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GLOBAL ASYMPTOTICAL STABILITY OF NON-STATIONARY DISCRETE SYSTEMS

Ferenc SZIDAROVSZKY and Dan LIU

Abstract: In this paper we generalize the global asymptotical stability conditions of Fujimoto (1987), Wu and Brown (1989) to non-stationary discrete systems.

1. INTRODUCTION

Several types of discrete physical and economic systems are characterized by difference equations of the form $\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k)$, where $\boldsymbol{f}: R^n \to R^n$ is a continuous mapping. The global asymptotic stability of discrete systems has an important role in analysing the long-term behavior of such systems, as well as in systems design. Fujimoto (1987), and Wu and Brown (1989) presented recently useful criteria for checking the global asymptotic stability of stationary discrete systems.

In many applications non-stationary discrete systems are analysed. For example, in the case of economic systems price changes, technical developments, inflation, etc. result in the changing in time of the iteration function $\boldsymbol{f}$. In this paper conditions will be derived for the global asymptotic stability of non-stationary discrete systems, which generalize the earlier results of Fujimoto (1987), Wu and Brown (1989). As we shall see, our conditions are easy to verify in practical applications.

2. GLOBAL STABILITY

Consider the iteration sequence

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k), \qquad k = 0, 1, 2, \dots \tag{1}$$

where for all k, $\boldsymbol{f}_k$: $R^n \to R^n$. Our analysis will be based on the following simple observations.

LEMMA. *Assume that for all* k, $\boldsymbol{f}_k(\boldsymbol{0}) = \boldsymbol{0}$ *and there is a continuous function* $\alpha: R^n \backslash \{0\} \to [0, 1)$ *such that for all* $k \geq 0$ *and* $\boldsymbol{x} \neq \boldsymbol{0}$,

$$\|\boldsymbol{f}_k(\boldsymbol{x})\| \leq \alpha(\boldsymbol{x}) \cdot \|\boldsymbol{x}\| . \tag{2}$$

Then the iteration sequence (1) *converges to* $\boldsymbol{0}$ *as* $k \to \infty$.

Proof. Note first, that from (2) we know that for all k, $\|\boldsymbol{x}_k\| \leq \|\boldsymbol{x}_0\|$. That is, the iteration sequence is bounded. If it does not converge to zero, then there is a subsequence $\{\boldsymbol{x}_{i_k}\}$ with nonzero elements which tends to a nonzero vector $\boldsymbol{x}^*$, and the subsequence $\{\boldsymbol{x}_{i_k+1}\}$ converges to an $\boldsymbol{x}^{**}$.

From (2) we have

$$\|\boldsymbol{x}_{i_{k+1}}\| \leq \|\boldsymbol{x}_{i_k+1}\| < \|\boldsymbol{x}_{i_k}\| ,$$

and by letting $k \to \infty$, we get the equality

$$\|\boldsymbol{x}^{**}\| = \|\boldsymbol{x}^{*}\| . \tag{3}$$

Since $\boldsymbol{x}^* \neq \mathbf{0}$, for sufficiently large value of k, $\boldsymbol{x}_{i_k}$ belongs to a closed ball B with centre $\boldsymbol{x}^*$ which does not contain the origin. Since B is compact, $\alpha(\boldsymbol{x}) \leq \alpha_B < 1$ for all $\boldsymbol{x} \in B$. Therefore, for large values of k,

$$\|\boldsymbol{x}_{i_k+1}\| = \|\boldsymbol{f}_{i_k}(\boldsymbol{x}_{i_k})\| \leq \alpha_B \cdot \|\boldsymbol{x}_{i_k}\| ,$$

and letting $k \to \infty$ yields to the relation

$$\|\boldsymbol{x}^{**}\| \leq \alpha_B \cdot \|\boldsymbol{x}^{*}\| < \|\boldsymbol{x}^{*}\| ,$$

which contradicts our earlier relation (3). Thus the proof is completed. □

Remark. If (2) is replaced by the weaker assumption that for all $k \geq 0$ and $\boldsymbol{x} \neq \mathbf{0}$,

$$\|\boldsymbol{f}_k(\boldsymbol{x})\| \leq \|\boldsymbol{x}\| , \tag{4}$$

the result may not hold, as the following example shows.

Example. Select $n = 1$, and for $k \geq 0$,

$$f_k(x) = \frac{(k+2)^2 - 1}{(k+2)^2} \cdot x .$$

If the initial term is choosen as $x_0 = 2$, then finite induction shows that

$$x_k = 1 + 1/(k+1) \to 1 \neq 0 \qquad \text{as} \quad k \to \infty .$$

Furthermore for all $k \geq 0$ and $x \neq 0$,

$$|f_k(x)| = ((k+2)^2 - 1)/(k+2)^2 \cdot |x| < |x| .$$

Our first result can be formulated as follows.

