Title	PARETO OPTIMALITY, CORE AND EQUILIBRIA IN A COOPERATIVE SUPER GAME WITHOUT SIDE PAYMENTS
Sub Title	
Author	塩澤, 修平(SHIOZAWA, Shuhei)
Publisher	Keio Economic Society, Keio University
Publication year	1987
Jtitle	Keio economic studies Vol.24, No.1 (1987.) ,p.25- 41
JaLC DOI	
Abstract	In this paper, we define a supergame payoff as the lim inf of average payoffs. We discuss the relationship between a single-period n-person cooperative game and the corresponding supergame, for notions of Pareto optimality, Nash-equilibrium, the α-core and the β-core.
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19870001-0025

PARETO OPTIMALITY, CORE AND EQUILIBRIA IN A COOPERATIVE SUPER GAME WITHOUT SIDE PAYMENTS

Shuhei SHIOZAWA*

Abstract. In this paper, we define a supergame payoff as the lim inf of average payoffs. We discuss the relationship between a single-period n-person cooperative game and the corresponding supergame, for notions of Pareto optimality, Nash-equilibrium, the α-core and the β-core.

1. INTRODUCTION

In this paper, we will discuss the relationship between a single-period n-person cooperative game and the corresponding supergame, that is, the game each play of which consists of an infinite sequence of plays of the single-period game.

Aumann [1959] proved that the β-core in a single-period game G coincided with the set of payoff vectors to stron equilibria in its supergame G^*. We do analogous things for Pareto optimality, Nash equilibria, the α-core and the β-core with a new definition of a payoff in G^*, namely the lim inf of average payoffs. One reason for using the lim inf definition is that it can be defined on the set of all possible outcomes in G^*. And with this definition some results which have been already established hold with appropriate modifications. Intuitively a payoff in G^* is determined so that average of payoffs in single-period games G's is at least as much as the payoff in G^* as number of periods goes to infinity.

The theory of supergames is concerned with the evolution of fundamental patterns of interaction between people. And supergames can be applied to resolve some problems which are difficult to solve in a single-period game. For example, a supergame can be used to solve the "free rider" problem in an economy with public goods. In such an economy public goods will tend to be undersupplied, and without any special taxation and allocation rule no optimal allocation can be achieved as a Nash equilibrium in a single-period situation. But in a supergame situation, by the use of dynamic feedback strategies, it has been shown that an optimal allocation can be achieved as a supergame Nash equilibrium (see McMillan [1979]).

In a supergame, the payoff is usually some kind of average of the payoff in the various stages. And in general, the set of feasible payoff vectors in a single-period

* I am grateful to Professors M. K. Richter and K. Kawamata for their valuable comments. Needless to say, I am solely responsible for any error.

game G is different from the set of feasible payoff vectors in its supergame G^*. In this paper we define in G and G^* notions of Pareto optimality, a Nash equilibrium, α-core, and β-core. We also study the properties and relations among these concepts in G and G^*.

Nash [1950, 1951] introduced an n-person noncooperative game and established an equilibrium concept. On the other hand, the concept of "supergame" was first introduced by Luce and Raiffa [1957]. It is known as a "Folk Theorem" that with an appropriate definition of a payoff in G^*, the set of payoff vectors to Nash equilibrium points in G^* coincides with the set of feasible and individually rational payoff vectors in the same G. The "Folk Theorem" has been known for twenty years but its authorship is obscure. The significance of the Folk theorem is that it relates cooperative behavior in G to non-cooperative behavior in G^*. This is one of the fundamental messages of the theory of repeated games of complete information; cooperation may be explained by the fact that the games people play are not one-time affairs, but are repeated over and over (see Aumann [1981]). The interest in the Folk theorem lies partly in its usefulness in solving the free rider problem in a noncooperative (Nash equilibrium) way.

Branching out from the Folk theorem, there was an attempt to refine somewhat the notion of "cooperative outcome" on the cooperative side of the Folk theorem. One would like a characterization, in terms of G^*, of more specific kinds of cooperative behavior in G. This is achieved by replacing the notion of equilibrium by "strong equilibrium". The concept of "strong equilibrium" was first introduced by Aumann [1959]. Strong equilibrium existence theorems are established in Ichiishi [1982]. Aumann [1961] also introduced the α-core and the β-core in a cooperative game without side payments, and discussed their properties. Scarf [1971] derived a sufficient condition for nonemptiness of the α-core. He also constructed a simple example in which the β-core is empty. From the definitions it is straightforward that the set of strong equilibrium payoff vectors is a subset of the β-core and the β-core is a subset of the α-core in an arbitrary cooperative game. The main result in Aumann is the theorem that the β-core in G coincides with the set of payoff vectors to strong equilibria in G^* (see Aumann [1959, 1967, 1981]).

Our main objectives in this paper are the following.

1) To prove that Pareto optimal solutions in G coincide with Pareto optimal solutions in G^*.

2) To prove the analogue of the Folk theorem, that is, the set of feasible individually rational payoffs coincides with the set of Nash equilibrium payoffs in G^*, with our lim inf definition of a payoff in G^* when the sets of feasible payoffs are equal.

3) To discuss the relation between the α-core in G and the α-core in G^*, and to characterize the $^+$-core in terms of Pareto optimality.

4) To prove that the β-core in G coincides with the β-core in G^* when the sets of feasible payoffs are equal, and to characterize the β-core in terms of Pareto

optimality.

In Section 2, the model is described. First we present an n-person single-period finite cooperative game G in normal form. Then we introduce G^*, the supergame of G. There are several ways of defining a payoff in G^*. For example Aumann [1959] and Owen [1982] defined a payoff in G^* as a limit of average payoffs. But in general the limit does not exist. Rubinstein [1979] introduced the "overtaking criterion". Unfortunately there is no utility function representing the "overtaking criterion" (see Rubinstein). In this paper a payoff in G^* is defined as the lim $\inf_{t\to\infty}$ of the average payoff. Of course our lim inf definition of a payoff in G^* ignores any finite time intervals. On the other hand it does give a supergame payoff function defined on the set of all possible supergame actions of the society. We define the set of feasible payoffs in G and in G^*.

And for our main objective we have the following results.

