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BERTRAND, COURNOT AND MIXED OLIGOPOLIES

Ferenc SZIDAROVSZKY and Sándor MOLNÁR

Abstract: This paper introduces a general oligopoly model when some of the firms follow price adjusting strategies and others follow quantity adjusting ones. The relation of the resulting noncooperative game to nonlinear complementarity problems is first examined, and the existence and uniqueness of the equilibrium is proven under realistic conditions.

1. INTRODUCTION

Since the appearance of the classic book by Cournot in 1838, an increasing attention has been given to oligopoly, which is a state of industry, where a small number of firms produce homogeneous goods or close substitutes competitively. A large class of models consider this situation as a static noncooperative game, which is not repeated in time. In these models the central problem is to find sufficient conditions which guarantee the existence of the Nash-equilibrium. Many variants of oligopoly models are known from the literature. A comprehensive survey of such models, existence results, and solution methods are presented in Okuguchi and Szidarovszky (1990). Many authors refer to quantity and price adjusting models as Cournot and Bertrand oligopolies, respectively. The comparison of equilibrium prices in Cournot and Bertrand oligopolies was investigated by Vives (1984), and more recently by Okuguchi (1987), who has proven that under realistic conditions the Cournot equilibrium prices are not lower than those in Bertrand oligopolies.

In this paper some classical results will be generalized to oligopolies when it is allowed that some firms follow quantity adjusting strategies and others follow simultaneously price setting strategies. This mixed oligopoly model is the common generalization of Cournot and Bertrand oligopolies.

2. THE GENERAL MODEL AND THE EXISTENCE OF THE EQUILIBRIUM

Assume that the market demand functions are denoted by

$$x_i = g_i(p_1, \cdots, p_n) \qquad (i = 1, 2, \cdots, n), \tag{1}$$

where n is the number of firms with product differentiation and p_i $(i = 1, 2, \cdots, n)$ is the unit price of item i. Let S be a subset of $\{1, 2, \cdots, n\}$, and assume that the firms in S are quantity adjusting and the firms from $\bar{S} = \{1, 2, \cdots, n\} - S$ are price adjusting. That is, the strategy of any firm $i \in S$ is its output x_i, and the strategy of any firm $i\in\bar{S}$ is the unit price p_i of item i. If C_i denotes the cost function of firm i, then the profit of this firm is given as

$$\pi_i = x_i \cdot p_i - C_i(x_i)\ . \tag{2}$$

The resulting n-person game with sets R_+ of strategies and payoff functions π_i is called the S-oligopoly. Note that Cournot and Bertrand oligopolies are special cases of this model if one selects $S=\{1, 2, \cdots, n\}$ and $S=\varnothing$, respectively. Introduce the notation

$$\boldsymbol{x}^S=(x_i)_{i\in S},\ \boldsymbol{p}^{\bar{S}}=(p_i)_{i\in\bar{S}},\quad \boldsymbol{x}^{\bar{S}}=(x_i)_{i\in\bar{S}},\quad \boldsymbol{p}^S=(p_i)_{i\in S},$$
$$\boldsymbol{g}^S=(g_i)_{i\in S},\quad \text{and}\quad \boldsymbol{g}^{\bar{S}}=(g_i)_{i\in\bar{S}}\ ,$$

and assume that

(A) For all S and nonnegative vectors $\boldsymbol{x}^S$ and $\boldsymbol{p}^{\bar{S}}$, equations

$$\begin{aligned}\boldsymbol{x}^S &= \boldsymbol{g}^S(\boldsymbol{p}^S, \boldsymbol{p}^{\bar{S}})\\ \boldsymbol{x}^{\bar{S}} &= \boldsymbol{g}^{\bar{S}}(\boldsymbol{p}^S, \boldsymbol{p}^{\bar{S}})\end{aligned} \tag{3}$$

have a unique nonnegative solution for $\boldsymbol{x}^{\bar{S}}$ and $\boldsymbol{p}^S$.

