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COLLECTIVE CHOICE RULES AND BARGAINING SOLUTIONS*

Takako FUJIWARA

Abstract. The Nash solution to bargaining problems is shown to induce a collective choice rule generating fully rational choice functions. This property is not shared by other solutions, including the Kalai-Smorodinsky, Gauthier, utilitarian, and egalitarian solution. An axiom of bargaining problems, Invariance under Affine Transformations of Utility, turns out to be crucially important for constructing a collective choice rule based on a bargaining solution. The Nash collective choice rule, however, violates the condition of Independence of Irrelevant Alternatives.

1. INTRODUCTION

Nash (1950) characterized a solution, now called the Nash solution, to bargaining problems with four axioms: Pareto Optimality, Invariance under Affine Transformations of Utility, Symmetry, and Independence Axiom. In the present paper, the last axiom will be referred to as Nash's Axiom. Some other solutions have been proposed by different authors: the Kalai-Smorodinsky solution (Kalai & Smorodinsky (1975)), the Gautheir solution (Gauthier (1985)), the utilitarian solution, the egalitarian solution, and so forth.

Based on the Nash solution to bargaining problems, Kaneko & Nakamura (1979) constructed the Nash social welfare function and gave a complete characterization of it. One may be inclined to conjecture that each of other solutions to bargaining problems may be employed to construct a social welfare function, as the Nash solution was employed to construct the Nash social welfare function. It will be shown that this conjecture is not true.

In view of the fact that, in most cases, social welfare functions can be expressed in terms of collective choice rules, we shall work exclusively with collective choice rules instead of social welfare functions. Our main conclusion is that the Nash solution induces a collective choice rule generating a rational choice function, while the other four solutions cited above do not. (Theorems 5.2–3.) Meanwhile we will show that a bargaining solution can be regarded as a choice function on some special choice space so that the properties of a solution are directly reflected in the social choice function induced by it. (Theorem 5.1.)

Finally, we clarify why Arrow's impossibility result does not apply to the Nash collective choice rule. (Theorem 5.4.)

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2. COLLECTIVE CHOICE RULES

Throughout this paper, we shall consider a society with n individuals, where n is a fixed integer not less than two. Put $N=\{1,2,\cdots,n\}$. A *choice space* is the ordered pair $(Z,\boldsymbol{H})$ of a nonempty set Z representing the set of conceivable states and a nonempty family $\boldsymbol{H}$ of nonempty subsets of Z, representing the set of conceivable opportunities or environments. A *choice function on* a choice space $(Z,\boldsymbol{H})$ is a function C of $\boldsymbol{H}$ into $\boldsymbol{P}(Z)-\{\phi\}$ such that $C(H)\subseteq H$ for every $H\in\boldsymbol{H}$, where $\boldsymbol{P}(Z)$ stands for the power set of Z.

Two rationality concepts for choice functions will be considered in the present paper. A choice function C on $(Z,\boldsymbol{H})$ is said to satisfy *Nash's Axiom* if $C(H)=C(H')$ for every $H,H'\in\boldsymbol{H}$ such that $H\subseteq H'$ and $C(H')\subseteq H$. To define the other kind of rationality, denote by $\Omega(Z)$ the set of complete, reflexive, and transitive binary relations on Z for each choice space $(Z,\boldsymbol{H})$. A choice function C on $(Z,\boldsymbol{H})$ is said to be *fully rational* if there is an element Q of $\Omega(Z)$ such that for every $H\in\boldsymbol{H}$, $C(H)=\{x\in H \mid xQy \text{ for every } y\in H\}$, and then Q is called a *rationalization* of C. The following relationship is known between these two rationality concepts.

LEMMA 2.1 *Every fully rational single-valued choice function satisfies Nash's Axiom.*

Proof. See Suzumura (1983, Theorems 2.1, 2.12, and A(3) on p. 26, p. 42, and p. 48.). Q.E.D.

Social choice theory considers a rule for defining a choice function based on preference profiles of the members of the society. Given a choice space $(Z,\boldsymbol{H})$, each individual's preference is assumed to be an element of $\Omega(Z)$. A *collective choice rule on* a choice space $(Z,\boldsymbol{H})$ is a function F whose domain, dom F, is a subset of $(\Omega(Z))^n$ and whose range is included in the set of fully rational choice functions on $(Z,\boldsymbol{H})$. We are interested in the following five conditions for a collective choice rule.

