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NON-MINORITY RULES: NECESSARY AND SUFFICIENT CONDITION FOR QUASI-TRANSITIVITY WITH QUASI-TRANSITIVE INDIVIDUAL PREFERENCES

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Abstract. It is shown that, under the assumption that individual weak preference relations are reflexive, connected and quasi-transitive, a necessary and sufficient condition for quasi-transitivity under every non-minority rule, which is different from weak Pareto-extension rule, is that the strict Latin Square unique value restriction holds over every triple of alternatives.

For several classes of social decision rules conditions have been formulated which ensure transitivity, quasi-transitivity, or acyclicity of social weak preference relation. Under the assumption that individual weak preference relations are reflexive, connected and transitive, i.e., are orderings, necessary and sufficient conditions for transitivity and quasi-transitivity under the method of majority decision have been derived by Inada [5] and Sen and Pattanaik [14], necessary and sufficient condition for transitivity under the simple non-minority rule by Fine [3], necessary and sufficient conditions for transitivity and quasi-transitivity under the class of non-minority rules by Jain [7], sufficient conditions for quasi-transitivity under the class of neutral and monotonic binary social decision rules by Sen [13], maximal sufficient conditions for quasi-transitivity under a subclass of simple games by Salles [12], and sufficient conditions for acyclicity under the simple non-minority rule by Dummett and Farquharson [2] and Pattanaik [11]. Under the assumption that individual weak preference relations are reflexive, connected and quasi-transitive, necessary and sufficient conditions for quasi-transitivity under the method of majority decision have been obtained by Inada [6] and Fishburn [4], necessary and sufficient condition for quasi-transitivity under the class of special majority rules by Jain [9], and sufficient conditions for quasi-transitivity under the class of neutral and monotonic binary social decision rules by Inada [6] and Pattanaik [10].

This paper is concerned with the derivation of necessary and sufficient condition for quasi-transitivity under the class of non-minority rules, for the case when individual weak preference relations are reflexive, connected and quasi-transitive. It is shown that a condition defined over triples of alternatives, called the strict Latin Square unique value restriction, is necessary and sufficient for

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quasi-transitivity under every non-minority rule which is different from weak Pareto-extension rule. The condition requires that there must not be an alternative in the triple such that it is uniquely proper medium in some individual weak preference relation R_i, uniquely proper best in some individual weak preference relation R_j, uniquely proper worst in some individual weak preference relation R_k; and R_i, R_j and R_k from a strict Latin Square.

I. NOTATION AND DEFINITIONS

The set of mutually exclusive social alternatives will be denoted by S. Throughout this paper, we assume that S is finite and $\#S \geq 3$. We designate by L the set of individuals and by N the number of individuals. Every individual $i \in L$ is assumed to have a binary weak preference relation R_i over S. Throughout this paper we shall assume that each R_i is reflexive, connected, and quasi-transitive. R will stand for the social weak preference relation. We denote the symmetric and asymmetric parts of R_i by I_i and P_i respectively and those of R by I and P. $N(\ )$ will denote the number of individuals having the preferences specified within the parentheses and N_k the number of individuals holding the k-th weak preference relation. We use the standard notation $[x]$ to denote the largest integer less than or equal to x. The class of non-minority rules is defined by

$$\forall x, y \in S: [xRy \text{ iff } \sim[N(yP_ix) > pN]]$$

where p is a fraction, $1/2 \leq p < 1$.

The above definition is equivalent to:

$$\forall x, y \in S: [[xPy \text{ iff } N(xP_iy) \geq [pN]+1] \ \& \ [xRy \text{ iff } \sim(yPx)]]$$

where p is a fraction, $1/2 \leq p < 1$. For $p = 1/2$ we obtain the familiar simple non-minority rule also known as the strict majority rule.

p-non-minority rule defined for the set of alternatives S and the set of individuals L will be written as NMR (S, L, p) in abbreviated form.

The weak Pareto-extension rule is defined as follows:

$$\forall x, y \in S: [xRy \text{ iff } \sim(\forall i \in L: yP_ix)]$$

Weak Pareto-extension rule defined for the set of alternatives S and the set of individuals L will be written as WPER (S, L) in abbreviated form.

