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NOTES ON THE STABILITY OF QUADRATIC GAMES

Ferenc SZIDAROVSZKY and Koji OKUGUCHI

Abstract: The stability of the equilibrium for quadratic games is examined under the assumption that each player forms adaptively expectations on other player's strategies and/or variables which are linear combinations of other players' strategies. In this paper a discrete time scale is assumed. As applications, the multiproduct oligopoly game is investigated under three different kinds of assumptions about the dynamic behaviour of the players.

1. INTRODUCTION

In this paper the stability of the equilibrium for an N-person noncooperative quadratic game is analysed under the assumption that each player forms adaptively expectations on other player's strategies and/or variables which are linear combinations of other players' strategies. A discrete time scale is assumed. Similar investigation assuming a continuous time scale has been performed by Szidarovszky and Okuguchi (1987). The structure of this paper is as follows. In Section 2 the general dynamic model is formulated, and stability conditions are derived. In Section 3 the implications of the stability conditions for a multiproduct oligopoly model are presented. Section 4 concludes.

2. THE MATHEMATICAL MODEL AND STABILITY CONDITIONS

In this section an N-person game

$$\Gamma=\{N;S_1,\cdots,S_N;\varphi_1,\cdots,\varphi_N\}$$

is examined, where

(A) For all k, the strategy set S_k of player k is a closed, convex, bounded subset of finite dimensional Euclidean space;

(B) For all k, the payoff function of player k is given as

$$\varphi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N)=\boldsymbol{x}^{(k)T}\begin{bmatrix} \boldsymbol{A}^{(k)}_{00} & \cdots & \boldsymbol{A}^{(k)}_{0n_k} \\ \vdots & & \vdots \\ \boldsymbol{A}^{(k)}_{n_k0} & \cdots & \boldsymbol{A}^{(k)}_{n_kn_k} \end{bmatrix}\boldsymbol{x}^{(k)}+\boldsymbol{b}^{(k)T}\boldsymbol{x}^{(k)}+c^{(k)}, \qquad (1)$$

where $x_k \in S_k$ $(\forall k)$,

$$x^{(k)} = \begin{bmatrix} x_k \\ s_{k1} \\ \vdots \\ s_{k,n_k} \end{bmatrix}, \qquad s_{kl} = \sum_{m \neq k} B_{lm}^{(k)} x_m \qquad (1 \leq l \leq n_k, 1 \leq k \leq N). \tag{2}$$

It is also assumed that matrix $A_{00}^{(k)} + A_{00}^{(k)T}$ is negative definite for $k = 1,2, \cdots, N$.

Note that payoff function (1) is a quadratic form of the strategy of player k and linear combinations of other players' strategies.

If a game is given in normal form, then φ_k is a function of all strategies $x_1, \cdots, x_k, \cdots, x_N$. In this case we set $x^{(k)} = (x_k^T, x_1^T, \cdots, x_{k-1}^T, x_{k+1}^T, \cdots, x_N^T)^T$, and therefore

$$B_{lm}^{(k)} = \begin{cases} I, & \text{if } l < k \text{ and } l = m, \text{ or } l \geq k \text{ and } l+1 = m, \\ 0 & \text{otherwise}. \end{cases}$$

Under assumptions (A) and (B) the quadratic game satisfies the conditions for the Nikaido-Isoda theorem (Nikaido and Isoda, 1955), therefore it has at least one equilibrium point.

First, we investigate a model in which no adaptive expectations are assumed. At time $t = 0$, let $x_k^{(0)}$ denote the strategy of player k. It is now assumed that at each time t, each player k maximises his own payoff value $\varphi_k(x_1^{(t)}, \cdots, x_{k-1}^{(t)}, x_k, x_{k+1}^{(t)}, \cdots, x_N^{(t)})$ as a function of x_k, and this optimal choice will be his strategy selection for the next time period $t+1$. By assuming that the optimal strategy is an interior point of S_k, the first order conditions for optimality imply that for all k,

$$(A_{00}^{(k)} + A_{00}^{(k)T}) x_k + \sum_{l=1}^{n_k} (A_{0l}^{(k)} + A_{l0}^{(k)T}) s_{kl}^{(t)} + b_0^{(k)} = 0,$$