DEFINITION 2.1. Let $(Z,\boldsymbol{H})$ be a choice space. A collective choice rule F on $(Z,\boldsymbol{H})$ is said to satisfy

Unrestricted Domain if dom $F=(\Omega(Z))^n$,

Full Rationality if $F(R)$ is fully rational for every $R\in\operatorname{dom} F$,

Unanimity if $y\in C_R(H)$ for every $R\in\operatorname{dom} F$, every $x,y\in Z$, and every $H\in\boldsymbol{H}$ such that $x\in H$ and $xP(R_i)y$ for every $i\in N$, where $C_R=F(R)$ and $P(R_i)$ stands for the asymmetric part of R_i,

Independence of Irrelevant Alternatives if $C_R(H)=C_Q(H)$ for every $R, Q\in\operatorname{dom} F$, and every $H\notin\boldsymbol{H}$ such that xR_iy if and only if xQ_iy for every $i\in N$ and every $x,y\in H$, where $C_R=F(R)$ and $C_Q=F(Q)$.

A choice space $(Z,\boldsymbol{H})$ is said to *admit binary choices* if $\boldsymbol{H}$ contains every two-element subset of Z.

DEFINITION 2.2. A collective choice rule F on a choice space $(Z,\boldsymbol{H})$ admitting binary choices is said to satisfy *Nondictatorship* if there is no $i\in N$ such that $\{x\}=C_R(\{x, y\})$ for every $R\in \operatorname{dom} F$, every $x, y\in Z$ such that $xP(R_i)y$, where $C_R=F(R)$.

Arrow (1963) proved that these five conditions are inconsistent as a whole if $\sharp Z\geqq 3$ and the choice space admits binary choices. (cf. Suzumura (1983), Theorem 3.4 on p. 76).

3. BARGAINING PROBLEMS AND SOLUTIONS

A *bargaining problem* is the ordered pair (S,d) of a compact convex subset S (called the set of feasible utility allocations) of $\boldsymbol{R}^n$ and a point d (called the disagreement point) of S such that $a>d$ for some $a\in S$, where $\boldsymbol{R}$ denote the set of all real numbers, and $a>d$ stands for $a_i>d_i$ for every $i\in N$. The set S is interpreted as the set of utilities which can be attained by a joint action of the members of N and the point d as the utility vector the individuals obtain when no agreement is reached. Denote by B^n the set of bargaining problems. A *solution* is a function f of a subset of B^n into $\boldsymbol{R}^n$ such that $f(S,d)\in S$ for every $(S,d)\in\operatorname{dom}f$. Given a subset Δ of B^n, a solution will be called a *solution on* Δ if $\operatorname{dom}f=\Delta$.

Since the appearance of the work by Nash (1950), many axioms a solution to bargaining problems should satisfy have been proposed, among which we shall be concerned with the following.

DEFINITION 3.1. A solution f is said to satisfy
Pareto Optimality if for every $(S,d)\in\operatorname{dom}f$ there is no $a\in S$ such that $a>f(S,d)$,
Symmetry if $f(\pi(S),\pi(d))$ for every $(S,d)\in\operatorname{dom}f$ and every permutation π on N, where $\pi(a)=(a_{\pi(1)},\dots,a_{\pi(n)})$ for each $a\in\mathbf{R}^n$ and $\pi(S)=\{\pi(a)\mid a\in S\}$,
Invariance under Affine Transformations of Utility if $f(T\cdot S,T\cdot d)=T(f(S,d))$ for every $(S,d)\in\operatorname{dom}f$ and every affine operator T on $\boldsymbol{R}^n$,
Nash's Axiom if $f(S,d)=f(S',d')$ for every (S,d), $(S',d')\in\operatorname{dom}f$ such that $d=d'$, $S\subseteq S'$, and $f(S',d')\in S$.

A solution f^N on B^n defined by $f^N(S,d)=\operatorname{argmax}_{s\in S}\prod_{i\in N}(s_i-d_i)$ for every $(S,d)\in B^n$ is called the *Nash solution*. Nash (1950) characterized it in terms of above four axioms in the spcial case $n=2$, and later, Roth (1979) extended the characterization to general cases. In the present paper, we shall consider four other solutions to be compared with the Nash solution.