In Section 3.1, existence theorems of Pareto optimal payoff vectors in G and in G^* are proved. Our main result here is that the set of Pareto optimal payoff vectors in G coincides with the set of Pareto optimal payoff vectors in G^* even though the set of feasible payoff vectors in G is different from that in G^*. Hence there may exist payoffs in G^* which are not feasible in G, but such payoffs are not Pareto optimal in G and in G^*, and so they are of less interest in a normative point of view. Thus a payoff allocation in G^* is Pareto optimal in G^* if and only if it is Pareto optimal in G, so that we can use Pareto optimal criterion in G to evaluate any payoff allocation in G^*. The above statement is no longer true for weak Pareto optimality, that is, there exists a payoff which is weakly Pareto optimal in G^* and is not feasible in G. Hence when the sets of feasible payoffs do not coincide, we can not apply weak Pareto optimality criterion in G to evaluate outcomes in G^*, but weak Pareto optimality in G is a sufficient condition for weak Pareto optimality in G^*. And in the framework of G^*, weak Pareto optimality is still a normative criterion, and it can be used as a target of the planner.

In Section 3.2, we define Nash equilibrium in G and in G^*. Existence of a Nash equilibrium in G^* is a straightforward consequence of the well-known existence theorem of a Nash equilibrium in G. When the sets of feasible payoffs are equal, we prove the analogue of the Folk theorem, that is, the set of feasible individually rational payoff vectors in G coincides with set of the set of payoff vectors supported by Nash equilibria in G^*. Since the set of feasible payoffs in G does not generally coincide with the set of feasible payoffs in G^*, there can exist a payoff vector which supported by a Nash equilibrium in G^* and is individually rational in G but is not feasible in G. Hence the analogue of the Folk theorem does not generally hold. But an important assertion of our version of the Folk theorem is that any payoff which is feasible and individually rational in G can be achieved as a Nash equilibrium in G^*. (This is usually all we need for a noncooperative G^* solution to a free-rider problem.)

In Section 3.3, we define the α-core in G and in G^*. Our definition is slightly different from usual ones. We prove that any coalition is α-effective in G^* for a payoff if it is α-effective in G for the payoff. For we can construct a supergame strategy for the coalition to be α-effective in G^*, depending on a single-period strategy which makes the coalition α-effective in G. Intuitively if a coalition can improve upon a payoff in G with a unique single-period strategy no matter what other players do, then the coalition can improve upon the payoff in G^* with a unique supergame strategy based on the single-period strategy no matter what other players do. Thus any threat by players outside the coalition is not valid in G^*. But the converse of the above statement is an open question. A difficulty lies in specifying a single-period strategy which makes the coalition α-effective in G. Hence the α-core in G^* is a subset of the α-core in G when the sets of feasible payoffs are equal. Therefore we can use the α-effective criterion in G to evaluate a payoff in G^*, in the sense that if some coalition is α-effective in G for a payoff then the same coalition is α-effective in G^* for the payoff, hence the payoff can not be stable in G^*, in the sense that it can not be in the α-core in G^*. And any payoff vector, which is in the α-core and is Pareto optimal in G^*, is in the α-core and is Pareto optimal in G. Payoff vectors in the α-core in G are not necessarily Pareto optimal in G.

In Section 3.4, we define the β-core in G and in G^*. We show that any coalition is β-effective in G^* for a payoff if and only if the coalition is β-effective in G for that payoff. In this case we do not have to specify a single-period strategy which makes the coalition β-effective in G. Intuitively if a payoff is not stable in G in the sense that some coalition can improve upon the payoff in G, then the payoff can not be stable in G^* in the sense that the same coalition can improve upon the payoff in G^*, and vice versa. Hence the β-core in G is a subset of the β-core in G^*, and when the sets of feasible payoffs are equal the β-core in G coincides with the β-core in G^*. By this theorem we can relate a cooperative concept in G to a corresponding cooperative concept in G^*, and we can use the β-effective criterion in G to evaluate a payoff allocation is G^*, in the sense that if no coalition is β-effective in G for a payoff then no coalition is β-effective in G^* for the payoff hence it is in the β-core in G^*, or in the sense that if some coalition is β-effective in G for a payoff then the coalition is β-effective in G^* for the payoff, hence it can not be in the β-core in G^*.

2. THE MODEL

2.1. The single-period game in normal form G

Let $G'=(\{X_i\}_{i\in N}, \{u_i\}_{i\in N})$ be an n-person game in normal form, where $N=\{1, \cdots, n\}$; the finite set of players

X_i: the finite set of pure strategies available to player i

u_i': $X \to R$; a payoff function of player i, where $X\equiv\Pi_{i\in N}X_i$. For such a game G', we also define the following concepts:

A: a partition of N, that is, a set of nonempty subsets (coalitions) $\{A_1, \cdots, A_k\}$ of N such that $A_j\cap A_{i'}=\emptyset, j\neq j'$, and $\cap_{j=1}^{k}A_j=N$;

$\mathscr{A}$: the set of all partitions of N;
S_{Aj}: the set of all mixed (correlated) strategies available to coalition $A_j \subset N$, that is, the set of all probability distributions defined on the finite set of pure strategies $X_{Aj} \equiv \Pi_{i \in Aj} X_i$ available to coalition A, and if a coalition consists of a single player i, we write S_i;
$S_A \equiv \Pi_{Aj \in} S_{Aj}$.

Given a partition $A = \{A_1, \cdots, A_k\}$ and strategies $s_{A_j} \in S_{A_j}$ of the coalitions, a probability distribution $s \in S$ on X is determined, where S is the set of all probability distributions defined on X. For each $s \in S$, expected payoff of each player is calculated by the function u_i'. Hence we can define an expected payoff function, and denote by $u_i(\{s_{A_j}\}_{A_j \in A})$ an expected payoff of player i of strategies $\{s_{A_j}\}_{A_j \in A}$. We write $u(\{s_{A_j}\}_{A_j \in A}) = [u_1(\{s_{A_j}\}_{A_j \in A}), \cdots, u_n(\{s_{A_j}\}_{A_j \in A})]$.

Then we define a mixed and cooperative extension $G = (\{S_A\}_{A \in \mathscr{A}}, \{u_i\}_{i \in N})$ of G'.

DEFINITION 2.1.1. Strategies $\{s_{A_j}\}_{A_j \in A}$ for a partition A is said to *support a payoff vector* $v = (v_1, \cdots, v_n) \in R^n$ *ib* G *if* $u(\{s_{A_j}\}_{A_j \in A}) = v$.