If this condition holds for only a convex, closed set for the parameters $(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})$, then we have to reduce the set of simultaneous strategies to this set.

Let the solution be denoted as

$$\boldsymbol{x}^{\bar{S}}=\boldsymbol{f}^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})$$

and (4)

$$\boldsymbol{p}^S=\boldsymbol{h}^S(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})\ .$$

For all $(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})\geqslant 0$, the profit of firm i can be rewritten as

$$\pi_i(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) = \begin{cases} x_i h_i^S(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) - C_i(x_i) & \text{if}\quad i\in S\\ p_i f_i^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) - C_i(f_i(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})) & \text{if}\quad i\in\bar{S}\ ,\end{cases} \tag{5}$$

where the components of $\boldsymbol{f}^{\bar{S}}$ and $\boldsymbol{h}^S$ are denoted by $f_i^{\bar{S}}$ and h_i^S, respectively.

Assume that

(B) Functions $\boldsymbol{f}^{\bar{S}}$, $\boldsymbol{h}^S$ and C_i are differentiable. Then the best response of firm i satisfies equation

$$h_i^S(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) + x_i \frac{\partial h_i^S}{\partial x_i}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) - C_i'(x_i) = 0 \qquad (i\in S)$$

or (6)

$$f_i^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) + p_i \frac{\partial f_i^{\bar{S}}}{\partial p_i}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) - C_i'(f_i^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}))\frac{\partial f_i^{\bar{S}}}{\partial p_i}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) = 0 \qquad (i\in\bar{S})\ ,$$

if we assume that the best response is positive for all i.

Assume furthermore that

(C) Function (5) is concave in x_i $(i \in S)$ or p_i $(i \in \bar{S})$ with fixed values of the other components of $(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})$.

The above derivation implies the following

THEOREM 1. *Under assumptions* (A), (B), *and* (C) *a positive vector* $(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})$ *is an equilibrium if and only if it satisfies equations* (6).

Remark. Note that (6) consists of n equations for the n unknowns (x_i, $i \in S$ and p_i, $i \in \bar{S}$). This system of nonlinear equations can be solved by standard numerical techniques. A survey of the relevant methodology is presented, for example, in Szidarovszky and Yakowitz (1978).

Consider next the general case, when $(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})$ is not necessarily an interior equilibrium. Then

$$\alpha_i^S(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) \begin{cases} \leqq 0 & \text{if} \quad x_i^S = 0 \\ = 0 & \text{if} \quad x_i^S > 0 \end{cases} \qquad (i \in S)$$

and

$$\beta_i^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) \begin{cases} \leqq 0 & \text{if} \quad p_i^{\bar{S}} = 0 \\ = 0 & \text{if} \quad p_i^{\bar{S}} > 0\,, \end{cases} \qquad (i \in \bar{S})$$

where $\alpha_i^S(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})$ and $\beta_i^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})$ denote the left-hand sides of equations (6) for $i \in S$ and $i \in \bar{S}$, respectively. Denote $\boldsymbol{\alpha}^S = (\alpha_i^S)_{i \in S}$ and $\boldsymbol{\beta}^{\bar{S}} = (\beta_i^{\bar{S}})_{i \in \bar{S}}$, then the above derivation implies the following

THEOREM 2. *Under assumptions* (A), (B), *and* (C) *a vector* $(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})$ *is an equilibrium of the S-oligopoly if and only if it satisfies the following nonlinear complementarity problem*:

$$\begin{pmatrix} -\boldsymbol{\alpha}^S(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) \\ -\boldsymbol{\beta}^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) \end{pmatrix} \geqq \boldsymbol{0}\,, \quad \begin{pmatrix} \boldsymbol{x}^S \\ \boldsymbol{p}^{\bar{S}} \end{pmatrix} \geqq \boldsymbol{0}\,, \quad \begin{pmatrix} \boldsymbol{x}^S \\ \boldsymbol{p}^{\bar{S}} \end{pmatrix}^T \begin{pmatrix} -\boldsymbol{\alpha}^S(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) \\ -\boldsymbol{\beta}^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) \end{pmatrix} = 0\,. \tag{7}$$

Remark 1. If one drops condition (C), then Theorems 1 and 2 are modified as follows. If a vector $(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}})$ is an equilibrium of the S-oligopoly, then it satisfies (7), and if in addition, the equilibrium is interior, then it satisfies equations (6).