A set of three distinct alternatives will be referred to as a triple of alternatives. An individual is defined to be concerned over a triple of alternatives iff it is not the case that the individual is indifferent over every pair of alternatives belonging to the triple; otherwise he is unconcerned. We define, for individual i, in the triple $\{x, y, z\}$, x to be best iff $(xR_iy \ \& \ xR_iz)$, medium iff $[(yR_ix \ \& \ xR_iz) \vee (zR_ix \ \& \ xR_iy)]$, worst iff $(yR_ix \ \& \ zR_ix)$, proper best iff $[(xP_iy \ \& \ xR_iz) \vee (xR_iy \ \& \ xP_iz)]$, proper medium iff $[(yP_ix \ \& \ xR_iz) \vee (yR_ix \ \& \ xP_iz) \vee (zP_ix \ \& \ xR_iy) \vee (zR_ix \ \& \ xP_iy)]$, and proper worst iff $[(yP_ix \ \& \ zR_ix) \vee (yR_ix \ \& \ zP_ix)]$.

It should be noted that in a triple, for an individual concerned over the triple and with reflexive, connected and transitive R_i over the triple, an alternative is best iff it is proper best, is medium iff it is proper medium and is worst iff it is proper worst. However, if R_i is intransitive over the triple, then an alternative can be best, medium, or worst without being proper best or proper medium or proper worst respectively. For instance, if R_i is (xP_iy, yI_iz, xI_iz) then z is best, medium and worst but is neither proper best, nor proper medium, nor proper worst.

Latin Square (LS): The set $\{R_i, R_j, R_k\}$ of individual weak preference relations over a triple $\{x, y, z\}$ forms a Latin Square iff R_i, R_j, R_k are concerned and there exist distinct $a, b, c \in \{x, y, z\}$ such that in R_i, a is best, b is medium and c is worst; in R_j, b is best, c is medium and a is worst; and in R_k, c is best, a is medium and b is worst.

The above Latin Square will be denoted by LS(*abca*).

Strict Latin Square (SLS): The set $\{R_i, R_j, R_k\}$ of individual weak preference relations over a triple $\{x, y, z\}$ forms a strict Latin Square iff there exist distinct $a, b, c \in \{x, y, z\}$ such that in R_i, a is best, b is proper medium and c is worst; in R_j, b is best, c is proper medium and a is worst; and in R_k, c is best, a is proper medium and b is worst.

The above strict Latin Square will be denoted by SLS(*abca*).

Strict Latin Square, as the name suggests, is a special case of Latin Square. Over a triple, if R_i, R_j, R_k are transitive then they form an LS iff they form an SLS. If individual weak preference relations are not necessarily transitive then it is possible that over a triple R_i, R_j, R_k form an LS but not an SLS. For example, $\{xP_iyP_iz, yP_jzP_jx, (zI_kx, xI_ky, zP_ky)\}$ forms an LS but not an SLS. It should be noted that R_i, R_j, R_k in the definitions of LS and SLS need not be distinct. For example, $\{R_i, R_j\}$ where $R_i = (xP_iy, yI_iz, xI_iz)$ and $R_j = (yP_jz, zI_jx, yI_jx)$ forms both LS(*xyzx*) and SLS(*xyzx*).

Now we define a restriction on individual preferences.

Strict Latin Square Unique Value Restriction (SLSUVR): *A* set Z of individual weak preference relations over a triple satisfies SLSUVR iff there does not exist an alternative belonging to the triple such that it is uniquely proper medium in some $R_i \in Z$, uniquely proper best in some $R_j \in Z$, uniquely proper worst in some $R_k \in Z$ and $\{R_i, R_j, R_k\}$ forms a strict Latin Square. More formally, a set Z of individual weak preference relations over $\{x, y, z\}$ satisfies SLSUVR iff $\sim[\exists a, b, c \in \{x, y, z\}$ and $R_i, R_j, R_k \in Z : [(aP_ib \ \& \ bP_ic \ \& \ aP_ic) \ \& \ (bP_jc \ \& \ cR_ja \ \& \ bR_ja) \ \& \ (cR_ka \ \& \ aP_kb \ \& \ cR_kb)]]$.