DEFINITION 2.1.2. The *set of feasible payoff vectors* V in G is defined by

$$V \equiv \{v \in R^n : v = u(\{s_{A_j}\}_{A_j \in A}) \text{ for some } A \in \mathscr{A} \text{ and some } \{s_{A_j}\}_{A_j \in A}\}.$$

DEFINITION 2.1.3. The *set of noncooperatively feasible payoff vectors* V^n in G is defined by

$$V^n \equiv \{v \equiv R^n : v = u(\{s_i\}_{i \in N}), \text{ for some } s_i \in S_i\} .$$

Remark 2.1.1. V can be expressed as the convex hull of the finite set $\bar{V} \equiv \{\bar{v} \in R^n : \bar{v} = u(x)$ for some pure strategy $x \in X\}$. Hence V is compact and convex in R^n. By definition $V^n \subset V$, and V^n is not convex in general.

2.2. The supergame G^* of G

For any game G we will denote by G^* the "supergame" of G, that is, the game each play of which consists of an infinite sequence of plays of G. Assume that all players exist for periods $t = 1, 2, 3, \cdots$.

Denote by G^t a constituent game G at period t, that is $G^t = G$, and $S_A{}^t = S_A$ for every t

Remark 2.2.1. We assume that the supergame is stationary, that is, all the constituent games are identical.

DEFINITION 2.2.1. A *supergame action* of player i is a sequence $s_i = \{s_i(1), s_i(2), \cdots) \in S_i^*$ of single-period strategies $s_i(t)$ where
$S_i^* \equiv \Pi_{t=1}^{\infty} S_i^t$, $S^* \equiv \Pi_{i \in N} S_i^*$, and denote $s(t) \equiv (s_1(t), \cdots, s_n\ (t))$ and $s \equiv s_1, \cdots, s_n) \in S^*$.

DEFINITION 2.2.2. A *supergame strategy* for player i is a sequence

$$f_i = \{f_i^t\}_{t=1}^{\infty}, \text{ where } f_1^t \in S_i \text{ and } f_i^t : \Pi_{k=1}^{t-1} S^k \to S_i \text{ for } t \geqq 2.$$

Thus a supergame strategy yields a choice $s_t \in S_t$ at every period t, where each choice is possibly dependent on the outcome of the preceding games, and where all players know all choices made by every player in the past.

Let F_i be the set of supergame strategies of player i, and F be the set of N-tuples of supergame strategies; $F \equiv \Pi_{i \in N} F_i$.

Given a supergame strategy $f \in F$, corresponding single-period strategies $s(f: t) = [s_1(f: t), \cdots, s_n(f: t)]$ at each period t can be defined recursively by

$$s(f; 1) = (f_1^1, \cdots, f_n^1)$$
$$s(f; t) = \{f_1^t[s(f; 1), \cdots, s(f; t-1)], \cdots, f_n^t[s(f; 1), \cdots, s(f; t-1)]\} .$$

Hence we can define a mapping

$$g: F \rightarrow S^* \quad \text{by} \quad g(f) = [s(f; 1), s(f; 2), \cdots] .$$

DEFINITION 2.2.3. A *supergame payoff function*

$$p_i: F \rightarrow R \quad \text{for player } i$$

is defined as follows; first define

$$p(_iT; f) \equiv 1/T \sum_{t=1}^{T} u_i[s(t)] ,$$

where $s(t)$ is given by the mapping g, and then define

$$p_i(f) \equiv \lim \inf_T p_i(T; f) .$$

Remark 2.2.2. One reason for not defining payoffs in terms of limit is that the $\lim_{T \rightarrow \infty} p_i(T; f)$ may not exist. By defining a payoff in G^* as the lim inf of average payoffs we have a supergame payoff function defined on the set of all possible outcomes in G^*. Of course when the $\lim_{T \rightarrow \infty} p_i(T; f)$ exists, it coincides with the $\lim \inf_T p_i(T; f)$.

Remark 2.2.3. Intuitively $p_i(f) = v_i$ implies that the player i's average payoff is at least very close to v_i as number of periods goes to infinity.

DEFINITION 2.2.4. A supergame strategy $f \in F$ is said to *support a payoff vector* $v \in R^n$ *in* G^* if $p(f) = v$.

DEFINITION 2.2.5. The *set of feasible payoff vectors* in G^* is defined by

$$V^* \equiv \{v \in R^n: v = p(f) \text{ for some } f \in F\} .$$

Remark 2.2.4. By definition the set of feasible payoff vectors in G is a subset of the set of feasible payoff vectors in G^*. But they are not equal in general. See the following example.

Example 2.2.1. Let $N = \{1, 2\}$, $X = \{x_1^1, x_1^2\} \times \{x_2^1, x_2^2\}$, and
$u_1(x_1^1, x_2^1) = u_1(x_1^2, x_2^2) = u_2(x_1^1, x_2^2) = u_2(x_1^2, x_2^1) = 1$,

$u_1(x_1^1, x_2^2)=u_1(x_1^2, x)_1^2=u_2(x_1^1, x_2^1)=u_2(x_1^2, x_2^2)=0$.
Then $V=\mathrm{co}\{(1, 0), (0, 1)\}$. Now we can choose a sequence of pure strategies $\{x^k(t)\}$ as a supergame strategy f such that $\lim\inf p_i(T;f)=0$, $i=1, 2$. Hence $V^*=\mathrm{co}\{(1, 0), (0, 1), (0, 0)\}$. See Fig. 2.2.1.

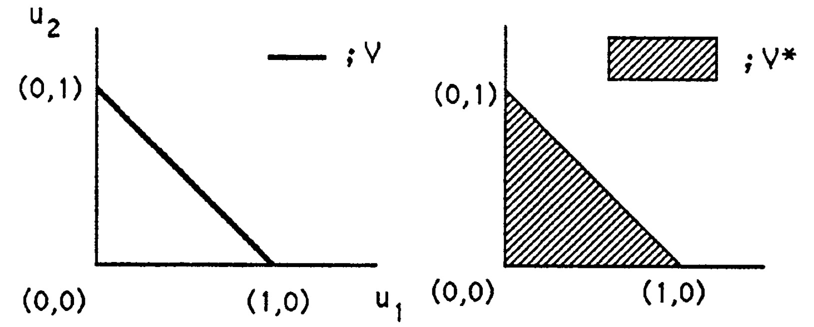

Fig. 2.2.1

DEFINITION 2.2.6. Let A be a coalition, that is $\emptyset\neq A\subset I$. Then a sequence of functions $f_A=\{f_A^1, f_A^1, \cdots\}$, where $f_A^1\in S_A$ and $f_A^t\colon \Pi_{k=1}^{t-1}S^k \to S_A$ for $t\geqq 2$, is said to be a *cooperative supergame strategy of A*. Denote by F_A the *set of all cooperative supergame strategies of A*.

