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ON THE LE CHÂTELIER-SAMUELSON PRINCIPLE*

Richard K. ANDERSON and Akira TAKAYAMA

Abstract: This paper obtains a method to derive the Le Châtelier-Samuelson Principle. Not only is it simpler and more straightforward than other methods in the literature, but also it enables us to obtain important results which are not readily accessible otherwise.

The purpose of this paper is to obtain a simple and clear method to derive the Le Châtelier-Samuelson (LeS) principle. In his work (1947), Samuelson observes that certain comparative statics properties can be obtained as a *general* feature of the solutions of the underlying extremum problem. In particular, he obtains the following results: "(1) the elasticity of factor demand and commodity supply are lower in the short-run than in the long-run, and (2) the elasticity of (compensated) demand for a product is lower with rationing than without" (Hatta, 1987, p. 155).

Hatta (1980, 1987) considers several types of extremum problems, and obtains the LeS principle in an ingenious way. He defines the "gain function" of a particular extremum problem, observing that it achieves an extremum with respect to parameters (as well as decision variables), and obtains the LeS principle from its first-order and second-order necessary conditions.

Not only is our method simple and transparent, but also it enables us to obtain important results which are not readily accessible by other methods including Hatta's. Namely, it yields the LeS principle with regard to the effect of parameter changes on Lagrangian multipliers as well as to the effect of parameter changes on decision variables, whereas Hatta's work (1980, 1987) is focused on the latter.[1] The crux of our method is to utilize the envelope theorem (e.g., Takayama, 1985, pp. 137–141).

We shall exposit our method in terms of specific examples. We begin our discussion with the familiar cost minimization problem and develop our analysis to more general results. Not only does this motivate our discussion, it will clarify the analytical simplicity of our method. The technique developed here can then be applied to other comparative statics problems.

Let x and z, respectively, denote vectors of variable and quasi-fixed factors, where $x \in R^n$ and $z \in R^m$. Let $f(x, z)$ be a production function, which we assume

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[1] Our results are also global. This is because we utilize the concavity of the maximum value function with respect to parameters (e.g., Otani 1982).

to satisfy the usual regularity conditions.[2] Let $w>0$ be the price vector of variable factors, and let $r>0$ be the vector of user's cost of quasi-fixed factors. The firm's "short-run" problem is written as,

$$\underset{x}{\text{Minimize}}: \quad w\cdot x \qquad \text{Subject to:} \quad f(x,z)\geqq y \text{ and } x\geqq 0\,,$$

where z is fixed. Write the unique solution and the Lagrangian multiplier as,

$$x^0(w,y;z)>0 \quad \text{and} \quad \lambda^0(w,y;z)>0\,.$$

Define $C^0(w,y;z)\equiv w\cdot x^0(w,y;z)$, which we assume to be twice continuously differentiable in w and y. By the envelope theorem, we at once obtain,

$$\partial x_i^0/\partial w_j=\partial x_j^0/\partial w_i, \quad \partial x_i^0/\partial y=\partial \lambda^0/\partial w_i, \quad i,j=1,2,\cdots,n\,. \tag{1}$$

We now proceed to the "long-run" problem. To ease the exposition, we first assume $m=1$, or z is a scalar signifying the "plant" of the firm, and write the firm's "long-run" problem as,

$$\underset{x,z}{\text{Minimize}}: \quad w\cdot x+rz \qquad \text{Subject to:} \quad f(x,z)\geqq y,\ x\geqq 0, \text{ and } z\geqq 0\,.$$

Write the unique solution and the Lagrangian multiplier as,

$$x^1(w,r,y)>0 \quad \text{and} \quad \lambda^1(w,r,y)>0\,.$$

Define $C^1(w,r,y)\equiv w\cdot x^1(w,r,y)+rz^1(w,r,y)$. Assume that C^1 is twice continuously differentiable in (w,r,y), then via the envelope theorem, we obtain,

$$\partial C^1/\partial w_i=x_i^1, \quad i=1,2,\cdots,n, \quad \partial C^1/\partial r=z^1, \quad \partial C^1/\partial y=\lambda^1\,. \tag{2}$$