Let $(S,d)\in B^n$. Define $b(S)\in\boldsymbol{R}^n$ by $b_i(S)=\max\{a\in R\mid \text{there is } c\in\boldsymbol{R}^{n-1} \text{ such that } (c_1,\cdots,c_{i-1},a,c_{i+1},\cdots,c_n)\in S\}$ for every $i\in N$. The i-th component of $b(S)$ is the maximal feasible utility in S of the i-th individual. The point $b(S)$ is called the *ideal point* of the bargaining problem (S,d). For each $x,y\in\boldsymbol{R}^n$, let $\boldsymbol{L}(x,y)$ denote the closed line segment joining x and y. A solution f^K defined by $f^K(S,d)=\max[\boldsymbol{L}(d,b(S))\cap S]$ for every $(S,d)\in B^n$ is called the *Kalai-Smorodinsky solution*, where $\max[\boldsymbol{L}(d,b(S))\cap S]$ denotes the maximal element of $\boldsymbol{L}(d,b(S))\cap S$ with respect to the usual partial ordering on $\boldsymbol{R}^n$ (see Kalai (1985) and Kalai and Smorodinsky (1975).) The Kalai-Smorodinsky solution is the maximal feasible element on the line segment joining the disagreement point and the ideal point. This solution is known to satisfy Pareto Optimality, Symmetry, and Invariance under Affine Transformations of Utility in the special case $n=2$, but not necessarily to satisfy Pareto Optimality when $n \geqq 3$. A solution f^G on B^n defined by

$$f^G(S,d) = \operatorname{argmax}_{s \in S} \min_{i \in N}(s_i - d_i)/(b_i(S) - d_i)$$

is called the *Gauthier solution* (see Gauthier (1985)). The Gauthier solution maximizes the minimum proportion of possible gain to any person. It is an extension of the Kalai-Smorodinsky solution for the purpose of satisfaction of Pareto Optimality even if $n > 2$.

Put $\Delta_C = \{(S,d) \in B^n \mid S \text{ is strictly convex}\}$. For each $\lambda \in \boldsymbol{R}^n{}_{++}$, a solution $f^{U\lambda}$ on Δ_C defined by $f^{U\lambda}(S,d) = \operatorname{argmax}_{s \in S} \sum_{i \in N} \lambda_i s_i$ for every $(S,d) \in \Delta_C$ is called the *utilitarian solution with weight* λ (see Kalai (1985)). For every $\lambda \in \boldsymbol{R}^n{}_{++}$, a solution $f^{E\lambda}$ on B^n defined by $f^{E\lambda}(S,d) = \max\{s \in S \mid \lambda_i(s_i - d_i) = \lambda_j(s_j - d_j) \text{ for every } i,j \in N\}$ is called the *egalitarian solution with weight* λ (see Kalai (1985)).

4. A SPECIAL CLASS OF BARGAINING PROBLEMS

To investigate the possibility of constructing a collective choice rule based on a solution to bargaining problems, it will turn out that we have to confine ourselves to a certain restricted class of bargaining problems. This section is devoted to defining the special subset of B^n in which we are interested.

Bargaining problems are expressed in terms of the choice of utilities, more specifically, von Neumann-Morgenstern utilities, while collective choice rules deal with the choice of states. We shall begin with filling this gap. Von Neumann-Morgenstern utility functions represent preferences on the space of lotteries over a nonempty finite set X of sure states. A lottery is a probability mixture of some sure states. In fact, a lottery which entitles the holder to x with probability $p(x)$ for every $x \in X$ can be specified by the probability measure p, where p is a function of X into $\boldsymbol{R}_+$ such that $\sum_{x \in X} p(x) = 1$.

Denote by L the set of functions p of X into $\boldsymbol{R}_+$ such that $\sum_{x \in X} p(x) = 1$. Since X is a finite set, each element p of L can be identified with a vector in the standard simplex of dimension $\sharp X - 1$. That is, the set L of lotteries can be regarded as the $(\sharp X - 1)$-dimensional standard simplex, which is a nonempty compact convex subset of $\sharp X$-dimensional Euclidian space. Each element x of X can be identified with the unit vector e_x defined by $e_x(x) = 1$ & $e_x(y) = 0$ for every $y \in X - \{x\}$. Thus X can be regarded as a subset of L. For each nonempty subset Y of X, denote by $L(Y)$ the set of elements p of L such that $\sum_{x \in Y} p(x) = 1$. Then every nonempty subset Y of X can be regarded as a subset of $L(Y)$, which can in turn be regarded as the $(\sharp Y - 1)$-dimensional standard simplex.

Given two lotteries p and q, let $tp*(1-t)q$ stand for the lottery which entitles the holder to p with probability t and q with probability $1-t$. Clearly, the ordered pair $(L,*)$ is a mixture set in the sense of Herstein & Milnor (1953). Denote by Θ the set of complete, reflexive, and transitive binary relations Q on L satisfying the following two axioms:

(I) For every $p,q,r \in L$, both $\{t \in [0,1] \mid (tp*(1-t)q)Qr\}$ and $\{t \in [0,1] \mid rQ(tp*(1-t)q)\}$ are closed in $[0,1]$.