3. PARETO OPTIMALITY, CORE AND EQUILIBRIA

3.1. *Pareto optimality in G and in G**

DEFINITION 3.1.1. A payoff $v\in V$ ($v\in V^*$) is said to be *Pareto optimal in G* (*in G**) if it is not true that there exists a payoff vector $v'\in V$ ($v'\in V^*$) such that

$$v'\geqq v_i \text{ for every } i\in N \text{ and}$$
$$v_i'>v_i \text{ for at least one } i\in N.$$

Remark 3.1.1. Some of Pareto optimal payoff vectors may not be achieved by noncooperative behavior in G.

DEFINITION 3.1.2. A payoff vector $v\in V$ ($v\in V^*$) is said to be *weakly Pareto optimal in G* (*in G**) if it is not true that there exists a payoff vector $v'\in V$ ($v'\in V^*$) such that

$$v'>v_i \quad \text{for every} \quad i\in N\,.$$

DEFINITION 3.1.3. A payoff vector $v\in V$ is said to be *Pareto superior in G* to $v'\in V$ if $v_i'\geqq v_i$ for every $i\in N$ and $v_i'>v_i$ for at least one $i\in N$.

LEMMA 3.1.1. *Let $v^*\in V^*$. Then there eixsts a payoff vector $v\in V$ with $v_i\geqq v_i^*$ for every $i\in N$.*

Proof. Since $v^* \in V^*$, there exists a supergame actions $s=(s(1), s(2), \cdots)$, such that $\liminf_T 1/T\sum_{t=1}^{T} u_i[s(t)]=v_i^*$ for every $i \in N$. Consider the sequence $\{1/T\sum_{t=1}^{T} u[s(t)]\}_{T=1}^{\infty} \subset V$. Since V is compact, there exists a subsequence $\{1/T_n\sum_{t=1}^{T_n} u[s(t)]\}_{T_n}$ which converges to some point $v' \in V$. Now suppose that $v_i < v_i^*$ for some i. Then there exist a real number $\epsilon < 0$ with $\epsilon < v_i^* - v_i'$ and a positive integer T^ϵ such that $|\sum_{t=1}^{T_n} u_i[s(t)] - v_i'| < \epsilon$ for all $T^n \geqq T^\epsilon$, which contradicts to the fact that $v_i^* = \liminf_T \sum_{t=1}^{T} u_i[s(t)]$. Therefore $v_i' \geqq v_i^*$ for every $i \in N$.
Q.E.D.

THEOREM 3.1.1. *There exists a Pareto optimal payoff vector in G and in G^*, and the set of Pareto optimal payoff vectors in G coincides with the set of Pareto optimal vectors in G^*.*

Proof. Let $\bar{V} \equiv \{v^1, \cdots, v^k\} \subset V$ be the set of payoff vectors supported by pure strategies in G. Since $\bar{V}$ is a finite set, we can choose an element $v^* \in \bar{V}$ such that $v_1^* \geqq v_1^k$ for all $k=1, \cdots, K$, and if $v_1^* = \bar{v}_1^k$ for some k we have $v_2^* \geqq v_2^k$ for such k, and if $v_1^* = \bar{v}_1^k$ and $v_2^* = \bar{v}_2^k$ for some k we have $v_3^* \geqq v_3^k$, and so on.

Since any payoff vector $v \in V$ is a convex combination of payoff vectors of $\bar{V}$, we have $v_1^* \geqq v_1$. If $v_1 > v_1$, v can not be Pareto superior to v^*. If we have $v_1^* = v_1$, v can be written as $v = \sum \alpha^j \bar{v}^j$ with $\sum \alpha^j = 1$ and $\alpha^j < 0$ and $v_1^* = \bar{v}_1^j$ for all j. Then we have $v_2^* \geqq v_2^j$ for all j. If we have $v_2^* > v_2^j$ for some j, v can not be Pareto superior to v^*. If we have $v_2^* = v_2^j$ for all j, we have $v_3^* \geqq v_3^j$ for all j, and we can apply the previous argument, and so on. Therefore *1* can not be Pareto superior to v^*. Hence v^* is Pareto optimal in G.

Let $v \in V$ be Pareto optimal in G. Suppose that *1* is not Pareto optimal in G^*. Then there exists a payoff vector $v' \in V^*$ such that

$$v' \geqq v_i \quad \text{for every} \quad i \in V, \quad \text{and}$$
$$v_i' > v_i \quad \text{for some } i\,.$$

By Lemma 3.1.1, there exists a payoff vector $v'' \in V$ with $v_n'' \geqq v_i'$ for every $i \in N$. Thus $v'' \geqq v_i$ for every $i \in N$ and $v_i'' < v_i$ for some i, which contradicts to the Pareto optimality of *1* in G.

On the other hand, let $v \in V^*$ be Pareto optimal in G^*. Suppose that v is not Pareto optimal in G. Then there exists a payoff vector $v' \in V$ such that

$$v_i' \geqq v_i \quad \text{for every} \quad i \in N, \quad \text{and}$$
$$v_i' > v_i \quad \text{for some } i\,.$$

Trivially, v' is an element of V^*, which contradicts to the Pareto optimality of v in G^*.
Q.E.D.

Remark 3.1.2. Theorem 3.1.1 does not hold if we define a payoff in G^* as a lim sup of average payoffs in stead of a lim inf. See Example 2.2.1. In that example, $V = \text{co}\{(1, 0), (0, 1)\}$ and any point in V is Pareto optimal in G. If we

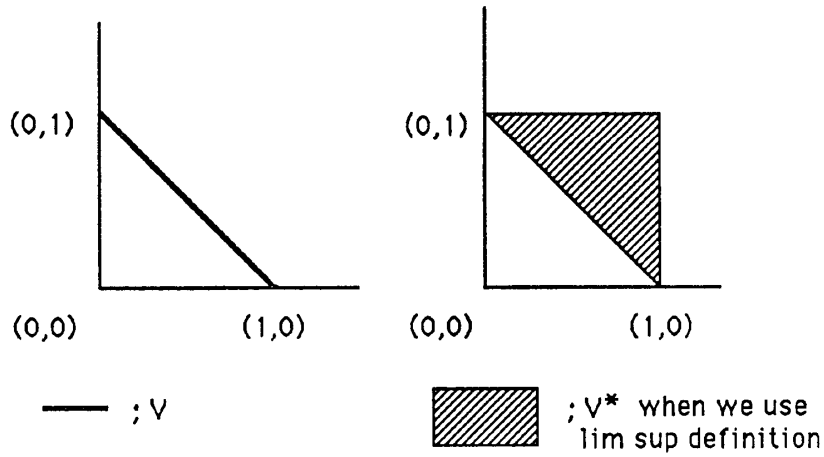

Fig. 3.1.1

define a payoff in G^* as a lim sup of average payoffs, then we have $V^*=co\{(1, 0), (0, 1), (1, 1)\}$. Hence (1, 1) is pareto optimal in G^*, and no point in V is Pareto optimal in G^*.