THEOREM 1. *Assume that* $\boldsymbol{f}_k \colon R^n \to R^n$ *is continuously differentiable, furthermore* $\boldsymbol{f}_k(\mathbf{0}) = \mathbf{0} (k \geq 0)$. *Let* $\boldsymbol{f}'_k$ *denote the Jacobian of* $\boldsymbol{f}_k$, *and assume that for all* $k \geq 0$ *and* $\boldsymbol{x} \neq \mathbf{0}$,

$$\|\boldsymbol{f}'_k(\boldsymbol{x})\| \leq \beta(\boldsymbol{x}) , \tag{5}$$

where $\beta \colon R^n \backslash \{\mathbf{0}\} \to [0, 1)$ *is a continuous function. Then the iteration sequence* (1) *converges to zero as* $k \to \infty$.

Proof. We will verify that the conditions of the lemma are satisfied, which will imply the assertion.

Note first that

$$\frac{d}{dt}f_k(tx)=f'_k(tx)\cdot x \quad (x\in R^n, 0\leq t\leq 1),$$

which implies that for all $k\geq 0$ and $x\neq \mathbf{0}$,

$$f_k(x)=\int_0^1 f'_k(tx)x\,dt .$$

Let $\delta\in(0, 1)$ be fixed, then

$$\|f_k(x)\|\leq\int_0^1 \|f'_k(tx)\|\cdot\|x\|dt\leq\left\{\int_0^\delta \|f'_k(tx)\|dt+\int_\delta^1 \|f'_k(tx)\|dt\right\}\cdot\|x\| . \tag{6}$$

The continuity of $f'_k(tx)$ implies that $\|f'_k\|\leq 1$ for $t\in[0, \delta]$. Furthermore

$$T(x)=\{tx \mid \delta\leq t\leq 1\}$$

is compact, and for $x\neq\mathbf{0}$, $\mathbf{0}\notin T(x)$. Therefore

$$\operatorname*{mas}_{z\in T(x)} \beta(z)=\beta_0(x)<1 ,$$

and from (6) we conclude that

$$\|f_k(x)\|\leq\{\delta+(1-\delta)\beta_0(x)\}\cdot\|x\| . \tag{7}$$

Since for fixed δ, $\beta_0(x)$ is continuous in x and $\delta+(1-\delta)\beta_0(x)<1$, the selection

$$\alpha(x)=\delta+(1-\delta)\beta_0(x)$$

satisfies the conditions of the Lemma. □

COROLLARY. *If process* (1) *is stationary, then* $f_k\equiv f$. *In this special case it is sufficient to assume that for all* $x\neq\mathbf{0}$, $\|f'(x)\|<1$, *since the selection* $\beta(x)=\|f'(x)\|$ *is satisfactory. This special result was introduced by Wu and Brown* (1989).

In our next result we show that if the condition of the Lemma holds in a neighborhood of $\mathbf{0}$, then outside this neighborhood condition (5) can be significantly relaxed.

THEOREM 2. *Assume that* f_k: $R^n\to R^n$ *is continuously differentiable, furthermore* $f_k(\mathbf{0})=\mathbf{0}$ $(k\geq 0)$. *Let B be an open neighborhood of* $\mathbf{0}$, *and assume that there exists a continuous function* α: $R^n\backslash\{\mathbf{0}\}\to[0, 1)$ *such that*

(i) $\|f_k(x)\|\leqq\alpha(x)\cdot\|x\|$ *for all k and* $\mathbf{0}\neq x\in B$;

(ii) *If* $x\notin B$ *and* $\|f_k(x)\|=\alpha(x)\cdot\|x\|$ *with some k, then* $\|f'_k(x)x\|<\alpha(x)\cdot\|x\|$.

Under these assumptions the iteration sequence (1) *converges to zero.*

Proof. We will prove that for all $k\geq 0$ and $x\neq\mathbf{0}$, relation (2) holds, which implies the assertion.