Remark 2. Any numerical method for solving nonlinear complementarity problems can be used to find the equilibrium of S-oligopolies.

Assume furthermore that

(D) $x_i = 0$ for $p_i \geqq \bar{p}_i$ and any $\boldsymbol{p}_{-i} = (p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_n)$ for a sufficiently large finite positive number $\bar{p}_i$, $i = 1, 2, \cdots, n$; and

(E) $p_i = 0$ for $x_i \geqq \bar{x}_i$ and any $\boldsymbol{x}_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ for a sufficiently large finite positive number $\bar{x}_i$, $i = 1, 2, \cdots, n$.

Under these assumptions we may assume that the set of strategies of any price adjusting firm i is the interval $[0, \bar{p}_i]$, and that of any quantity setting firm i is interval $[0, \bar{x}_i]$. Since all conditions of the Nikaido-Isoda theorem (see Nikaido and Isoda, 1955) are satisfied, we have the following

THEOREM 3. *Under assumptions* (A)–(E) *the S-oligopoly has at least one equilibrium.*

Remark. The assertion remains true in the more general case when only the continuity of functions $\boldsymbol{f}^{\bar{S}}$, h^S and C_i is assumed.

The uniqueness of the equilibrium is discussed next. In addition to the above conditions assume that

(F) Function $\begin{pmatrix} -\boldsymbol{\alpha}^S \\ -\boldsymbol{\beta}^{\bar{S}} \end{pmatrix}$ is strictly monotone, that is, for all $x_i, x_i' \in [0, \bar{x}_i]$ $(i \in S)$ and $p_i, p_i' \in [0, \bar{p}_i]$ $(i \in \bar{S})$,

$$(\boldsymbol{x}^S - \boldsymbol{x}'^S)^T(\boldsymbol{\alpha}^S(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) - \boldsymbol{\alpha}^S(\boldsymbol{x}'^S, \boldsymbol{p}'^{\bar{S}})) + (\boldsymbol{p}^{\bar{S}} - \boldsymbol{p}'^{\bar{S}})^T(\boldsymbol{\beta}^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) - \boldsymbol{\beta}^{\bar{S}}(\boldsymbol{x}'^S, \boldsymbol{p}'^{\bar{S}})) < 0 . \tag{8}$$

THEOREM 4. *Under assumptions* (A)–(F) *the S-oligopoly has exactly one equilibrium.*

Proof. Conditions (A)–(E) imply that there is at least one equilibrium and from Karamardian (1969) we know that there is at most one solution of the nonlinear complementarity problem (7).

Remark. If functions $\boldsymbol{\alpha}^S$ and $\boldsymbol{\beta}^{\bar{S}}$ are continuously differentiable and the Jacobian of $\begin{pmatrix} \boldsymbol{\alpha}^S \\ \boldsymbol{\beta}^{\bar{S}} \end{pmatrix}$ is negative definite, then condition (F) holds.

3. ECONOMIC INTERPRETATION

In this section assumptions (A)–(F) will be interpreted. For the sake of simplicity assume that the market demand function is linear:

$$x_i = \sum_{j=1}^{n} a_{ij} p_j + b_i \qquad (i = 1, 2, \cdots, n) .$$

The general case can be discussed similarly by replacing the coefficient matrix $A = (a_{ij})$ by the Jacobian of $\boldsymbol{g} = (g_i)$. Decomposing matrix A as

$$A = \begin{pmatrix} \boldsymbol{A}^{SS} & \boldsymbol{A}^{S\bar{S}} \\ \boldsymbol{A}^{\bar{S}S} & \boldsymbol{A}^{\bar{S}\bar{S}} \end{pmatrix}$$

according to the firms from S and $\bar{S}$, equations (3) can be reduced to the following:

$$\boldsymbol{x}^S = \boldsymbol{A}^{SS}\boldsymbol{p}^S + \boldsymbol{A}^{S\bar{S}}\boldsymbol{p}^{\bar{S}}$$
$$\boldsymbol{x}^{\bar{S}} = \boldsymbol{A}^{\bar{S}S}\boldsymbol{p}^S + \boldsymbol{A}^{\bar{S}\bar{S}}\boldsymbol{p}^{\bar{S}} .$$