THEOREM 3.1.2. *The set of weakly Pareto optimal payoff vectors in G is a subset of weakly Pareto optimal payoff vectors in G^*.*

Proof. Let $v \in V$ be a weakly Pareto optimal payoff vector in G. Obviously v is an element of V^*. Suppose that v is not weakly Pareto optimal in G^*. Then there exists a payoff vector $v' \in V^*$ such that $v_i' > v_i$ for every $i \in N$. By Lemma 3.1.1, there exists a payoff vector $v_i'' \in V$ with $v_i'' \geqq v_i$ for every $i \in N$. Thus we have $v_i'' > v_i$ for every $i \in N$, which contradicts to the weak Pareto optimality of v in G. Q.E.D.

Remark 3.1.3. In general the set of weakly Pareto optimal payoff vectors in G does not coincide with the set of weakly Pareto optimal payoff vectors in G^*. See the following example.

Rxample 3.1.1. Let $N=\{1, 2, 3\}$ and $V=co\{(1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1)\}$. Then $V^*=co\{(1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1), (0, 0, 1), (0, 0, 0)\}$. And any payoff vector in V is weakly Pareto optimal in G and in G^*. But there exist payoff vector which are weakly Pareto optimal in G^* and do not belong to V, say (0, 0, 1). See Fig. 3.1.2.

*3.2. Nash equilibrium in G and in G**

DEFINITION 3.2.1. A noncooperative strategy $s^*=(s_1^*, \cdots, s_n^*)$, $s_i^* \in S_i^*$, is said to be a *Nash equilibrium in G* if

$$u_i(s^*) \geqq u_i(s_{-i}^*, s_i) \quad \text{for all} \quad s_i \in S_i \quad \text{and for every} \quad i \in N\,,$$

where $s_{-i} \equiv (s_i, \cdots, s_{i-1}, s_{1+1}, \cdots, s_n)$.

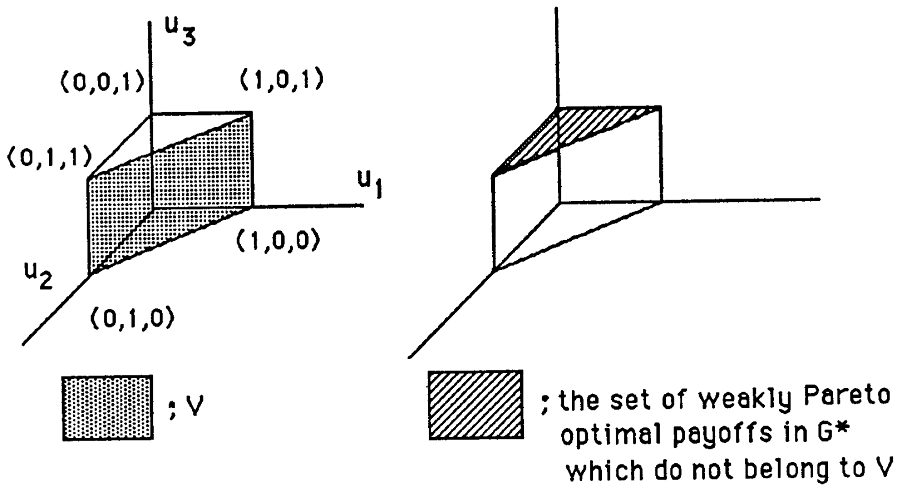

Fig. 3.1.2.

DEFINITION 3.2.2. A noncooperative strategy $f^* \in F$ is said to be a *Nash equilibrium in G^** if

$$p_i(f^*) \geqq p_i(f^*_{-i}, f_i) \quad \text{for all} \quad f_i \in F_i \quad \text{and for every} \quad i \in N .$$

DEFINITION 3.2.3. A payoff $v \in V$ is *individually rational in G* if for each player i

$$v_i \geqq \min_{s_{-i}} \max_{s_i} u_i(s_{-i}, s_i) \equiv m_i .$$

Remark 3.2.1. Since the number of pure strategies is finite there exists the min max for each player.

Remark 3.2.2. Existence of a Nash equilibrium in G^* is a straightforward consequence of the well-known existence theorem of a Nash equilibrium in G, since any payoff vector supported by a Nash equilibrium in G can be supported by a Nash equilibrium in G^*.

THEOREM 3.2.1. *The set of noncooperatively feasible and individually rational payoff vectors in G is a subset of the set of payoff vectors supported by Nash equilibria in G^*.*

Proof. Let $v \in V^n$ be individually rational payoff vector in G. Then there is a strategy $s^* = (s^*_1, \cdots, s^*_n)$, $s^*_i \in S_i$, which supports v, that is $u(s^*) = v$. Now define a supergame strategy $f^* \in F$ as follows; the players start by s^*. If at any period player i does not play the prescribed choice s^*_i, then from next period, other players play the strategy s'_{-i} with

$$u_i(s'_{-i}, s_i) \leqq m_i \leqq v_i \quad \text{for any} \quad s_i \in S_i .$$

Hence we have

$$\liminf p_i(f^*_{-i}, f_i) \leqq m_i \leqq v_i \quad \text{for any} \quad f_i \in F_i ,$$

for every $i \in N$. And we have $p(f^*)=v$. Therefore $f^* \in F$ is a Nash equilibrium in G^* and it supports v. Q.E.D.