Applying Young's theorem to the Hessian of C^1, and noting that it is negative semidefinite (since C^1 is concave in w and r), we obtain from (2):

$$\partial x_i^1/\partial r=\partial z^1/\partial w_i, \quad i=1,2,\cdots,n, \quad \partial z^1/\partial y=\partial \lambda^1/\partial r\,, \tag{3-a}$$

$$\partial x_i^1/\partial w_i\leqq 0, \quad i=1,2,\cdots,n, \quad \partial z^1/\partial r\leqq 0\,. \tag{3-b}$$

Next, we assume the following identities:

$$x_i^1(w,r,y)\equiv x_i^0[w,y;z^1(w,r,y)], \quad i=1,2,\cdots,n\,, \tag{4-a}$$

$$\lambda^1(w,r,y)\equiv \lambda^0[w,y;z^1(w,r,y)]\,, \tag{4-b}$$

which are crucial to obtain the LeS results. For example, (4-a) states that the short-run and the long-run demands for variable factors are identical when the optimal value of z in the long-run, $z^1(w,r,y)$, is substituted into the short-run demand function x^0.[3] From (4), we obtain,

[2] Namely, f is positive, finite, twice continuously differentiable, strictly monotone, and strictly quasi-concave.

[3] For this assumption, Samuelson (1947, p. 36) writes, "the short-run condition holds in the long-run as well... since long-run total costs cannot be at a minimum unless short-run total costs are as low as possible."

$$\partial x_i^1/\partial w_i = \partial x_i^0/\partial w_i + (\partial x_i^0/\partial z)(\partial z^1/\partial w_i), \quad i=1, 2, \cdots, n, \tag{5-a}$$

$$\partial \lambda^1/\partial y = \partial \lambda^0/\partial y + (\partial \lambda^0/\partial z)(\partial z^1/\partial y). \tag{5-b}$$

Then, recalling $\partial z^1/\partial w_i = \partial x_i^1/\partial r$ from (3-a), and noting $\partial x_i^1/\partial r = (\partial x_i^0/\partial z)(\partial z^1/\partial r)$ from (4-a), we may rewrite (5-a) as,

$$\partial x_i^1/\partial w_i = \partial x_i^0/\partial w_i + (\partial x_i^0/\partial z)^2(\partial z^1/\partial r), \quad i=1, 2, \cdots, n. \tag{6}$$

Since $\partial z^1/\partial r \leqq 0$ by (3-b) we may conclude from this,

$$\partial x_i^1/\partial w_i \leqq \partial x_i^0/\partial w_i \ (\leqq 0), \quad i=1, 2, \cdots, n. \tag{7}$$

Eq. (5-a) can be rewritten as,

$$\partial x_i^1/\partial w_i = \partial x_i^0/\partial w_i + (\text{adjustment effect}), \tag{8}$$

where (adjustment effect) $\equiv (\partial x_i^0/\partial z)^2(\partial z^1/\partial r) \leqq 0$. Eq. (8) relates the long-run input demand function to its short-run counterpart with respect to own price changes, where the adjustment effect will never be positive.

Also, recall $\partial z^1/\partial y = \partial \lambda^1/\partial r$ from (3-a). Then noting $\partial \lambda^1/\partial r = (\partial \lambda^0/\partial z)(\partial z^1/\partial r)$ from (4-b), we may rewrite (5-b) as,

$$\partial \lambda^1/\partial y = \partial \lambda^0/\partial y + (\partial \lambda^0/\partial z)^2(\partial z^1/\partial r). \tag{9}$$