Assume that for some k, (2) does not hold in the entire set $R^n\backslash\{\mathbf{0}\}$, then

$$r^* = \inf\{\|x\| \mid x \neq 0 \text{ and (2) does not hold}\}$$

exists and is positive. If for all vectors satisfying $\|x\| = r^*$, $\|f_k(x)\| > \alpha(x) \cdot \|x\|$, then the continuity of functions f_k and α implies that r^* can be reduced, which contradicts the definition of r^*. Therefore for at least one x^*,

$$\|x^*\| = r^* \quad \text{and} \quad \|f_k(x^*)\| = \alpha(x^*) \cdot \|x^*\| . \tag{8}$$

Since f_k is differentiable, we know that for any $\varepsilon > 0$,

$$\|f_k((1-s)x^*) - f_k(x^*) - s f_k'(x^*) x^*\| < \varepsilon s \|x^*\|$$

with $s \in (0, 1)$ being large enough (see Ortega and Rheinboldt, 1970, p. 61), which inequality and (ii) imply that

$$\|f_k((1-s)x^*) - f_k(x^*)\| < s[\|f_k'(x^*)x^*\| + \varepsilon \|x^*\|] = s[\beta(x^*) + \varepsilon] \|x^*\| ,$$

where

$$\beta(x^*) = \frac{\|f_k'(x^*)x^*\|}{\|x^*\|} < \alpha(x^*) .$$

From this and equality (8) we conclude that

$$\|f_k((1-s)x^*)\| > \|f_k(x^*)\| - s[\beta(x^*) + \varepsilon] \|x^*\| = \|x^*\|(\alpha(x^*) - s\beta(x^*) - s\varepsilon)$$
$$\geq \|x^*\| \alpha(x^*)(1-s) = \|(1-s)x^*\| \alpha(x^*) ,$$

when ε is small enough. Since α is continuous, with sufficiently large value of $s \in (0, 1)$,

$$\|f_k((1-s)x^*)\| > \|(1-s)x^*\| \alpha((1-s)x^*) .$$

This inequality contradicts again the definition of r^*, which completes the proof. □

COROLLARY. *Consider aginst the special case, when* $f_k \equiv f$. *Condition* (i) *and* (ii) *can now be subsituted by the assumptions*:

There exists an $\varepsilon > 0$ *and a* $0 \leq q < 1$ *such that*

(i)′ *For all* $x \neq 0$ *and* $\|x\| < \varepsilon$,

$$\|f(x)\| \leq q \cdot \|x\| ;$$

(ii)′ *If* $\|x\| \geq \varepsilon$ *and* $\|f(x)\| = \|x\|$, *then*

$$\|f'(x)x\| < \|x\| .$$

Proof. Define

$$Q_k = \max\{\|f'(x)x\| / \|x\| \mid \|f(x)\| = \|x\|, k\varepsilon \leq \|x\| \leq (k+1)\varepsilon\}$$

for $k = 1, 2, \ldots$. Obviously $Q_k < 1$ for all k. Introduce now the constants

$$R_k = \max\{q; Q_1; Q_2; \cdots; Q_k\},$$

and the piece-wise linear function $s(t)$ with vertices $(0, q), (\varepsilon, R_1), (2\varepsilon, R_2), (3\varepsilon, R_3), \ldots$. Then all conditions of the theorem are satisfied with $B = \{x \mid \|x\| < \varepsilon\}$ and $\alpha(x) = s(\|x\|)$. □

Remark. The mean-value theorem of derivatives imply that if $\|f'(\mathbf{0})\| < 1$, then there exist $\varepsilon > 0$ and $0 \le q < 1$ which satisfy condition (i)′. Assume further that if $x \ne \mathbf{0}$ and $\|f(x)\| = \|x\|$, then $\|f'(x) \cdot x\| < \|x\|$. In this case condition (ii)′ is also satisfied. Hence the iteration sequence (1) converges to zero. This special result was first introduced by Fujimoto (1987).

The conditions of the above theorems can be further relaxed in the following ways.

1. In the Lemma the continuity of $\alpha(x)$ can be relaxed as follows. For all closed balls B such that $\mathbf{0} \notin B$ there exists a constant α_B such that

$$\alpha(x) \le \alpha_B < 1 \qquad \text{for all} \quad x \in B.$$

2. It is easy to verify that the Lemma remains true in the more general case when $f_k: R^n \to 2^{R^n}$ is a point-to-set mapping, and condition (2) is replaced by the following:

$$\|y\| \le \alpha(x) \cdot \|x\| \qquad \text{for all} \quad y \in f_k(x).$$

3. Since all iterates x_k remain in the ball $B_0 = \{x \mid \|x\| \le \|x_0\|\}$, it is sufficient to assume that the conditions of the theorems hold in a certain subset of R^n which contains B_0.
4. Let $K > 0$ be a fixed integer, and assume that the conditions hold for all $k \ge K$. Then the assertions remain true.
5. Define function

$$f_{km} = f_{k+m} \circ f_{k+m-1} \circ \cdots \circ f_k,$$

and assume that sequence $f_{0,m}, f_{m,m}, \ldots, f_{im,m}, \ldots$ satisfies the conditions. Then the iteration sequence (1) converges to zero as $k \to \infty$. The details are left to the reader.

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