THEOREM 3.2.2. *Any payoff vector which is supported by a Nash equilibrium in G^* is individually rational in G.*

Proof. Let $f^* \in F$ be a Nash equilibrium in G^* with $p(f^*)=v$. Suppose that v^* is not individually rational in G, that is, there exists a player i with $v_i^* < m_i$. Then define a supergame strategy $f'_i \in F_i$ for player i as follows; at $t=1$, for f^{*1}_{-1} choose a strategy $s_i(1) \in S_i$ with $u_i[f^{1*}_{-i}, s_i(1)] \geqq m_i > v_i^*$, at $t=2$, for $f^2_{-i}[f^{1*}_{-1}, s_i(1)]$ choose a strategy $s_i(2) \in S_i$ with $u_i\{f^2_{-i}[f^1_{-i}, s_i(1)], s_i(2)\} \geqq m_i > v_i^*$, and so on.

Then $\liminf p_i(f^*_{-i}, f'_i) \geqq m_i > v_i^{**}$, which contradicts to the fact that f^* is a Nash equilibrium in G^*. Q.E.D.

Remark 3.2.3. Since V does not coincide with V^*, there can exist a payoff vector which is supported by a Nash equilibrium in G^* and is not feasible in G. But by Theorem 3.1.1. such a payoff vector is not Pareto optimal in G^*, hence it is of less interest in a normative point of view. See the following example.

Example 3.2.1. Let $N=\{1, 2, 3\}$, $X_1=\{A^1, A^2\}$, $X_2=\{B^1, B^2\}$, $X_3=\{C^1, C^2\}$ and

$$u(A^1, B^1, C^2)=u(A^2, B^2, C^1)=(1, 0, 1)$$
$$u(A^1, B^1, C^2)=u(A^2, B^2, C^1)=(0, 1, 1)$$
$$u(A^1, B^2, C^1)=u(A^2, B^1, C^2)=(1, 0, 0)$$
$$u(A^1, B^2, C^2)=u(A^2, B^1, C^1)=(0, 1, 0) .$$

Then $V=co\{(1, 0, 1), (0, 1, 1), (1, 0, 0), (0, 1, 0)\}$. And $m_1=m_2=m_3=0$, since

$$u_1(s_1, B^2, C^2)=0 \quad \text{for any} \quad s_1 \in S_1$$
$$u_2(A^2, s_2, C^1)=0 \quad \text{for any} \quad s_2 \in S_2$$
$$u_3(A^1, B^2, s_3)=0 \quad \text{for any} \quad s_3 \in S_3 .$$

Now choose a payoff vector which is in V^* but is not in V, say (1/3, 1/3, 1). Let $s^1=(A^1, B^1, C^1)$ and $s^2=(A^2, B^2, C^2)$.

Define a supergame strategy $f^* \in F$ as follows;

$s(f^*)=\{s^1, s^2, s^2, s^1, s^1, s^1, s^2, s^2, s^2, s^2, s^2, s^2, \cdots\}$, and if player i deviates from this strategy, then from next period other two players take a strategy s_{-i} with $u_i(s_{-i}, s_i)=0$ for any $s_i \in S_i$. Then we have

$$p(f^*)=(1/3, 1/3, 1)$$

and f^* is a Nash equilibrium in G^*. See Fig. 3.2.1.

3.3. *The α-core in G and in G**

DEFINITION 3.3.1. A coalition $A \subset N$ is said to be *α-effective in G (in G*)* for the payoff vector v if there exist a cooperative strategy $s_A^* \in S_A$ ($f_A^* \in F_A$) for A, and a constant number $\epsilon^* > 0$ such that

$$u_i(s_A, s_{N\setminus N}) \geqq v_i + \epsilon^* \quad (p_i(f_A^*, f_{N\setminus A}) \geqq v_i + \epsilon^*) \quad \text{for every } i \in A$$

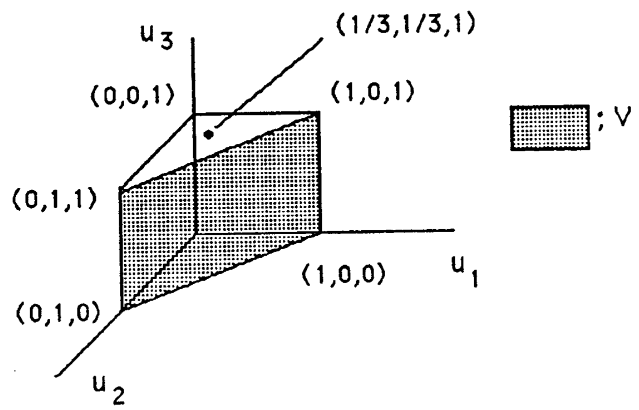

Fig. 3.2.1.

for any cooperative strategy $s_{N\setminus A}\in S_{N\setminus A}$ ($f_{N\setminus A}\in F_{N\setminus A}$) for the coalition $N\setminus A$.

LEMMA 3.3.1. *A coalition $A\subset N$ is α-effective in G^* for the payoff vector v if the coalition A is α-effective in G for the payoff vector v.*

Proof. Let A be α-effective in G for v. Then there exist a strategy $s_A^*\in S_A$ and a real number $\epsilon^*>0$ such that for any strategy $s_{N\setminus A}\in S_{N\setminus A}$,

$$u_i(s_A^*, s_{N\setminus A})\geqq v_i+\epsilon^* \quad \text{for every } i\in A .$$

Now define a supergame strategy f_A^* for the coalition A by

$$f_A=(s_A^*, s_A^*, \cdots) .$$

Then for any supergame strategy $f_{N\setminus A}$ for $N\setminus A$ and for any positive integer T, we have

$$p_i(T; f_A^*, f_{N\setminus A})\geqq v_i+\epsilon^* \quad \text{for every } i\in A .$$

Hence we have

$$p_i(f_A^*, f_{N\setminus A})\geqq v_i+\epsilon^* \quad \text{for every } i\in A .$$

Threfore the coalition A is α-effective in G^* for v. Q.E.D.

Remark 3.3.1. The converse of Lemma 3.3.1 is an open question. A difficulty lies in specifying a single-period strategy which makes the coalition α-effective in G.

DEFINITION 3.3.2. *A* payoff vector $v\in V$ ($v\in V^*$) is in the *α-core in G* (*in G^**) if no coalition is α-effective in G (in G^*) for the payoff vector v.

DEFINITION 3.3.3. A payoff vector $v\in V^*$ is in the *α-core in G^** if no coalition is α-effective in G^* or the payoff vector v.