that is,

$$x_k^{(t+1)} = -(A_{00}^{(k)} + A_{00}^{(k)T})^{-1} \sum_{l=1}^{n_k} (A_{0l}^{(k)} + A_{l0}^{(k)T}) s_{kl}^{(t)} + \alpha, \tag{3}$$

where α is a constant vector. By using the definition of vectors s_{kl}, this relation can be rewritten as

$$\begin{aligned} x_k^{(t+1)} &= -(A_{00}^{(k)} + A_{00}^{(k)T})^{-1} \sum_{l=1}^{n_k} (A_{0l}^{(k)} + A_{l0}^{(k)T}) \sum_{m \neq k} B_{lm}^{(k)} x_m^{(t)} + \alpha \\ &= -(A_{00}^{(k)} + A_{00}^{(k)T})^{-1} \sum_{m \neq k} \left(\sum_{l=1}^{n_k} (A_{0l}^{(k)} + A_{l0}^{(k)T}) B_{lm}^{(k)} \right) x_m^{(t)} + \alpha. \end{aligned}$$

By introducing the notations

$$H_{km} = -(A_{00}^{(k)} + A_{00}^{(k)T})^{-1} \sum_{l=1}^{n_k} (A_{0l}^{(k)} + A_{l0}^{(k)T}) B_{lm}^{(k)},$$

$$x^{(t)} = \begin{bmatrix} x_1^{(t)} \\ \vdots \\ x_N^{(t)} \end{bmatrix} \qquad H = \begin{bmatrix} 0 & H_{12} & \cdots & H_{1N} \\ H_{21} & 0 & \cdots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1} & H_{N2} & \cdots & 0 \end{bmatrix}$$

we have

$$x^{(t+1)} = Hx^{(t)} + \alpha. \tag{4}$$

Next, the adjustment mechanisms for players' strategies and expectations are introduced. Let s_{kl}^E be the kth player's expectations on s_{kl}. It is assumed that expectations are formed adaptively, that is,

$$s_k^{E(t+1)} = s_{kl}^{E(t)} + M_{kl}(s_{kl}^{(t)} - s_{kl}^{E(t)}). \tag{5}$$

It is also assumed that strategy $x_k^{(t+1)}$ of player k is obtained from equation (3), where $s_{kl}^{(t)}$ is replaced by his expectation $s_{kl}^{E(t+1)}$. Hence,

$$x_k^{(t+1)} = -(A_{00}^{(k)} + A_{00}^{(k)T})^{-1} \sum_{l=1}^{n_k} (A_{0l}^{(k)} + A_{l0}^{(k)T}) \cdot \left[\sum_{m \neq k} M_{kl} B_{lm}^{(k)} x_m^{(t)} + (I - M_{kl}) s_{kl}^{E(t)} \right] + \alpha,$$

which can now be rewritten in matrix form as

$$\begin{bmatrix} x_1^{(t+1)} \\ \vdots \\ x_N^{(t+1)} \\ s_{11}^{E(t+1)} \\ \vdots \\ s_{1,n_1}^{E(t+1)} \\ \vdots \\ s_{N1}^{E(t+1)} \\ \vdots \\ s_{N,n_N}^{E(t+1)} \end{bmatrix} = \begin{bmatrix} H_{00} & H_{01} & \cdots & H_{0N} \\ H_{10} & H_{11} & \cdots & H_{1N} \\ \vdots & \vdots & & \vdots \\ H_{N0} & H_{N1} & \cdots & H_{NN} \end{bmatrix} \begin{bmatrix} x_1^{(t)} \\ \vdots \\ x_N^{(t)} \\ s_{11}^{E(t)} \\ \vdots \\ s_{1,n_1}^{E(t)} \\ \vdots \\ s_{N1}^{E(t)} \\ \vdots \\ s_{N,n_N}^{E(t)} \end{bmatrix} + \alpha \tag{6}$$

where the blocks are as follows:

$$H_{00} = \begin{bmatrix} K_{11} & \cdots & K_{1N} \\ \vdots & & \vdots \\ K_{N1} & \cdots & K_{NN} \end{bmatrix}$$

with

$$K_{km}=\begin{cases}\mathbf{0}, & \text{if } m=k\\ -(A_{00}^{(k)}+A_{00}^{(k)T})^{-1}\sum_{l=1}^{n_k}(A_{0l}^{(k)}+A_{l0}^{(k)T})M_{kl}B_{lm}^{(k)}, & \text{if } m\neq k;\end{cases}$$