II. NECESSARY AND SUFFICIENT CONDITION FOR QUASI-TRANSITIVITY

LEMMA 1. *Let individual weak preference relations be reflexive, connected and quasi-transitive. Then NMR(S, L, p) violates quasi-transitivity for some configuration of individual weak preference relations iff*

$$[pN]+1<N.$$

Proof. Suppose $[pN]+1<N$. As $1/2\leq p<1$, it follows that $N\geq 3$. Let $x, y, z\in S$. Consider the following assignment of preferences:

$$N(xP_iyP_iz)=[pN]$$

$$N(yP_izP_ix)=N-([pN]+1)$$

$$N(zP_ixP_iy)=1$$

As $N(xP_iy)=[pN]+1$ and $N(yP_iz)=N-1\geq[pN]+1$, we obtain xPy and yPz. $N(xP_iz)=[pN]$ implies $\sim(xPz)$. Thus quasi-transitivity is violated.

Next suppose that $[pN]+1=N$. Suppose xPy and yPz, where $x, y, z\in S$.

$$xPy\rightarrow N(xP_iy)>pN$$

$$\rightarrow N(xP_iy)=N$$

Similarly, $yPz\rightarrow N(yP_iz)=N$.
$\forall i\in L$: $(xP_iy\wedge yP_iz)$ implies $\forall i\in L: xP_iz$ by quasi-transitivity of individual weak preference relations. So we must have xPz. Thus violation of quasi-transitivity is impossible. This establishes the lemma.

Remark 1. An equivalent way to state Lemma 1 is as follows: Let individual weak preference relations be reflexive, connected and quasi-transitive. Then NMR(S, L, p) yields quasi-transitive social weak preference relation for every configuration of individual weak preference relations iff it is identical to WPER(S, L).

LEMMA 2. *A set Z of individual reflexive, connected and quasi-transitive weak preference relations over a triple $\{x, y, z\}$ violates strict Latin Square unique value restriction iff Z contains one of the following* 9 *sets of weak preference relations, except for a formal interchange of alternatives*:

A.	1.	xP_iyP_iz	*B.*	1.	xP_iyP_iz	*C.*	1.	xP_iyP_iz
	2.	yP_jzP_jx		2.	yP_jzP_jx		2.	yP_jzI_jx
	3.	zP_kxP_ky		3.	zI_kxP_ky		3.	zP_kxP_ky
D.	1.	xP_iyP_iz	*E.*	1.	xP_iyP_iz	*F.*	1.	xP_iyP_iz
	2.	yP_jzI_jx		2.	yP_jzP_jx		2.	yP_jzI_jx
	3.	zI_kxP_ky		3.	zI_kx, xP_ky, zI_ky		3.	zI_kx, xP_ky, zI_ky
G.	1.	xP_iyP_iz	*H.*	1.	xP_iyP_iz	*I.*	1.	xP_iyP_iz
	2.	yP_jz, zI_jx, yI_jx		2.	yP_jz, zI_jx, yI_jx		2.	yP_jz, zI_jx, yI_jx
	3.	zP_kxP_ky		3.	zI_kxP_ky		3.	zI_kx, xP_ky, zI_ky

Proof. By definition, Z violates SLSUVR over $\{x, y, z\}$ iff there exist $a, b, c\in\{x, y, z\}$ and R_i, R_j, $R_k\in Z$ such that $(aP_ib$ & bP_ic & $aP_ic)$, $(bP_jc$ & cR_ja

& bR_ja) and (cR_ka & aP_kb & cR_kb). As individual weak preference relations are quasi-transitive, (bP_jc & cR_ja & bR_ja) is equivalent to $[bP_jcP_ja \vee bP_jcI_ja \vee (bP_jc, cI_ja, bI_ja)]$ and (cR_ka & aP_kb & cR_kb) is equivalent to $[cP_kaP_kb \vee cI_kaP_kb \vee (cI_ka, aP_kb, cI_kb)]$. By taking assignment $(a, b, c) = (x, y, z)$ one obtains the nine sets of the lemma.