Remark 3.3.2. By Lemma 3.3.1 we can say that, if some coalition is α-effective in G for a given payoff hence the payoff is not in the α-core in G, then the same coaltion is α-effective in G^* for the payoff, hence it can not be in the α-core in G^*. But in general Lemma 3.3.1 does not imply that the α-core in G^* is a subset of the α-core in G, since V does not coincide with V^*. Hence there can exist a payoff vector, which is in the α-core in G^* and is not in V hence is not in the α-core in G. See Example 3.2.1.

Example 3.2.1. Let $N=\{1, 2, 3\}$, $X_1=\{A^1, A^2\}$, $X_2=\{B^1, B^2\}$, $X_3=\{C^1, C^2\}$ and

$$u(A^1, B^1, C^1)=u(A^2, B^2, C^1)=(1, 0, 1)$$
$$u(A^1, B^2, C^2)=u(A^2, B^2, C^2)=(0, 1, 1)$$
$$u(A^1, B^2, C^1)=u(A^2, B^1, C^1)=(1, 0, 0)$$
$$n(A^1, B^2, C^2)=u(A^2, B^1, C^2)=(0, 1, 0)\ .$$

Then $V=co\{(1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1)\}$, and $V^*=co\{(1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1), (0, 0, 1), (0, 0, 0)\}$. A payoff vector $(1/3, 1/3, 1)$ is in V^*, and no coalition is α-effective in G^* for that payoff vector, hence it is in the α-core in G^*. But it is not in V, so is not in the α-core in G.

THEOREM 3.3.1. *The α-core in G^* is a subset of the α-core in G if V coincides with V^*.*

Proof. Let $v \in V$ be in the α-core in G^*, that is no coalition is α-effective in G^* for v. Then by Lemma 3.3.1, no coalition is α-effective in G for v. And by hypothesis *1* is an element of V. Hence v is in the α-core in G. Q.E.D.

Remark 3.3.3. Payoff vectors in the α-core in G are not necessarily Pareto optimal in G, and payoff vectors in the α-core in G^* are not necessarily Pareto optimal in G^*, though they are weakly Pareto optimal. See the following counter-example.

Example 3.3.1. Let $N=\{1, 2\}$, $X_i=\{E, T\}$, $i=1, 2$, and the payoff matrix be given by the following;

Player 1's choice

E	1, 1	0, 1
T	1, 0	0, 0
	E	T

Player 2's choice

Then the payoff vectors $(1, 0)$ and $(0, 1)$ are in the α-core in G since no coalition is α-effective in G for these payoff vectors. But they are not Pareto optimal in G since there exists the payoff vector $(1, 1)$. Similarly $(1, 0)$ and $(0, 1)$ are in the α-core in G^* since no coalition is α-effective in G^* for them. In this example V coincides with V^*, and the α-core in G coincides with the α-core in G^* and it is given by

$$[(1, 0), (1, 1)] \cup [(1, 1), (0, 1)]\ .$$

See Fig. 3.3.1.

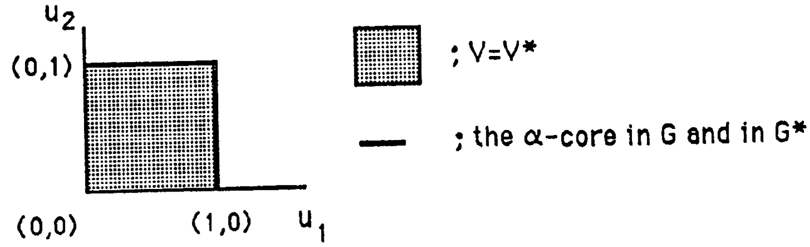

Fig. 3.3.1.

3.4. *The β-core in G and in G**

DEFINITION 3.4.1. A coalition $A \subset N$ is said to be *β-effective in G* (*in G**) for the payoff vector v if for any cooperative strategy $s_{N\setminus A} \in S_{N\setminus N}(f_{N\setminus N} \in F_{N\setminus A})$ there exists a cooperative strategy $s_A \in S_A$ for which

$$u_i(s_A, s_{N\setminus A}) \geqq v_i + \epsilon^*$$

$$(p_i(f_A, f_{N\setminus A}) \geqq v_i + \epsilon^*)$$

for some constant number $\epsilon^* > 0$, and for every $i \in A$.

LEMMA 3.4.1. *A coalition $A \subset N$ is β-effective in G for the payoff vector v if and only if the coalition A is β-effective in G* for the payoff vector v.*

Proof. Let A be β-effective in G for the payoff vector v. Then for each strategy $s_{N\setminus A}$ for $N\setminus A$, there exists a strategy s_A for A for which

$$u_i(s_A, s_{N\setminus A}) \geqq v_i + \epsilon^* \quad \text{for some constant } \epsilon^* > 0, \text{ and for every } i \in A .$$

Then for any supergame strategy $f_{N\setminus A} \in F_{N\setminus A}$, we can define a supergame strategy f_A for A as follows; at $t=1$, find a strategy $s_A(1)$ for which

$$u_i(s_A(1), f^1_{N\setminus A}) \geqq v_i + \epsilon^* \quad \text{for every } i \in A .$$

and at $t=2$, find a strategy $s_A(2)$ for which

$$u_i[s_A(2), f^2_{N\setminus A}(s_A(1), f^1_{N\setminus A}(1)] \geqq v_i + \epsilon^* \quad \text{for every } i\ A .$$

We can continue in this way; no matter $f_{N\setminus A}$ dictates, there exists a strategy $s_A(t)$ for A that yields at least $v_i + \epsilon^*$ at every period. Hence we have

$$p_i(f_A, f_{N\setminus A}) \geqq v_i + \epsilon^* \quad \text{for some constant } \epsilon^* > 0, \text{ and for every } i \in A .$$

Therefore the coalition A is β-effective in G^* for the payoff vector v.