(II) For every $p,q,r \in L$, if $pI(Q)q$, then $({}^1/_2p*{}^1/_2r)I(Q)({}^1/_2q*{}^1/_2r)$, where $I(Q)$ is the symmetric part of Q.

For every $Q \in \Theta$, the set of *von Neumann-Morgenstern utility functions for* Q is defined by

$U(Q)=\{u:L\to R \mid \text{for every } p,q \in L, pQq \text{ if and only if } u(p) \geqq u(q)\}$. Herstein and Milnor (1953) proved that for every $Q \in \Theta$, (a) $U(Q)$ is not empty, (b) for every $u \in U(Q)$, $u(tp*(1-t)q)=tu(p)+(1-t)u(q)$, for every $p,q \in L$, and every $t \in [0,1]$, and (c) every element of $U(Q)$ is an affine transformation of every other element of $U(Q)$.

For every $R \in \Theta^n$, the set of *continuous utility profiles for* R is defined by

$U^C(R)=\{u:L\to \boldsymbol{R}^n \mid u \text{ is continuous and } u_i \in U(R_i) \text{ for every } i \in N\}$. Following Kaneko & Nakamura (1979), we choose an element x_0 of X to play the role of the disagreement point. Put

$$\boldsymbol{Y}=\{Y \in \boldsymbol{P}(X)-\{\phi\} \mid x_0 \in Y \text{ and } \# Y \geqq 2\},$$

and

$$\Xi=\{R \in \Theta^n \mid (1)\ pR_ie_{x0} \text{ for every } i \in N \text{ and every } p \in L, \text{ and } (2) \text{ for every } Y \in \boldsymbol{Y}, \text{ there is } q \in L(Y) \text{ such that } qP(R_i)e_{x0} \text{ for every } i \in N\}.$$

For every $u \in \cup\{U^C(R) \mid R \in \Theta^n\}$ and every nonempty subset Y of X, define $S(u,Y)=u(L(Y))$.

REMARK 4.1. *$(S(u,Y),u(e_{x0})) \in B^n$ for every $u \in \{U^C(R) \mid R \in \Xi\}$ and every nonempty subset Y of X.*

Proof. See Appendix.

Define $\Delta_B=\{(S(u,Y), u(e_{x0})) \mid Y \in \boldsymbol{Y} \text{ and } u \in U^C(R) \text{ for some } R \in \Xi\}$. This is the desired subset of B^n we shall be concerned with.

5. CONSTRUCTING A COLLECTIVE CHOICE RULE

When bargaining problems are restricted to Δ_B, the choice in a bargaining set $S(u,Y)$ can be converted into that in a space $L(Y)$ of lotteries so that one can construct a choice function on $L(Y)$ based on a bargaining solution. Some care should be taken, however, in regarding the choice function as specified by a collective choice rule. A collective choice rule should, by definition, generate a choice function depending solely on preference profiles but not on particular choices of utility profiles.

REMARK 5.1. *For every* $\lambda \in \boldsymbol{R}^n{}_{++}$, *neither the utilitarian solution nor the egalitarian solution with weight* λ *satisfy Invariance under Affine Transformations of Utility.*

Proof. Obvious.

Define $\boldsymbol{H}_0 = \{L(Y) \mid Y \in \boldsymbol{Y}\}$. For every solution f satisfying Invariance under Affine Transformations of Utility, and every $(R,H) \in \Xi \times \boldsymbol{H}_0$ define $C^f{}_R(H) = u^{-1}(f(S(u,L^{-1}(H)),u(e_{x0}))$, where $u \in U^C(R)$. We call $C^f{}_R$ the *choice function induced by f at R*. The following remark guarantees that $C^f{}_R$ is unambiguously defined.

REMARK 5.2. *For every* $R \in \Xi$, *every solution f satisfying Invariance under Affine Transformations of Utility, and every* $Y \in Y$, $u^{-1}(f(S(u,Y),u(e_{x0}))) = v^{-1}(f(S(v,Y),v(e_{x0})))$ *for every* $u,v \in U^C(R)$. *Moreover,* $C^f{}_R$ *is a choice function on* (L,H_0) *for every* $R \in \Xi$.

Proof. See Appendix.

Given a choice space $(L,\boldsymbol{H})$, admitting binary choices, such that $\boldsymbol{H}_0 \subseteq \boldsymbol{H}$ and a collective choice rule F on $(L,\boldsymbol{H})$, a solution f satisfying Invariance under Affine Transformations of Utility is said to *partially induce* F if, for every $R \in \Xi$, $F(R)$ is an extension of the choice function $C^f{}_R$ induced by f at R. This is the way we combine bargaining solutions and collective choice rules.