Assuming that A^{SS} is invertible, from the first equation we have

$$p^S = (A^{SS})^{-1}(x^S - A^{S\bar{S}} p^{\bar{S}}) , \tag{9}$$

and by substituting this relation into the second equation we get

$$x^{\bar{S}} = A^{\bar{S}S}(A^{SS})^{-1} x^S + [A^{\bar{S}\bar{S}} - A^{\bar{S}S}(A^{SS})^{-1} A^{S\bar{S}}] p^{\bar{S}} . \tag{10}$$

The right hand sides of equations (9) and (10) give functions h^S and $f^{\bar{S}}$, respectively. The linearity of these functions implies that the set of the parameter values $(x^S, p^{\bar{S}})$, for which equations (9) and (10) give nonnegative vectors p^S and $x^{\bar{S}}$, is a nonempty polyhedron P. Note that A^{SS} is invertible, for example under the following assumptions:

(G) For all i, $a_{ii} < 0$, $a_{ij} \geqq 0$ $(i \neq j)$ and $-a_{ii} > \sum_{j \neq i} a_{ij}$.

Note that these conditions are essentially the same as those introduced in Okuguchi (1987), and they imply that $(A^{SS})^{-1} \leqq 0$, and matrices $A^{S\bar{S}}$ and $A^{\bar{S}S}$ are nonnegative.

Since $f^{\bar{S}}$ and h^S are linear, they are obviously differentiable. Therefore condition (B) holds, if C_i is differentiable for all i.

Consider next the payoff functions (5), and assume that the cost functions C_i are all convex. Then π_i is concave in x_i $(i \in S)$ and p_i $(i \in \bar{S})$, if the first terms are concave. If $i \in S$, then the first term of π_i reduces to a quadratic function, and the coefficient of x_i^2 is the i^{th} diagonal element of matrix $(A^{SS})^{-1}$. Under assumption (G) this diagonal element is nonpositive, so π_i is concave in x_i. Assume next that $i \in \bar{S}$. Then π_i is again quadratic in p_i, and the coefficient of p_i^2 is the i^{th} diagonal element of $A^{\bar{S}\bar{S}} - A^{\bar{S}S}(A^{SS})^{-1} A^{S\bar{S}}$. Observe that under condition (G) the diagonal elements of the first term are all negative, however the second term is a nonnegative matrix. Therefore the signs of the diagonal elements of the sum are undetermined. Hence, in order to guarantee that the payoff function π_i is concave for $i \in \bar{S}$ we have to assume that

(H) Matrix $A^{\bar{S}\bar{S}} - A^{\bar{S}S}(A^{SS})^{-1} A^{S\bar{S}}$ has negative diagonal.

This additional condition can be interpreted as the negative diagonal elements of $A^{\bar{S}\bar{S}}$ must be sufficiently large in absolute value.

In the linear case assumptions (D) and (E) are not satisfied. However, assuming discontinuity in the market demand and price functions these conditions may be enforced.