Remark 2. If individual weak preference relations are reflexive, connected and transitive, then a set Z of individual weak preference relations over a triple $\{x, y, z\}$ violates SLSUVR iff Z contains one of the four sets A–D of Lemma 2, except for a formal interchange of alternatives.

THEOREM. *Given that individual weak preference relations are reflexive, connected and quasi-transitive, for every non-minority rule which is different from weak Pareto-extension rule, a necessary and sufficient condition for quasi-transitivity of the social weak preference relation R is that the strict Latin Square unique value restriction holds over every triple of alternatives.*

Proof. Suppose quasi-transitivity is violated. Then for some $x, y, z \in S$ we must have xPy, yPz and zRx.

$$xPy \rightarrow N(xP_iy) > pN \tag{1}$$

$$yPz \rightarrow N(yP_iz) > pN \tag{2}$$

$$zRx \rightarrow N(xP_iz) \leq pN$$
$$\rightarrow N(zR_ix) \geq (1-p)N \tag{3}$$

(1) & (2) $\rightarrow \exists i \in L: (xP_iy \;\&\; yP_iz)$, as $1/2 \leq p < 1$

$$\rightarrow \exists i \in L: (xP_iyP_iz), \text{ by quasi-transitivity of } R_i \tag{4}$$

(2) & (3) $\rightarrow \exists j \in L: (yP_jz \;\&\; zR_jx)$

$$\rightarrow \exists j \in L: [yP_jzP_jx \vee yP_jzI_jx \vee (yP_jz, zI_jx, yI_jx)], \tag{5}$$

by quasi-transitivity of R_j.

(3) & (1) $\rightarrow \exists k \in L: (zR_kx \;\&\; xP_ky)$

$$\rightarrow \exists k \in L: [zP_kxP_ky \vee zI_kxP_ky \vee (zI_kx, xP_ky, zI_ky)], \tag{6}$$

by quasi-transitivity of R_k.

In R_i, x is best, y proper medium and z worst; in R_j, y is best, z proper medium and x worst; and in R_k, z is best, x proper medium and y worst. Thus the set of individual weak preference relations $\{R_i, R_j, R_k\}$ forms SLS($xyzx$). Furthermore, y is uniquely proper medium in R_i, uniquely proper best in R_j and uniquely proper worst in R_k. Thus (4), (5) and (6) imply that SLSUVR is violated. This proves that the violation of quasi-transitivity implies the violation of SLSUVR, i.e., SLSUVR is sufficient for quasi-transitivity.

SLSUVR is violated iff the set of individual weak preference relations contains one of the sets A–I of Lemma 2, except for a formal interchange of alternatives. Therefore for proving the necessity of SLSUVR, it suffices to show that for every non-minority rule which is different from weak Pareto-extension rule and for each set A–I there exists an assignment of individuals which results in violation of quasi-transitivity.

Let f be any non-minority rule different from weak Pareto-extension rule. So by Lemma 1 and Remark 1, $[pN]+1<N$; consequently, $N\geq 3$ and $[pN]\geq 1$. For each set A through I, take $N_1=[pN]$, $N_2=N-([pN]+1)$ and $N_3=1$. Then for each set A through I, we have $N(xP_iy)=[pN]+1$, $N(yP_iz)=N-1\geq[pN]+1$, and $N(xP_iz)=[pN]$. This results, for each set, in xPy & yPz & $\sim(xPz)$, which violates quasi-transitivity.