Put $\boldsymbol{H}_S = \{S \mid (S,d) \in \Delta_B \text{ for some } d \in \boldsymbol{R}^n\}$. Then a solution on Δ_B can be regarded as a choice function on the choice space $(\boldsymbol{R}^n,\boldsymbol{H}_S)$. We can characterize the full rationality of a bargaining solution f as a choice function on the choice space $(\boldsymbol{R}^n,\boldsymbol{H}_S)$ with that of the choice function $C^f{}_R$ on $(L,\boldsymbol{H}_0)$ at $R \in \Xi$.

THEOREM 5.1. *For every solution satisfying Invariance under Affine Transformations of Utility, f, regarded as a choice function on* $(R^n,\boldsymbol{H}_S)$, *is fully rational if and only if for every* $R \in \Xi$, *the choice function* $C^f{}_R$ *on* (L,H_0) *induced by f at R is fully rational.*

Proof. See Appendix.

Note that a solutionf f on Δ_B satisfies Nash's Axiom if and only if f, regarded as a choice function on the choice space $(\boldsymbol{R}^n,\boldsymbol{H}_S)$ satisfies Nash's Axiom. Hence, by Lemma 2.1, no solution on Δ_B which violates Nash's Axiom induces a collective choice rule. (Notice that collective choice rules, by definition, generates fully rational choice functions.) The Kalai-Smorodinsky solution and its extension, the Gauthier solution, belong to this category. For completeness, we prove it formally.

REMARK 5.3. *The Kalai-Smorodinsky solution and the Gauthier solution do not* *satisfy Nash's Axiom on* Δ_B *so that they, regarded as choice functions on* $(\boldsymbol{R}^n, \boldsymbol{H}_S)$, *fail to satisfy Nash's Axiom of choice functions.*

Proof. See Appendix.

THEOREM 5.2. *The Kalai-Smorodinsky solution, the Gauthier solution, the utilitarian solution, and the egalitarian solution do not induce a collective choice rule.*

Proof. From Lemma 2.1, Remarks 5.1. and 5.3, and Theorem 5.1. Q.E.D.

We shall now show that the Nash solution does induced a collective choice rule. The *Nash social welfare function* Q^N is defined by

$$Q^N(R) = \{(p,q) \in L \times L \mid \prod_{i \in N}(u_i(p) - u_i(e_{x0})) \geqq \prod_{i \in N}(u_i(q) - u_i(e_{x0}))\}$$

for each $R \in \Theta^n$, where $u_i \in U(R_i)$ for every $i \in N$. Note that this does not depend on the particular choice of U. For each choie space $(L, \boldsymbol{H})$, admitting binary choices, such that $\boldsymbol{H}_0 \subseteq \boldsymbol{H}$, the Nash collective choice rule F^N on $(L, \boldsymbol{H})$ is defined by

$$F^N(R) = C^N{}_R \qquad \text{for every} \quad R \in \Xi,$$

where, $C^N{}_R(H) = \{p \in H \mid pQ^N(R)q \text{ for every } q \in H\}$ for every $H \in \boldsymbol{H}$.

THEOREM 5.3. *For every choice space* $(L, \boldsymbol{H})$, *admitting binary choices, such that* $\boldsymbol{H}_0 \subseteq \boldsymbol{H}$, *the Nash solution* f^N *on* Δ_B *partially induces the Nash collective choice rule* F^N *on* $(L, \boldsymbol{H})$. *That is,* $C^N{}_R(H) = u^{-1}(f^N(S(u, L^{-1}(H)), u(e_{x0})))$ *for every* $(R, H) \in \Xi \times \boldsymbol{H}_0$, *where* $u \in U^C(R)$.

Proof. See Appendix.

Finally, we inquire how the Nash collective choice rule escapes Arrow's impossibility. The condition of Unrestricted Domain is obviously violated, and Independence of Irrelevant Alternatives is also violated, as we shall see below. In general, we shall say that a collective choice rule F on $(L, \boldsymbol{H})$ satisfies the *Von Neumann-Morgenstern Domain* if $\operatorname{dom} F = \Xi$. We have the following theorem.

THEOREM 5.4. *For every choice space* $(L, \boldsymbol{H})$, *admitting binary choices, the Nash collective choice rule* F^N *on* $(L, \boldsymbol{H})$ *satisfies Von Neumann-Morgenstern Domain, Full Rationality, Unanimity, and Nondictatorship but does not satisfy Independence of Irrelevant Alternatives.*

Proof. See Appendix.