$$H_{0k}=\begin{bmatrix}L_{11} & \cdots & L_{1N}\\ \vdots & & \vdots\\ L_{N1} & \cdots & L_{NN}\end{bmatrix}$$

with

$$L_{ml}=\begin{cases}\mathbf{0}, & \text{if } m\neq k\\ -(A_{00}^{(k)}+A_{00}^{(k)T})^{-1}(A_{0l}^{(k)}+A_{l0}^{(k)T})(I-M_{kl}) & \text{if } m=k;\end{cases}$$

$$H_{k0}=\begin{bmatrix}M_{k1}B_{11}^{(k)} & \cdots & M_{k1}B_{1,k-1}^{(k)} & \mathbf{0} & M_{k1}B_{1,k+1}^{(k)} & \cdots & M_{k1}B_{1,N}^{(k)}\\ \vdots & & \vdots & \vdots & \vdots & & \vdots\\ M_{k,n_k}B_{n_k1}^{(k)} & \cdots & M_{k,n_k}B_{n_k,k-1}^{(k)} & \mathbf{0} & M_{k,n_k}B_{n_k,k+1}^{(k)} & \cdots & M_{k,n_k}B_{n_k,N}^{(k)}\end{bmatrix};$$

and

$$H_{km}=\begin{cases}\mathbf{0} & \text{if } m\neq k\\ \begin{bmatrix}I-M_{k1} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & I-M_{kn_k}\end{bmatrix}, & \text{if } m=k.\end{cases}$$

If H denotes the matrix of coefficients of the linear difference equation (6), then it has a special form:

$$H=\begin{bmatrix}H_{00} & H_{01} & \cdots & H_{0N}\\ H_{10} & H_{11} & & \mathbf{0}\\ & & \ddots & \\ H_{N0} & \mathbf{0} & & H_{NN}\end{bmatrix}.$$

The system (6) is asymptotically stable if and only if the eigenvalues of matrix H are inside the unit circle. In the following part of this section sufficient conditions will be derived for the stability.

Consider the eigenvalue problem of H:

$$\begin{aligned}&H_{00}u+\sum_{k=1}^{N}H_{0k}v_k=\lambda u\\ &H_{k0}u+H_{kk}v_k=\lambda v_k \qquad (k=1,2,\cdots,N)\end{aligned}\tag{7}$$

The second equations imply that

$$(\boldsymbol{H}_{kk}-\lambda\boldsymbol{I})\boldsymbol{v}_k=-\boldsymbol{H}_{k0}\boldsymbol{u}. \tag{8}$$

Assume that

(C) Eigenvalues of matrix $\boldsymbol{H}_{kk}$ are all inside the unit circle for $k=1,2,\cdots,N$.

Note that this condition is equivalent to the condition that the eigenvalues of matrices $\boldsymbol{M}_{kl}$ are inside the circle with centre and radius being equal to unity. If in addition, matrices $\boldsymbol{M}_{kl}$ are positive definite—which is the usual assumption in adaptive expectations—, then condition (C) is equivalent to the condition that the eigenvalues of $\boldsymbol{M}_{kl}$ are all less than two.

If in (8), λ is an eigenvalue of $\boldsymbol{H}_{kk}$, then the stability condition is satisfied for this eigenvalue. Otherwise

$$\boldsymbol{v}_k=-(\boldsymbol{H}_{kk}-\lambda\boldsymbol{I})^{-1}\boldsymbol{H}_{k0}\boldsymbol{u}, \tag{9}$$

and by simple substitution into the first equation of (7) we obtain the relation

$$\left(\boldsymbol{H}_{00}-\sum_{k=1}^{N}\boldsymbol{H}_{0k}(\boldsymbol{H}_{kk}-\lambda\boldsymbol{I})^{-1}\boldsymbol{H}_{k0}-\lambda\boldsymbol{I}\right)\boldsymbol{u}=\boldsymbol{0}. \tag{10}$$

If $\boldsymbol{u}=\boldsymbol{0}$, then for all k, $\boldsymbol{v}_k=0$. This is a contradiction, since eigenvectors differ from zero. Hence $\boldsymbol{u}\neq\boldsymbol{0}$, which proves the following

THEOREM. *If conditions* (A), (B) *and* (C) *are satisfied, then system* (6) *is asymptotically stable if and only if all roots of equation*

$$\det\left[\boldsymbol{H}_{00}-\sum_{k=1}^{N}\boldsymbol{H}_{0k}(\boldsymbol{H}_{kk}-\lambda\boldsymbol{I})^{-1}\boldsymbol{H}_{k0}-\lambda\boldsymbol{I}\right]=0 \tag{11}$$

are inside the unit circle.