Since $\partial z^1/\partial r \leqq 0$ by (3-b), we then obtain,

$$\partial \lambda^1/\partial y \leqq \partial \lambda^0/\partial y. \tag{10}$$

We assume $\partial \lambda^1/\partial y > 0$ at the optimum.[4] Since λ is marginal cost (MC), we may then obtain the following relation from (10),

$$0 < \partial MC^1/\partial y \leqq \partial MC^0/\partial y. \tag{11}$$

Corresponding to (8), we may obtain the following relation,

$$\partial MC^1/\partial y = \partial MC^0/\partial y + (\text{adjustment effect}) \ [\equiv (\partial \lambda^0/\partial z)^2(\partial z^1/\partial r)] \leqq 0,$$

which again relates the long-run effect to the short-run effect.

Assume that the firm is competitive, so that its output price $p > 0$ as well as factor price vector $w > 0$ are given exogenously. Then the firm's supply response is obtained by solving $p = MC$ for y. Thus, the supply response is the reciprocal of the marginal cost output response, so that (10) implies,

$$\partial y^1/\partial p \geqq \partial y^0/\partial p. \tag{12}$$

If we impose Samuelson's "regularity condition",[5] then the own price effects are obtained in strict inequalities. In other words,

[4] This is the usual assumption in the literature. It can also be shown that $\partial \lambda^1/\partial y \geqq 0$ for *all* y iff f is concave. See Marino-Otani-Sicilian (1981).

[5] See Samuelson (1947, p. 68).

$$\partial x_i^0/\partial w_i<0, \quad \partial x_i^1/\partial w_i<0, \quad \partial z^1/\partial r<0, \quad i=1, 2, \cdots, n\,.$$

Then (7), (11), and (12) can, respectively, be written as,

$$\partial x_i^1/\partial w_i<\partial x_i^0/\partial w_i<0, \quad i=1, 2, \cdots, n\,, \tag{7'}$$

$$0<\partial MC^1/\partial y<\partial MC^0/\partial y\,, \tag{11'}$$

$$\partial y^1/\partial p>\partial y^0/\partial p>0\,. \tag{12'}$$

Relations (7), (11), (12), (7′), (11′), and (12′) constitute the LeS results for the present problem. (7′) states that *the long-run input demand function will be more responsive to own price changes than will be its short-run counterpart.* (11′) states that *the long-run marginal cost curves are flatter than their short-run counterparts.* (12′) states that *the long-run supply response is greater than the short-run response.* In terms of elasticities, (7′) means that *the long-run own price elasticities of variable input demands are greater than their short-run counterparts*, and (11′) means that *the long-run supply elasticity is greater than its short-run counterpart.* Hatta's (1980, 1987) method via the gain function enables us to obtain (7) and (7′), but not (11), (11′), (12), and (12′).

Now we relax the assumption that z is a scalar, and we suppose $z\in R^m$. Let $z^s=(z_1, \cdots, z_s)$, $z_{(s)}=(z_{s+1}, \cdots, z_m)$, $r^s=(r_1, \cdots, r_s)$, and $r_{(s)}=(r_{s+1}, \cdots, r_m)$. Obviously, $z=(z^s, z_{(s)})$ and $r=(r_1, \cdots, r_m)=(r^s, r_{(s)})$. We then consider the following minimization problem for $0\leqq s\leqq m$.

$$\underset{x,\, z^s}{\text{Minimize:}} \quad w\cdot x+r^s\cdot z^s \qquad \text{Subject to:} \quad f(x, z)\geqq y, x\geqq 0, \text{and } z^s\geqq 0\,,$$

where $z_{(s)}$ is a fixed vector. Let $x^s(w, r^s, y\,;\, z_{(s)})>0$ and $z^s(w, r^s, y\,;\, z_{(s)})>0$ be its unique solution,[6] and let $\lambda^s(w, r^s, y\,;\, z_{(s)})>0$ be the Lagrangian multiplier. Define the function C^s by $C^s(w, r^s, y\,;\, z_{(s)})\equiv w\cdot x^s(\cdot)+r^s\cdot z^s(\cdot)$, where $[C^s+r_{(s)}\cdot z_{(s)}]$ signifies the minimum total cost.