The condition of Independence of Irrelevant Alternatives is appropriate for ordinal preferences and too weak for cardinal preferences, as can be seen from the argument by Kalai & Schmeidler (1977), which proved a cardinal impossibility with a finite set of alternatives. The Nash collective choice rule violates *a fortiori* Kalai-Schmeidler's Cardinal Independence of Irrelevant Alternatives, although it is still an open question if their impossibility theorem can be extended in the case of infinite alternatives. Hence their impossibility theorem either does not apply to this collective choice rule even if it is extended.

6. CONCLUDING REMARKS

Theorem 5.1. and Lemma 2.1. indicate that Nash's Axiom of bargaining solutions is closely related to the full rationality of the choice functions induced by them. Throughout the present paper, we have used the term *collective choice rule* in the narrow sense to mean a rule that generates fully rational choice functions. If we use the term in a broader sense to include a rule that generates a choice function which is not necessarily fully rational, then we can say that the Kalai-Smorodinsky slution and the Gauthier solution induce collective choice rules which satisfy Unanimity (as for the Kalai-Smorodinsky solution, only when $n=2$) and Nondicatatorship.

But, for two reasons stated below, we abstain from stating that the Kalai-Smorodinsky solution or the Gauthier solution induces a collective choice rule. One is that these solutions cannot be evaluated as good as the Nash solution from the viewpoint of social choice. In fact, they violate Full Rationality in addition to Unrestricted domain and Independence of Irrelevant Alternatives.

The second reason is more fundamental and related to the admissibility of binary choices. In general, social welfare functions and collective choice rules are related via a rationalization or the base relation only when the choice space admits binary choices. The proof of Arrow's impossibility theorem for collective choice rules is crucially dependent on binary choices. (See Suzumura (1983).) Collective choice rules partially induced by the Kalai-Smorodinsky or the Gauthier solution, however, cannot have a choice space which admits binary choices, because the choice functions induced by them are defined only on some sets of infinite lotteries. Then the definition of our Nondictatorship and, therefore, the impossibility theorem itself turn out to be irrelevant to such collective choice rules.

The main concern of the present paper has been the possibility of constructing a collective choice rule, based on a solution to bargaining problems, to which Arrow's impossibility theorem does not apply. So we have not dealt with collective choice rules induced by irrational solutions.

As was mentioned, the restriction of a choice space into $(L,\boldsymbol{H}_0)$ obscures the relationship between social welfare functions and collective choice rules but, technically speaking, there remains a question if one really cannot replace the choice space admitting binary choices by $(L,\boldsymbol{H}_0)$ in the impossibility theorem.

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APPENDIX

Proof of Remark 4.1. Clearly $S(u,Y)\subseteq \boldsymbol{R}^n$, $u(e_{x0})\in S(u,Y)$, and as noted,

$S(u,Y)$ is compact. Since $R \in \Xi$, there is $q \in L(Y)$ such that for every $i \in N$, $qP(R_i)e_{x0}$. Hence there is $u(q) \in S(u,Y)$ such that $u(q) > u(e_{x0})$. Take any $a,b \in S(u,Y)$ and any $t \in [0,1]$. Since $a, b \in S(u,Y)$, there are $p,q \in L(Y)$ such that $u(p)=a$ and $u(q)=b$.

Then $ta+(1-t)b = tu(p)+(1-t)u(q) = u(tp*(1-t)q)$. $(L(Y),*)$ is also a mixture set so that $tp*(1-t)q \in L(Y)$. Therefore $u(tp*(1-t)q) \in S(u,Y)$. Q.E.D.

Proof of Remark 5.2. Since $u,v \in U^C(R)$, there is an affine operator T such that $u = T \cdot v$. By Invariance under Affine Transformations of Utility,

$$\begin{aligned}
u^{-1}(f(S(u,Y),u(e_{x0}))) &= (T\cdot v)^{-1}[f(S(T\cdot v,Y),T\cdot v(e_{x0}))] \\
&= (T\cdot v)^{-1}[f(T\cdot v(L(Y)),T\cdot v(e_{x0}))] \\
&= v^{-1}\cdot T^{-1}[f(T\cdot S(v,Y),T\cdot v(e_{x0}))] \\
&= v^{-1}\cdot T^{-1}[T(f(S(v,Y),v(e_{x0})))] \\
&= v^{-1}(f(S(v,Y),v(e_{x0}))).
\end{aligned}$$

To assure that $C^f{}_R$ is a choice function on $(L,\boldsymbol{H}_0)$, take any $H \in \boldsymbol{H}_0$. Then, since $f(S(u,L^{-1}(H)),u(e_{x0}))$ is not empty, $C^f{}_R(H) \neq \phi$. Clearly, $C^f{}_R(H) \subseteq H$ for every $H \in \boldsymbol{H}_0$. Q.E.D.