REMARK 1. In the special case when no adaptive expectations are assumed, we may select $\boldsymbol{M}_{kl}=\boldsymbol{I}$, which implies that $\boldsymbol{H}_{0k}=\boldsymbol{0}$ for all k. $\boldsymbol{H}_{00}$ equals the matrix of coefficients of system (4). Hence for $\boldsymbol{H}_{kl}=\boldsymbol{I}$ ($\forall\, k,l$) the Theorem gives the stability condition for the dynamic game without adaptive expectations, as well.

REMARK 2. In the general case it is a very difficult task to check the validity of the condition (11) of the Theorem. In special cases anyhow it might not be so difficult as the following case shows. Assume that λ is real and for all k, $\boldsymbol{H}_{kk}=\alpha\boldsymbol{I}$, where α does not depend on k. Then (11) can be rewritten as

$$\det\left[\boldsymbol{H}_{00}-\frac{1}{\alpha-\lambda}\sum_{k=1}^{N}\boldsymbol{H}_{0k}\boldsymbol{H}_{k0}-\lambda\boldsymbol{I}\right]=0,$$

or

$$\det\left[\boldsymbol{S}-\frac{1}{\alpha-\lambda}\boldsymbol{T}-\lambda\boldsymbol{I}\right]=0,$$

where

$$\boldsymbol{S}=\boldsymbol{H}_{00} \quad \text{and} \quad \boldsymbol{T}=\sum_{k=1}^{N} \boldsymbol{H}_{0k}\boldsymbol{H}_{k0}\,.$$

This condition is satisfied if there exists a real vector $\boldsymbol{u}\neq\boldsymbol{0}$ such that

$$\boldsymbol{Su}-\frac{1}{\alpha-\lambda}\boldsymbol{Tu}-\lambda\boldsymbol{u}=\boldsymbol{0}\,.$$

Premultiplying by $\boldsymbol{u}^T$ and multiplying by $\alpha-\lambda$ we have that

$$s(\alpha-\lambda)-t-\lambda(\alpha-\lambda)=0,$$

where

$$s=\frac{\boldsymbol{u}^T\boldsymbol{Su}}{\boldsymbol{u}^T\boldsymbol{u}} \quad \text{and} \quad t=\frac{\boldsymbol{u}^T\boldsymbol{Tu}}{\boldsymbol{u}^T\boldsymbol{u}}\,.$$

Hence

$$\lambda^2-\lambda(s+\alpha)+(s\alpha-t)=0,$$

which has roots inside the unit circle if and only if

$$s\alpha-t-1<0,$$

$$s+\alpha-s\alpha+t-1<0$$

and

$$-s-\alpha-s\alpha+t-1<0.$$

By using the definition of s and t we may conclude that these inequalities hold if matrices

$$\alpha\boldsymbol{S}-\boldsymbol{T}-\boldsymbol{I}, \qquad (1-\alpha)\boldsymbol{S}+\boldsymbol{T}+(\alpha-1)\boldsymbol{I} \quad \text{and} \quad -(1+\alpha)\boldsymbol{S}+\boldsymbol{T}-(\alpha+1)\boldsymbol{I}$$

are all negative definite.

In the next section the special case of the multiproduct oligopoly game will be investigated.