On the other hand, let A be a coalition which is β-effective in G^* for v, that is, for any supergame strategy $f_{N\setminus A} \in F_{N\setminus A}$ for $N\setminus A$, there exists a supergame strategy $f_A \in F_A$ for A for which

$p_i(f_A, f_{N\setminus A}) \geqq v_i + \epsilon^*$ for some constant $\epsilon^* > 0$, and for every $i \in A$, that is

$$\liminf_T p_i(T; f_A, f_{N\setminus A}) \geqq v_i + \epsilon^* .$$

Hence for any $\epsilon > 0$, there exists a positive integer T^ϵ such that for any positive integer $T \geqq T^\epsilon$

$$p_i(T; f_A, f_{N\setminus A}) > v_i + \epsilon^* - \epsilon \quad \text{for every } i \in A .$$

Now let $s'_{N\setminus A} \in S^{t'}_{N\setminus A}$ be any arbitrary strategy for $N\setminus A$. Define a supergame strategy $f'_{N\setminus A}$ by $f^{t'}_{N\setminus A} \equiv s'_{N\setminus A}$. Let $\epsilon' = 1/2 \cdot \epsilon^*$. Then there exist a super game strategy $f'_A \in F_A$ and a positive integer $T^{\epsilon'}$ such that for any positive integer $T \geqq T^{\epsilon'}$

$p_i(T; f'_A, f'_{N\setminus A}) > v_i + \epsilon^* - \epsilon'$ for every $i \in A$, that is
$1/T \cdot \sum_{t=1}^T u_i[s_A(f'_A, f'_{N\setminus A}; t), s'_{N\setminus A}] > v_i + 1/2 \cdot \epsilon^*$, hence
$u_i[1/T \cdot \sum_{t=1}^T s_A(f'_A, f'_{N\setminus A}; t), s'_{N\setminus A}] > v_i + 1/2 \cdot \epsilon^*$.

Thus for any strategy $s'_{N\setminus A} \in S_{N\setminus A}$ there exists a strategy $s'_A \equiv 1/T \cdot \sum_{t=1}^T s_A(f'_A, f'_{N\setminus A}; t) \in S_A$, for which

$u_i(s'_A, s'_{N\setminus A}) \geqq v_i + \epsilon'$ for some constant number $\epsilon' > 0$ and for every $i \in A$.

Therefore the coalition A is β-effective in G for v. Q.E.D.

Remark 3.4.1. In any cooperative game, the α-effectiveness implies the β-effectiveness.

DEFINITION 3.4.2. A payoff vector $v \in V$ ($v \in V^*$) is in the *β-core in G (in G^*)* if no coalition is β-effective in G (in G) for the payoff vector v.

Remark 3.4.2. By Lemma 3.4.1 we can use the β-effective criterion in G to evaluate a payoff allocation in G^* in the sense that if no coalition is β-effective in G for a payoff vector in V^* then no coalition is β-effective in G^* for the payoff vector, hence it is in the β-core in G^*, or in the sensethat if some coalition is β-effective in G for a payoff vector then the same coalition is β-effective in G^* for the payoff vector, hence it can not be in the β-core in G^*. But Lemma 3.4.1 does not imply that the β-core in G coincides with the β-core in G^*, because V does not coincide with V^*. See Example 3.2.1. In that example no coalition is β-effective in G and in G^* for (1/3, 1/3, 1). Since (1/3, 1/3, 1) is an element of V^*, it is in the β-core in G^*, but it is not feasible in G, hence it can not be in the β-core in G.

Remark 3.4.3. In a 2-person game, the α-core coincides with the β-core. See Aumann [1961].

THEOREM 3.4.1. *The β-core in G is a subset of the β-core in G^*.*

Proof. Let $v \in V$ be in the β-core in G, that is no coalition is β-effective in G for v. Then by Lemma 3.4.1, no coalition is β-effective in G^* for v. And trivially v is an element of V^*. Hence v is in the β-core in G^*. Q.E.D.

THEOREM 3.4.2. *If V coincides with V^*, then the β-core in G coincides with the β-core in G^*.*

Proof. Let $v \in V^* = V$ be in the β-core in G^*, that is no coalition is β-effective in G^* for v. Then by Lemma 3.4.1, no coalition is β-effective in G for v. Hence v is in the β-core in G. Q.E.D.

COROLLARY 3.4.1. *Payoff vectors, which are in the β-core in G^* and the Pareto optimal in G^*, are in the β-core in G.*

Prood. Let $v \in V^*$ be a payoff vector which is in the β-core in G^* and is Pareto optimal in G^*. Then v is an element of V by Theorem 3.1.1. Since no coalition is β-effective in G^* for the payoff vector v, no coalition is β-effective in G by Lemma 3.4.1. Hence v is in the β-core in G. Q.E.D.

4. CONCLUDING REMARK

We have discussed fundamental notions in a single-period n-person game G and in the corresponding super game G^*, with a new definition of a payoff in G^*, namely the lim inf of average payoffs. If preference relations of players in a single-period game can not be represented by real valued functions, we can not use such a definition nor any other definitions with numerical representations of a single-period outcome. In that case, we have to derive a super game preference relation from a single-period preference relation in some reasonable way. This question deserves further research.

Keio University

REFERENCES

Aumann, R. J., "Acceptable Points in General Cooperative n-Person Games," in R. D. Luce and A. W. Tucker (Eds.), *Contributions to the Theory of Games* IV, Ann. of Math. Study 40 [1959], Princeton University Press.

———, "The Core of a Cooperative Game Without Side Payments," *Transactions of the American Mathematical Society* [1961], 539–552.

———, "A Survey of Cooperative Games Without Side Payments," in M. Shubik (Ed.), *Essays in Mathematical Economics In Honor of Oskar Morgenstern* [1967], Princeton University Press.

———, "Survey of Repeated Games," in R. J. Aumann et al. *Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern* [1981], B. I.-Wissenschaftsverlag.

Friedman, J. W., *Oligopoly and Theory of Games* [1977], North-Holland.

Ichiishi, T. "Non-cooperation and Cooperation," in M. Deistler, E. Fürst and G*i* Schwödiauer (Eds.), *Games, Economic Dynamics and Time Series Analysis* [1982], Physica-Verlag.

———, *Game Theory for Economic Analysis* [1983], Academic Press.

Luce, R. D. and D. Raiffa, *Games and Decisions* [1957], Wiley.

McMillan, J., "Individual Incentives in the Supply of Public Inputs," *Journal of Public Economics*, **12**, 87–98 [1979].

Nash, J., "The Bargaining Problem," *Econometrica*, **18**, 155–162 [1950].

———, "Non-Cooperative Games," *Ann. of Math.*, **54**, 286–295 [1951].

Owen, G., *Game Theory*, 2nd ed. [1982], Academic Press.

Rubinstein, A., "Equilibrium in Supergame with the Overtaking Criterion," *Journal of Economic*

Theory, **21**, 1–9 [1979].
Scarf, H., "On the Existence of a Cooperative Solution for a General Class of N-person games," *Journal of Economic Theory*, **3**, 169–181 [1971].