Proof of Theorem 5.1. Suppose f is fully rational. Fix any $R \in \Xi$, $H \in \boldsymbol{H}_0$, and $u \in U^C(R)$. Then there is a binary relation Q on $\boldsymbol{R}^n$ such that $f(S(u,L^{-1}(H)),u(e_{x0}))Qa$ for every $a \in S(u,L^{-1}(H))$. Take any $x^* \in C^f{}_R(H)$. Then $u(x^*)Qa$ for every $a \in S(u,L^{-1}(H))$. Define a binary relation Q^* on L by

$$\begin{aligned}
Q^* = \{(x,y) \in L \times L \mid &(1)\ (x,y) \in H^2 \ \text{ and } \ u(x)Qu(y), \\
&(2)\ x \in H \ \text{ and } \ y \in L-H, \\
&(3)\ (x,y) \in [L-H]^2\}.
\end{aligned}$$

Since for every $y \in H$, $u(y) \in S(u,L^{-1}(H))$, $u(x^*)Qu(y)$ for every $y \in H$. Therefore x^*Q^*y for every $y \in H$. Furthermore, if x^*Qy for every $y \in H$, then by the definition of Q^*, $u(x^*)Qa$ for every $a \in S(u, L^{-1}(H))$ so that $u(x^*) \in f(S(u,L^{-1}(H)), u(e_{x0}))$. Hence $x^* \in C^f{}_R(H)$.

Conversely, suppose $C^f{}_R$ is fully rational and let Q^* be its rationalization. Define Q by

$$\begin{aligned}
Q = \{(p,q) \in [\boldsymbol{R}^n]^2 \mid &(1)\ (p,q) \in [S(u,Y)]^2 \text{ and there are } x \in u^{-1}(p) \text{ and} \\
&\quad y \in u^{-1}(q) \text{ such that } xQ^*y, \\
&(2)\ p \in S(u,Y) \text{ and } q \in \boldsymbol{R}^n - S(u,Y), \\
&(3)\ (p,q) \in [\boldsymbol{R}^n - S(u,Y)]^2\}.
\end{aligned}$$

Then Q is a binary relation on $[\boldsymbol{R}^n]^2$. Denote $p^* = f(S(u,Y),u(e_{x0}))$, then $u^{-1}(p^*) = C^f{}_R(L(Y))$. Take any $x^* \in C^f{}_R(L(Y))$. Since x^*Q^*y for every $y \in L(Y)$, p^*Qq for every $q \in S(u,Y)$. Q.E.D

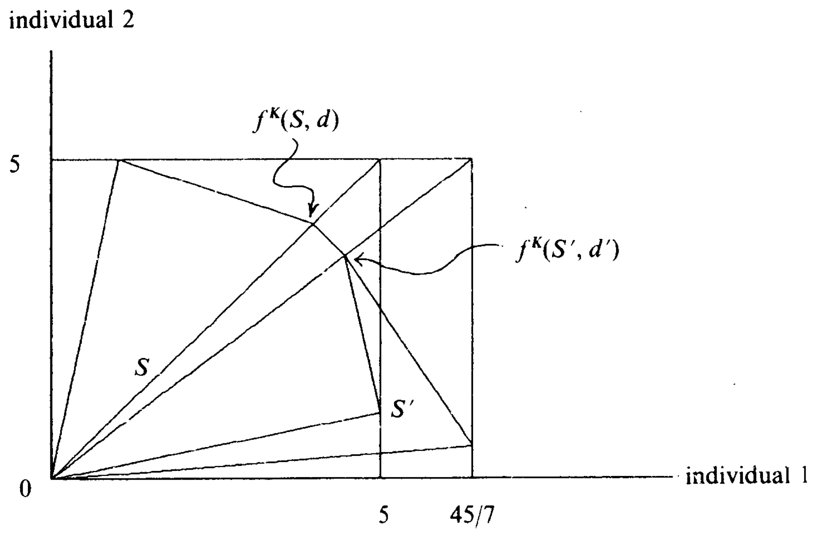

Fig. 1.