3. APPLICATION TO OLIGOPOLY WITH MULTIPRODUCT FIRMS

Let N and M denote the numbers of players (firms) and products. The kth firm's output vector is $\boldsymbol{x}_k=(x_k^{(1)},\cdots,x_k^{(M)})$, where $x_k^{(m)}$ denotes the output in product m of firm k. The inverse demand function vector is given by

$$p(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N) = \boldsymbol{A}\left(\sum_{k=1}^{N} \boldsymbol{x}_k\right) + \boldsymbol{b}\,.$$

It is also assumed that the cost function of firm k is as follows:

$$C_k(\boldsymbol{x}_k) = \boldsymbol{b}_k^T \boldsymbol{x}_k + c_k \qquad (k = 1, 2, \cdots, N)$$

Hence the kth firm's profit equals

$$\varphi_k(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N) = \boldsymbol{x}_k^T\left(\boldsymbol{A}\left(\sum_{k=1}^{N} \boldsymbol{x}_k\right) + \boldsymbol{b}\right) - (\boldsymbol{b}_k^T \boldsymbol{x}_k + c_k)\,. \tag{12}$$

1. First the case *without adaptive expectation* is examined. In this case we may select

$$\boldsymbol{x}^{(k)} = \begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{s}_{k1} \end{bmatrix}, \qquad \boldsymbol{s}_{k1} = \sum_{m \neq k} \boldsymbol{x}_k\,,$$

and so $n_k = 1$,

$$\boldsymbol{B}_{1m}^{(k)} = \boldsymbol{I} \qquad (m \neq k).$$

Thus the matrix of coefficients of the system (4) is as follows:

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{H}_{12} & \cdots & \boldsymbol{H}_{1N} \\ \boldsymbol{H}_{21} & \boldsymbol{0} & \cdots & \boldsymbol{H}_{2N} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{H}_{N1} & \boldsymbol{H}_{N2} & \cdots & \boldsymbol{0} \end{bmatrix} \tag{13}$$

with

$$\boldsymbol{H}_{kl} = -(\boldsymbol{A} + \boldsymbol{A}^T)^{-1}\boldsymbol{A} \qquad (\forall\, k \neq l).$$

In our earlier paper (Okuguchi and Szidarovszky, 1987) we have proved that the eigenvalues of matrix (13) are inside the unit circle if and only if $N = 2$ provided that the eigenvalues of matrix $\boldsymbol{A}^{-1}\boldsymbol{A}^T$ are real.

2. Consider next the case when *adaptive expectations* are assumed *on individual firm's outputs*. Let $\boldsymbol{x}_{kl}^E$ denote the kth firm's adaptive expectations on the lth firm's output vector. Then

$$\boldsymbol{x}^{(k)} = \begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_{k-1} \\ \boldsymbol{x}_{k+1} \\ \vdots \\ \boldsymbol{x}_N \end{bmatrix},$$

and

$$\varphi(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N)=\boldsymbol{x}^{(k)T}\begin{bmatrix} A & A & \cdots & A \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}\boldsymbol{x}^{(k)}+((\boldsymbol{b}-\boldsymbol{b}_k)^T,\mathbf{0},\cdots,\mathbf{0})\boldsymbol{x}^{(k)}-c_k.$$

Hence by using the notation for the blocks of system (6),

$$\boldsymbol{K}_{km}=\begin{cases} \mathbf{0}, & \text{if} \quad k=m \\ -(A+A^T)^{-1}A\boldsymbol{M}_{km}, & \text{if} \quad m<k \\ -(A+A^T)^{-1}A\boldsymbol{M}_{k,m-1}, & \text{if} \quad m>k; \end{cases}$$

$$\boldsymbol{L}_{ml}=\begin{cases} \mathbf{0}, & \text{if} \quad m\neq k \\ -(A+A^T)^{-1}A(\boldsymbol{I}-\boldsymbol{M}_{kl}), & \text{if} \quad m=k; \end{cases}$$

$$\boldsymbol{H}_{k0}=\begin{bmatrix} \boldsymbol{M}_{k1} & & & & \mathbf{0} & & & \\ & \boldsymbol{M}_{k2} & & & \mathbf{0} & & & \\ & & \ddots & & \vdots & & & \\ & & & \boldsymbol{M}_{k,k-1} & \mathbf{0} & & & \\ & & & & \mathbf{0} & \boldsymbol{M}_{k,k+1} & & \\ & & & & \vdots & & \ddots & \\ & & & & \mathbf{0} & & & \boldsymbol{M}_{k,N-1} \end{bmatrix};$$

and matrices $\boldsymbol{H}_{km}$ are the same as in the original case:

$$\boldsymbol{H}_{km}=\begin{cases} \mathbf{0} \quad \text{if} \quad m\neq k, \\ \begin{bmatrix} \boldsymbol{I}-\boldsymbol{M}_{k1} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{I}-\boldsymbol{M}_{k,N-1} \end{bmatrix}, & \text{if} \quad m=k. \end{cases}$$

