ whenever $y \geq 0$. Q.E.D.

4. NONSUBSTITUTION

To proceed further, it is assumed that an activity, to be operational, requires a positive quantity of a primary (i.e. nonproducible) factor (say, labor) as input. Wlg, each activity is supposed to require exactly one unit of this factor at unit level of operation.

To see the connection of the Theorem with the nonsubstitution result, consider, as in Chander (1974), the following linear programming problem:

$$\text{Min.} \sum_{i=1}^m x_i \qquad \text{subject to} \quad \sum_{i=1}^m x_i z_i \geq c \gg 0\,,$$

where, $c = (c_1, c_2, \cdots, c_n)$ is a *feasible* final demand vector. Assume, moreover, that $\boldsymbol{Z}$ satisfies (P). The optimal basis for the problem must then consist of exactly n vectors, say $z_1, z_2, \cdots, z_n$. This basis may be denoted by the $n \times n$ productive matrix $\boldsymbol{Z}^{n*}$.

The strategy of Chander's proof of the nonsubstitution theorem (as applied to the present situation) is to show that $\boldsymbol{Z}^{n*}$ is an optimal basis for the n final demand vectors $\boldsymbol{u}_i$, $i = 1, \cdots, n$ also, where $\boldsymbol{u}_i$ represents the $n \times 1$ vector $(0, \cdots, 0, 1, 0, \cdots, 0)$ with unity appearing in the i-th position. Indeed, if this happens to be the case, then by linearity it is an optimal basis for *all* nonnegative final demand vectors, guaranteeing thereby that the nonsubstitution property is valid.

The first step in showing the required optimality of $\boldsymbol{Z}^{n*}$ is to demonstrate its feasibility for all $\boldsymbol{u}_i$ and it is *here* that the Theorem of the previous section plays a major role. For the property in question follows straightaway from the productivity of $\boldsymbol{Z}^{n*}$ noted above. The second step consists of arriving at a contradiction based on the supposition that $\boldsymbol{Z}^{n*}$, though feasible, is not *optimal* for all $\boldsymbol{u}_i$. Indeed, if it is not so, then there is at least one $\boldsymbol{u}_i$, say $\boldsymbol{u}_i$, for which the optimal basis is (say) $\boldsymbol{Z}' \neq \boldsymbol{Z}^{n*}$. By the linearity of the input-output system, $\boldsymbol{Z}'$ is an optimal basis for the final demand vector $(c_1, 0, \cdots, 0)$ also.

Assuming wlg that the optimal bases for $\boldsymbol{u}_2, \cdots, \boldsymbol{u}_n$ continue to be $\boldsymbol{Z}^{n*}$, it follows from linearity once again, that a combination of $\boldsymbol{Z}'$ and $\boldsymbol{Z}^{n*}$ can realize $\boldsymbol{c}$ more cheaply than $\boldsymbol{Z}^{n*}$ alone. This yields the required contradiction to the hypothesis that $\boldsymbol{Z}^{n*}$ is an optimal basis for $\boldsymbol{c}$.

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