III. CONCLUDING REMARKS

As transitivity is a special case of quasi-transitivity, it follows that the above theorem is valid also for the case when the individual weak preference relations are assumed to be orderings (reflexive, connected and transitive). Of course, as pointed out in Remark 2, when individual weak preference relations are orderings, SLSUVR is violated if and only if the set of individual orderings contains one of the sets A–D of Lemma 2, except for a formal interchange of alternatives. Under the assumption that individual weak preference relations are orderings, transitivity and quasi-transitivity under the class of non-minority rules have been discussed in [7]. The degenerate cases where non-minority rules coincide with weak Pareto-extension rules have been implicitly ruled out in [7]. It is shown in the paper that a necessary and sufficient condition for quasi-transitivity under any non-minority rule which is different from weak Pareto-extension rule is that the set of individual orderings satisfies at least one of the three conditions of value restriction (VR), weakly conflictive preferences (WCP) and unique value restriction (UVR) over every triple of alternatives. Therefore it follows that, given that individual weak preference relations are orderings, strict Latin Square unique value restriction is logically equivalent to the union of value-restriction, weakly conflictive preferences and unique value restriction. The logical equivalence of SLSUVR and (VR $\vee$ WCP $\vee$ UVR) can also be established directly. The proof is quite straightforward.

There are two distinct advantages in stating the quasi-transitivity theorem for non-minority rules in terms of SLSUVR condition. First, as has been argued in [8], stating the theorem in terms of a single condition results not only in gain in clarity but also in considerable simplification of proof, particularly of the necessity part. Secondly, it becomes much easier to compare conditions for different classes of social decision rules. In [8] and [9], it has been shown that strict Latin Square partial agreement (SLSPA) is necessary and sufficient for quasi-transitivity under the method of majority decision and under the class of special majority rules. In [8], SLSPA condition has been given a partial unanimity interpretation. SLSUVR can also be given partial unanimity interpretation. The kind of partial unanimity required by SLSUVR, of course, differs from the kind required by SLSPA. SLSUVR prohibits disagreement of the kind in which a particular alternative is regarded as uniquely proper medium by some one, uniquely proper best by someone else and uniquely proper worst by another, where these individuals' weak preference relations belong to the same strict Latin Square.

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REFERENCES

[1] Arrow, K. J., Social Choice and Individual Values, IInd Edition, Wiley, New York, 1963.
[2] Dummett, M. and R. Farquharson, "Stability in Voting", *Econometrica*, 29, 1961, 33–43.
[3] Fine, K., "Conditions for the Existence of Cycles under Majority and Non-minority Rules", *Econometrica*, 41, 1973, 889–899.
[4] Fishburn, P. C. "Conditions for Simple Majority Decision Functions with Intransitive Individual Indifference", *Journal of Economic Theory*, 2, 1970, 354–67.
[5] Inada, K., "On the Simple Majority Decision Rule", *Econometrica*, 37, 1969, 490–506.
[6] Inada, K., "Majority Rule and Rationality", *Journal of Economic Theory*, 2, 1970, 27–40.
[7] Jain, S. K., "Non-minority Rules: Characterisation of Configurations with Rational Social Preferences", *Keio Economic Studies*, 21, 1984, 45–54.
[8] Jain, S. K., "A Direct Proof of the Inada-Sen-Pattanaik Theorem on Majority Rule", *The Economic Studies Quarterly*, 36, 1985, 209–215.
[9] Jain, S. K., "Special Majority Rules: A Necessary and Sufficient Condition for Quasi-Transitivity with Quasi-Transitive Individual Preferences", *Social Choice and Welfare*, 3, 1986, 99–106.
[10] Pattanaik, P. K., "On Social Choice with Quasi-transitive Individual Preferences", *Journal of Economic Theory*, 2, 1970, 267–275.
[11] Pattanaik, P. K., *Voting and Collective Choice*, Cambridge University Press, London, 1971.
[12] Salles, M., "Characterisation of Transitive Individual Preferences for Quasi-Transitive Collective Preference under Simple Games", *International Economic Review*, 17, 1976, 308–318.
[13] Sen, A. K., *Collective Choice and Social Welfare*, Holden-Day, San Francisco, 1970.
[14] Sen, A. K. and P. K. Pattanaik, "Necessary and Sufficient Conditions for Rational Choice Under Majority Decision, *Journal of Economic Theory*, 1, 1969, 178–202.