Consider next the special case, when for all k,l, $\boldsymbol{M}_{kl}=\alpha\boldsymbol{I}$, furthermore $A=A^T$. Then $\boldsymbol{H}$ is the Kronecker product of matrix

$$
\left[\begin{array}{cccc|ccc|c|ccc}
0 & -\frac{\alpha}{2} & \cdots & -\frac{\alpha}{2} & -\frac{1}{2}(1-\alpha) & \cdots & -\frac{1}{2}(1-\alpha) & \cdots\vdots\cdots & 0 & \cdots & 0 \\
-\frac{\alpha}{2} & 0 & \cdots & -\frac{\alpha}{2} & 0 & \cdots & 0 & \vdots & \vdots & & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & 0 & \cdots & 0 \\
-\frac{\alpha}{2} & -\frac{\alpha}{2} & \cdots & 0 & 0 & \cdots & 0 & \cdots\vdots\cdots & -\frac{1}{2}(1-\alpha) & \cdots & -\frac{1}{2}(1-\alpha) \\
\hline
0 & \alpha & & & 1-\alpha & & & \vdots & & & \\
\vdots & & \ddots & & & \ddots & & \vdots & & & \\
0 & & & \alpha & & & 1-\alpha & \vdots & & & \\
\hline
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\alpha & & & 0 & & & & \vdots & 1-\alpha & & \\
 & \ddots & & \vdots & & & & \vdots & & \ddots & \\
 & & \alpha & 0 & & & & \vdots & & & 1-\alpha
\end{array}\right] \qquad (14)
$$

and the identity matrix. Hence the eigenvalues of $\boldsymbol{H}$ coincide with the eigenvalues of matrix (14), the eigenvalue equation of which has the form

$$\begin{aligned} -\frac{\alpha}{2}\sum_{m\neq k}\boldsymbol{u}_m-\frac{1-\alpha}{2}\sum_{l=1}^{N-1}\boldsymbol{v}_{kl}&=\lambda\boldsymbol{u}_k \qquad (k=1,2,\cdots,N) \\ \alpha\boldsymbol{u}_{k(l)}+(1-\alpha)\boldsymbol{v}_{kl}&=\lambda\boldsymbol{v}_{kl} \qquad (\forall\, k,l), \end{aligned} \tag{15}$$

where

$$k(l)=\begin{cases} l+1, & \text{if } k\leq l \\ l, & \text{if } k>l. \end{cases}$$

By summing up the second equations of (15) for l we obtain the identity

$$\alpha\sum_{m\neq k}\boldsymbol{u}_m+(1-\alpha)\sum_{l=1}^{N-1}\boldsymbol{v}_{kl}=\lambda\sum_{l=1}^{N-1}\boldsymbol{v}_{kl}. \tag{16}$$

By multiplying (16) by 1/2 and adding to the first equation of (15) we have

$$\lambda\left(\boldsymbol{u}_k+\frac{1}{2}\sum_{l=1}^{N-1}\boldsymbol{v}_{kl}\right)=\boldsymbol{0}.$$

If $\lambda=0$, then $|\lambda|<1$, so this eigenvalue belongs to the interior of the unit circle. If $\lambda\neq 0$, then

$$\sum_{l=1}^{N-1}\boldsymbol{v}_{kl}=-2\boldsymbol{u}_k.$$

By substituting this relation into (16) we get

$$-\frac{\alpha}{2}\sum_{m\neq k}\boldsymbol{u}_m+(1-\alpha)\boldsymbol{u}_k=\lambda\boldsymbol{u}_k \qquad (k=1,2,\cdots,N),$$

which is equivalent to the eigenvalue problem of matrix

$$\begin{aligned} &\begin{bmatrix} +(1-\alpha) & -\frac{\alpha}{2} & \cdots & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & +(1-\alpha) & \cdots & -\frac{\alpha}{2} \\ \vdots & \vdots & \ddots & \\ -\frac{\alpha}{2} & -\frac{\alpha}{2} & \cdots & +(1-\alpha) \end{bmatrix} = \\ &= -\frac{\alpha}{2}\cdot\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}+\left(1-\frac{\alpha}{2}\right)\cdot\boldsymbol{I}. \end{aligned} \tag{17}$$