We may call this problem "problem s" and interpret the distinction between problems "$(s-1)$" and "s" as caused by "time." Namely, in problem s, the $(m-s)$ quasi-fixed factors are fixed, whereas in problem $(s-1)$, the $(m-s+1)$ quasi-fixed factors are fixed because time is not long enough to allow the adjustment of the s-th quasi-fixed factor.

Assume that C^s is twice continuously differentiable in (w, r^s, y). The envelope results may be stated as,

$$\partial C^s/\partial w_i=x_i^s, \quad \partial C^s/\partial r_k=z_k^s, \quad \partial C^s/\partial y=\lambda^s\,. \tag{13}$$

Since C^s is concave in (w, r^s), its Hessian with respect to (w, r^s) is negative semidefinite as well as symmetric. Thus using (13), we have,

$$\partial z_k^s/\partial w_i=\partial x_i^s/\partial r_k, \quad \partial z_k^s/\partial r_j=\partial z_j^s/\partial r_k, \quad \partial z_k^s/\partial y=\partial\lambda^s/\partial r_k\,, \tag{14-a}$$

[6] The demand function $x_i(w, r^s, y\,;\, z_{(s)})$ corresponds to Pollak's (1969) "conditional demand function." His formulation is a special case of ours.

$$\partial x_i^s/\partial w_i \leqq 0, \quad \partial z_k^s/\partial r_k \leqq 0 . \tag{14-b}$$

Corresponding to (4), we assume the following identities, for $1 \leqq s \leqq m$,

$$x_i^s(w, r^s, y; z_{(s)}) \equiv x_i^{s-1}[w, r^{s-1}, y; z_s(\cdot), z_{(s)}] , \tag{15-a}$$

$$z_j^s(w, r^s, y; z_{(s)}) \equiv z_j^{s-1}[w, r^{s-1}, y; z_s(\cdot), z_{(s)}] , \tag{15-b}$$

$$\lambda^s(w, r^s, y; z_{(s)}) \equiv \lambda^{s-1}[w, r^{s-1}, y; z_s(\cdot), z_{(s)}] , \tag{15-c}$$

where $z_s(\cdot) \equiv z_s^s(w, r^s, y; z_{(s)})$. Carrying out the procedure which is similar to that used to derive (6) and (9), we obtain from (14) and (15),

$$\partial x_i^s/\partial w_i = \partial x_i^{s-1}/\partial w_i + (\partial x_i^{s-1}/\partial z_s)^2(\partial z_s^s/\partial r_s), \quad i=1, \cdots, n ,$$

$$\partial z_j^s/\partial r_j = \partial z_j^{s-1}/\partial r_j + (\partial z_j^{s-1}/\partial z_s)^2(\partial z_s^s/\partial r_s), \quad j=1, \cdots, s-1 ,$$

$$\partial \lambda^s/\partial y = \partial \lambda^{s-1}/\partial y + (\partial \lambda^{s-1}/\partial z_s)^2(\partial z_s^s/\partial r_s) .$$

Since $\partial z_s^s/\partial r_s \leqq 0$, we obtain the following relations from this:

$$0 \geqq \partial x_i^0/\partial w_i \geqq \partial x_i^1/\partial w_i \geqq \cdots \geqq \partial x_i^m/\partial w_i, \quad i=1, \cdots, n , \tag{16-a}$$

$$0 \geqq \partial z_j^{s-1}/\partial r_j \geqq \partial z_j^s/\partial r_j, \quad j=1, \cdots, s-1, \quad 1<s \leqq m , \tag{16-b}$$

$$\partial \lambda^0/\partial y \geqq \partial \lambda^1/\partial y \geqq \cdots \geqq \partial \lambda^m/\partial y > 0 . \tag{16-c}$$