Proof of Remark 5.3. Let $n=2$. Note that then the Kalai-Smorodinsky solution coincides with the Gauthier solution. Construct (S,d) and $(S',D')\in\Delta_B$ as follows. Let $Y=\{x_0,x_1,x_2,x_3,x_4\}$. There are (R_1,R_2) and $(Q_1,Q_2)\in\Xi$ such that e_{x4} $P(R_1)$ e_{x3} $P(R_1)$ e_{x2} $P(R_1)$ e_{x1} $P(R_1)$ e_{x0}, e_{x1} $P(R_2)$ e_{x2} $P(R_2)$ e_{x3} $P(R_2)$ e_{x4} $P(R_2)$ e_{x0}, and $R_i\cap[Y]^2=Q_i\cap[Y]^2$ for every $i\in\{1,2\}$. There is $u\in U^C(R)$ such that $u_1(e_{x4})=5$, $u_1(e_{x3})=9/2$, $u_1(e_{x2})=4$, $u_1(e_{x1})=1$, $u_1(e_{x0})=0$, $u_2(e_{x4})=1$, $u_2(e_{x3})=7/2$, $u_2(e_{x2})=4$, $u_2(e_{x1})=5$, and $u_2(e_{x0})=0$. Define $S=S(u,Y)$, then S is the convex hull of the points (0,0), (1,5), (4,4), (9/2,7/2), (5,1).

There is $v\in U^C(Q)$ such that $v_1(e_{x4})=45/7$, $v_1(e_{xt})=u_1(e_{xt})$ for every $t\in\{0,1,2,3\}$, $v_2(e_{x4})=1/2$, $v_2(e_{xt})=u_2(e_{xt})$ for every $t\in\{0,1,2,3\}$. Define $S'=S(v,Y)$, then S' is the convex hull of the points (0,0), (1,5), (4,4), (9/2,7/2), (45/7,1/2). Define $d=u(e_{x0})$ and $d'=v(e_{x0})$. Then (S,d), $(S',d')\in\Delta_B$, $d=d'$, $S\subseteq S'$, and $f^K(S',d')=f^G(S',d')=(9/2,7/2)\in S$. But $f^K(S,d)=f^G(S,d)=(4,4)\neq f^K(S',d')$. (See Figure 1.) Q.E.D.

Proof of Theorem 5.3. Let $u\in U^C(R)$ and $p\in C^N{}_R(H)$. Then $p\in H$ and $\Pi_{i\in N}(u_i(p)-u_i(e_{x0}))\geqq\Pi_{i\in N}(u_i(q)-u_i(e_{x0}))$ for every $q\in H$. Hence $u(p)\in f^N(S(u,L^{-1}(H)),u(e_{x0}))$, i.e. $p\in u^{-1}(f^N(S(u,L^{-1}(H)),u(e_{x0})))$ so that $C^N{}_R(H)\subseteq u^{-1}(f^N(S(u,L^{-1}(H)),u(e_{x0})))$. The converse inclusion follows similarly. Q.E.D.

Proof of Theorem 5.4. Von Neumann-Morgenstern Domain, Full Rationality, Unanimity, and Nondictatorship are obviously satisfied. There are $R,Q\in\Xi$, $p,q\in L$, $j\in N$, $u\in\{U(R_i)\}^n$, and $v\in\{U(Q_i)\}^n$ such that $R_i=Q_i$ for every $i\in N-\{j\}$, $R_j\cap\{p,q\}^2=Q_j\cap\{p,q\}^2$, $u_i=v_i$, $u_i(p)=3$, $u_i(q)=1$, and $u_i(e_{x0})=0$ for every $i\in N-\{j\}$, $u_j(p)=1$, $u_j=3$, $u_j(e_{x0})=0$, $v_j(r)=3^n u_j(r)$ for every $r\in L(\{q,q\})$, and $v_j(e_{x0})=3^n-1$. (Note that R_j and Q_j can be different on $L-\{p,q\}$ so that u_j and v_j can be chosen affinely nontransferable on L.)

Then $\Pi_{i\in N}(u_i(p)-u_i(e_{x0}))=3^{n-1}\times 1\geqq 1^{n-1}\times 3=\Pi_{i\in N}(u_i(q)-u_i(e_{x0}))$ so that $pQ^N(R)q$ and $p\in C^N{}_R(\{p,q\})$. (Note that $n\geqq 2$ and $\{p,q\}\in H$.) But

$$\Pi_{i\in N}(v_i(p)-v_i(e_{x0}))=3^{n-1}<2\times 3^n+1=\Pi_{i\in N}(v_i(q)-v_i(e_{x0}))$$

so that we have $qP(Q^N(R))p$ and $p\notin C^N{}_Q(\{p,q\})$. Therefore F^N does not satisfy

Independence of Irrelevant Alternatives. Q.E.D.

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