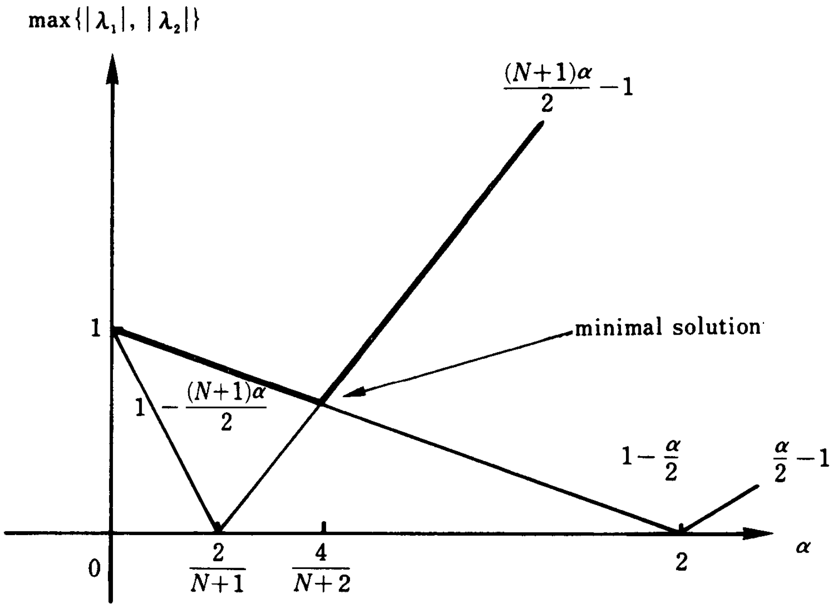

Fig. 1. The Optimal Speed of Adjustment.

The eigenvalues of this matrix are either $\lambda_1 = 1 - \alpha/2$ or $\lambda_2 = -\alpha/2 \cdot N + 1 - \alpha/2 = 1 - (N+1)\alpha/2$. They are inside the unit circle if and only if

$$0 < \alpha < \frac{4}{N+1}. \tag{18}$$

The optimal value of the adjustment speed α can be determined as follows:

$$\text{Minimize } \max\{|\lambda_1|, |\lambda_2|\}.$$

In our case

$$|\lambda_1| = \begin{cases} 1 - \dfrac{\alpha}{2} & \text{if } \alpha < 2 \\ \dfrac{\alpha}{2} - 1 & \text{if } \alpha \geq 2 \end{cases}$$

$$|\lambda_2| = \begin{cases} 1 - \dfrac{(N+1)\alpha}{2}, & \text{if } \alpha < \dfrac{2}{N+1} \\ \dfrac{\alpha(N+1)}{2} - 1, & \text{if } \alpha \geq \dfrac{2}{N+1}. \end{cases}$$

Function $\max\{|\lambda_1|, |\lambda_2|\}$ is shown in Fig. 1. The minimal value of α can be obtained as the intersection of the lines $1 - \alpha/2$ and $(N+1)\alpha/2 - 1$, that is,

$$\alpha_{\text{opt}} = \frac{4}{N+2}. \tag{19}$$

3. Consider now the case when *adaptive expectations* are assumed on the *rest of the industry outputs*. In this case

$$\boldsymbol{x}^{(k)} = \begin{bmatrix} x_k \\ s_k \end{bmatrix}, \qquad s_k = \sum_{m \neq k} x_m ,$$

$$\varphi_k(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N) = \boldsymbol{x}^{(k)} \begin{bmatrix} A & A \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{x}^{(k)} + ((\boldsymbol{b} - \boldsymbol{b}_k)^T, \mathbf{0}) \boldsymbol{x}^{(k)} - c_k .$$