Again, assume that the firm is competitive, so that its output price and factor price vectors are given exogenously. Since the firm's supply response is then the reciprocal of the marginal cost output response, (16-c) implies,

$$0 < \partial y^0/\partial p \leqq \partial y^1/\partial p \leqq \cdots \leqq \partial y^m/\partial p . \tag{17}$$

Assume again Samuelson's regularity condition. Then we have,

$$\partial x_i^s/\partial w_i < 0, \quad i=1, \cdots, n, \quad \partial z_k^s/\partial r_k < 0, \quad k=1, \cdots, s . \tag{14'-b}$$

Thus, (16) and (17) can be sharpened as,

$$0 > \partial x_i^0/\partial w_i > \partial x_i^1/\partial w_i > \cdots > \partial x_i^m/\partial w_i, \quad i=1, \cdots, n , \tag{16'-a}$$

$$0 > \partial z_j^{s-1}/\partial r_j > \partial z_j^s/\partial r_j, \quad j=1, \cdots, s-1, \quad 1<s \leqq m , \tag{16'-b}$$

$$\partial MC^0/\partial y > \partial MC^1/\partial y > \cdots > \partial MC^m/\partial y > 0 , \tag{16'-c}$$

$$0 < \partial y^0/\partial p < \partial y^1/\partial p < \cdots < \partial y^m/\partial p . \tag{17'}$$

The economic interpretation of (16′-a), (16′-c), and (17′) can be obtained in a manner similar to that of (7′), (11′), and (12′), respectively, whereas (16′-b) means that the "longer-run" demand for quasi-fixed factors will be more responsive to own price changes than will be its shorter-run counterpart.[7]

The above method of obtaining the LeS principle results can also be applied to the theory of consumption. To this end, let $u(x)$ be an individual's utility function, where $x=(x_1, \cdots, x_n)$ is his (or her) consumption bundle. Let $p=(p_1, \cdots, p_n)$ be the price vector, and let $p^s=(p_1, \cdots, p_s)$ and $p_{(s)}=(p_{s+1}, \cdots, p_n)$. Similarly, partition $x=(x^s, x_{(s)})$, where $x^s=(x_1, \cdots, x_s)$ and $x_{(s)}=(x_{s+1}, \cdots, x_n)$. We then consider the following problem.

$$\underset{x^s}{\text{Minimize:}} \quad p^s \cdot x^s \qquad \text{Subject to:} \quad u(x) \geqq u \text{ and } x^s \geqq 0\,,$$

where $x_{(s)}$ is a fixed vector. In the theory of rationing, this is a form of "point rationing" in which $x_{(s)}$ is a vector of rationed commodities. There are other instances in which $x_{(s)}$ is fixed. As Pollak (1969, p. 63) writes, "consumers, like firms, have commitments which are fixed in the short-run. For example, if an individual signs a lease to rent an apartment for twelve months, his consumption of housing services during any month is fixed. . . . ."

Let $x^s(p^s, u; x_{(s)})>0$ be the unique solution of the above problem, and let $\lambda^s(p^s, u; x_{(s)})>0$ be the Lagrangian multiplier associated with it. Note that this problem is formally identical to the cost minimization problem discussed earlier by considering the change of notations, $r \to p$ and $z \to x$. Then, the results which correspond to (16) and (17) follow at once. In particular, from (16-b), we obtain,

$$0 \geqq \partial x_i^{s-1}/\partial p_i \geqq \partial x_i^s/\partial p_i\,, \quad i=1, \cdots, s-1, \quad 1<s\leqq n\,. \tag{18}$$

This generalizes formula (6.4), the LeS result, obtained by Pollak (1969), where he assumes $s=n$. Under the Samuelson regularity condition, (18) is sharpened as,

$$0>\partial x_i^{s-1}/\partial p_i>\partial x^s/\partial p_i, \quad i=1, \cdots, s-1, \quad 1<s\leqq n\,. \tag{18'}$$