As we noted earlier, assumption (F) holds if the Jacobian of function $\begin{pmatrix} \alpha^S \\ \beta^{\bar{S}} \end{pmatrix}$ is negative definite. As the conclusion of this section this condition is examined. Observe first that for $i \in S$,

$$\alpha_i^S(x^S, p^{\bar{S}}) = \{(A^{SS})^{-1}(x^S - A^{S\bar{S}} p^{\bar{S}})\}_i + x_i \{(A^{SS})^{-1}\}_{ii} - C_i'(x_i) ,$$

where $\{\cdot\}_i$ and $\{\cdot\}_{ii}$ denote the i^{th} element of a vector and the (i, i) element of a matrix, respectively. Similarly,

$$\beta_i^{\bar{S}}(\boldsymbol{x}^S, \boldsymbol{p}^{\bar{S}}) = \{A^{\bar{S}S}(A^{SS})^{-1}\boldsymbol{x}^S + [A^{\bar{S}\bar{S}} - A^{\bar{S}S}(A^{SS})^{-1}A^{S\bar{S}}]\boldsymbol{p}^{\bar{S}}\}_i$$
$$+ \{A^{\bar{S}\bar{S}} - A^{\bar{S}S}(A^{SS})^{-1}A^{S\bar{S}}\}_{ii} \cdot (p_i - C_i'(\{A^{\bar{S}S}(A^{SS})^{-1}\boldsymbol{x}^S$$
$$+ [A^{\bar{S}\bar{S}} - A^{\bar{S}S}(A^{SS})^{-1}A^{S\bar{S}}]\boldsymbol{p}^{\bar{S}}\}_i)) .$$

For the sake of simplicity assume furthermore that

(I) For $i=1, 2, \cdots, n$, C_i is linear.

Then the Jacobian can be written as

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}^{SS} & \boldsymbol{J}^{S\bar{S}} \\ \boldsymbol{J}^{\bar{S}S} & \boldsymbol{J}^{\bar{S}\bar{S}} \end{pmatrix} \tag{11}$$

with

$$\boldsymbol{J}^{SS} = (A^{SS})^{-1} + \mathrm{diag}(\{(A^{SS})^{-1}\}_{ii};\ i \in S) ,$$
$$\boldsymbol{J}^{S\bar{S}} = -(A^{SS})^{-1}A^{S\bar{S}} ,$$
$$\boldsymbol{J}^{\bar{S}S} = A^{\bar{S}S}(A^{SS})^{-1} ,$$

and

$$\boldsymbol{J}^{\bar{S}\bar{S}} = A^{\bar{S}\bar{S}} - A^{\bar{S}S}(A^{SS})^{-1}A^{S\bar{S}} + \mathrm{diag}(\{A^{\bar{S}\bar{S}} - A^{\bar{S}S}(A^{SS})^{-1}A^{S\bar{S}}\}_{ii},\ i \in \bar{S}) .$$

In the general case, when C_i is not linear, multiples of C_i'' add to the diagonal elements of $\boldsymbol{J}^{SS}$ and $\boldsymbol{J}^{\bar{S}\bar{S}}$. Under assumptions (G) and (H), the diagonal terms of $\boldsymbol{J}^{SS}$ and $\boldsymbol{J}^{\bar{S}\bar{S}}$ are negative definite, so a sufficient condition for $\boldsymbol{J}$ being negative definite is the following:

(J) Matrix

$$\begin{pmatrix} (A^{SS})^{-1} & -(A^{SS})^{-1}A^{S\bar{S}} \\ A^{\bar{S}S}(A^{SS})^{-1} & A^{\bar{S}\bar{S}} - A^{\bar{S}S}(A^{SS})^{-1}A^{S\bar{S}} \end{pmatrix} \tag{12}$$

is negative semidefinite.

A possible interpretation of this last condition is the following. Notice first that in the case when A is a diagonal matrix with negative diagonal elements, the same holds for matrix (12), hence it is negative definite. The continuity of the eigenvalues on the matrix elements implies that it is sufficient that the off-diagonal elements of A are sufficiently small compared to the negative diagonal elements.

4. CONCLUSIONS

In this paper a common generalization of Cournot and Bertrand oligopolies is introduced. The equivalence of the equilibrium problem of this n-person noncooperative game and the solution of a nonlinear complementarity problem is shown. Using this equivalence, sufficient conditions are given for the existence of a unique equilibrium. And finally, these conditions are interpreted in the case of linear demand and cost functions.

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