With the notations of the blocks of the matrix of coefficients in (6) and defining $\boldsymbol{E} = (\boldsymbol{A} + \boldsymbol{A}^T)^{-1}\boldsymbol{A}$, $\boldsymbol{M}_k = \boldsymbol{M}_{k1}$,

$$\boldsymbol{K}_{km} = \begin{cases} \mathbf{0}, & \text{if } m = k \\ -\boldsymbol{E}\boldsymbol{M}_k, & \text{if } m \neq k ; \end{cases}$$

$$\begin{bmatrix} \boldsymbol{H}_{10} \\ \vdots \\ \boldsymbol{H}_{N0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{M}_1 & \cdots & \boldsymbol{M}_1 \\ \boldsymbol{M}_2 & \mathbf{0} & \cdots & \boldsymbol{M}_2 \\ \vdots & \vdots & & \vdots \\ \boldsymbol{M}_N & \boldsymbol{M}_N & \cdots & \mathbf{0} \end{bmatrix} ;$$

$$(\boldsymbol{H}_{01}, \cdots, \boldsymbol{H}_{0N}) = \begin{bmatrix} -\boldsymbol{E}(\boldsymbol{I} - \boldsymbol{M}_1) & & 0 \\ & \ddots & \\ 0 & & -\boldsymbol{E}(\boldsymbol{I} - \boldsymbol{M}_N) \end{bmatrix} ;$$

$$\begin{bmatrix} \boldsymbol{H}_{11} & \cdots & \boldsymbol{H}_{1N} \\ \vdots & & \vdots \\ \boldsymbol{H}_{N1} & \cdots & \boldsymbol{H}_{NN} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{M}_1 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{I} - \boldsymbol{M}_N \end{bmatrix} .$$

The eigenvalue problem of matrix $\boldsymbol{H}$ has now the special form:

$$\begin{aligned} &\sum_{l \neq k} -\boldsymbol{E}\boldsymbol{M}_k \boldsymbol{u}_l - \boldsymbol{E}(\boldsymbol{I} - \boldsymbol{M}_k)\boldsymbol{v}_k = \lambda \boldsymbol{u}_k \qquad (\forall\, k) \\ &\sum_{l \neq k} \boldsymbol{M}_k \boldsymbol{u}_l + (\boldsymbol{I} - \boldsymbol{M}_k)\boldsymbol{v}_k = \lambda \boldsymbol{v}_k \qquad (\forall\, k) . \end{aligned} \tag{20}$$

By adding the $\boldsymbol{E}$-multiple of the second equation of (20) to the first equation of (20) we get

$$\lambda(\boldsymbol{E}\boldsymbol{v}_k + \boldsymbol{u}_k) = \mathbf{0}. \tag{21}$$

We may assume that $\lambda \neq 0$. Then $\boldsymbol{u}_k = -\boldsymbol{E}\boldsymbol{v}_k$, and by substituting this relation into the second equation of (20), the equations

$$\sum_{l \neq k} (-\boldsymbol{M}_k \boldsymbol{E} \boldsymbol{v}_k) + (\boldsymbol{I} - \boldsymbol{M}_k)\boldsymbol{v}_k = \lambda \boldsymbol{v}_k \qquad (\forall\, k)$$

are obtained, which are equivalent to the eigenvalue problem of matrix

$$\begin{bmatrix} I-M_1 & -M_1E & \cdots & -M_1E \\ -M_2E & I-M_2 & \cdots & -M_2E \\ \vdots & \vdots & \ddots & \vdots \\ -M_NE & -M_NE & \cdots & I-M_N \end{bmatrix} \tag{22}$$

In the special case of $A=A^T$ and $M_k=\alpha I$ ($\forall k$) this matrix is the Kronecker product of matrix (17) and the identity matrix. Hence the stability condition is again relation (18), and the optimal adjustment speed is given again by (19).

4. CONCLUSIONS

After formulating a general quadratic model with and without adaptive expectations a general stability condition was derived, which reduced the problem to the examination of a nonlinear eigenvalue problem (11).

As applications, the multiproduct oligopoly game was investigated. If no adaptive expectation is assumed, then the equilibrium is stable for only $N=2$, which gives the multiproduct generalization of the classical result of Theocharis (1959). If adaptive expectations are assumed, then in both cases examined stability can be assured for arbitrary numbers of players provided that the speed of adjustment is sufficiently small. Concerning this result, two comments are in order. Observe first, that the case without adaptive expectations is equivalent to the case of adaptive expectation with $\alpha=1$. The inequality (18) holds only for $N=2$. This observation gives not only a new generalization of the famous theorem of Theocharis but also gives an explanation why $N=2$ is the border line for stability in the classical case. Besides presenting the stability conditions we have shown how the optimal adjustment speed can be determined, optimal in the sense that the convergence is fastest.

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