In particular, if $s=n$ as in Pollak (1969), we obtain,

$$0>\partial x_i^{n-1}/\partial p_i>\partial x_i^n/\partial p_i, \quad i=1, \cdots, n-1\,. \tag{18''}$$

Thus, if the n-th commodity is rationed, then the own price elasticity of (compensated) demand for commodity i is lower with rationing than without.[8]

We now obtain the LeS results for a general formulation which encompasses the above studies. Let $f(x, z)$ be a real-valued function where $x \in R^n$ and $z \in R^n$. As before, let us partition z by $z=(z^s, z_{(s)})$, where $z^s=(z_1, \cdots, z_s)$ and $z_{(s)}=(z_{s+1}, \cdots, z_m)$. We then consider the following problem.

$$\underset{x,z}{\text{Maximize:}} \quad f(x, z)-\alpha \cdot x-\beta^s \cdot z^s \quad \text{Subject to:} \quad g(x, z)\leqq \gamma,\ x\geqq 0, \text{ and } z^s \geqq 0\,,$$

[7] This result corresponds to the following by Samuelson (1947, pp. 38–39): "A lengthening of the time period so as to permit new factors to be varied will result in *greater* changes in the factor whose price has changed, . . ." However, as Pollak (1969, p. 76) remarks, "Samuelson's proof of the theorem requires the manipulation of cumbersome Jacobian and the use of Jacobi's theorem."

[8] See Samuelson (1947, p. 168). See also Pollak (1969).

where $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta^s=(\beta_1, \cdots, \beta_s)$, and $\gamma\equiv(\gamma_1, \cdots, \gamma_\ell)$ are vectors of parameters. Let $\delta^s\equiv(\alpha, \beta^s, \gamma; z_{(s)})$. Let $[x(\delta^s), z^s(\delta^s)]$ be a solution of the above problem, which is assumed to be unique, and $x(\cdot)>0$ and $z^s(\cdot)>0$. Let $\lambda^s\equiv(\lambda_1^s, \cdots, \lambda_\ell^s)>0$ be the Lagrangian multiplier associated with the solution, where $\lambda^s=\lambda^s(\delta^s)$. Define the maximum value function by $F^s(\delta^s)\equiv f([x(\delta^s), z^s(\delta^s)]-\alpha\cdot x(\delta^s)-\beta^s\cdot z^s(\delta^s)$. Then F^s is convex in α and β^s. Assume that F^s is twice continuously differentiable in δ^s.

Then repeating the analysis similar to that which gave (16) and (16′), and assuming the Samuelson regularity condition, we obtain, for $1\leqq s\leqq m$,[9]

$$0>\partial x_i^0/\partial\alpha_i>\partial x_i^1/\partial\alpha_i>\cdots>\partial x_i^m/\partial\alpha_i, \quad i=1, \cdots, n, \tag{19-a}$$

$$0>\partial z_j^{s-1}/\partial\beta_j>\partial z_j^s/\partial\beta_j, \quad j=1, \cdots, s-1, \quad 1\leqq s\leqq m, \tag{19-b}$$

$$\partial\lambda_k^0/\partial\gamma_k>\partial\lambda_k^1/\partial\gamma_k>\cdots>\partial\lambda_k^m/\partial\gamma_k, \quad k=1, \cdots, \ell. \tag{19-c}$$

(19-a) and (19-b) state that a lengthening of the time period so as to allow an additional z variable to adjust will result in greater own response effect on the decision variables than without (19-c) states that a lengthening of the time period will result in a fall in the shadow prices of resources.[10]

Texas A & M University
Southern Illinois University

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[9] This extends Otani's (1982) result. See also Samuelson (1966).

[10] By the envelope theorem, we have $\partial F^s/\partial\gamma_k=\lambda_k^s$, so that λ_k^s signifies the "shadow price" of the k-th resource, in which $g(x, z)\leqq\gamma$ is considered a set of l "resource constraints."