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CONTRIBUTION TO OLIGOPOLY THEORY: THE CASE OF UNCERTAIN COLLUSIONS

Zsuzsanna FLUCK, Koji OKUGUCHI and Ferenc SZIDAROVSZKY

Abstract. Several studies dealt with uncertainty on games from different points of view. Uncertainty of strategy sets, pay-off functions, and uncertainty in the number of players were investigated by many authors, but no one focused on uncertainty in players' collusion. To represent uncertainty certain deterministic and stochastic information formulas are used. We show how the strategy selection process of a certain player depends on the type of information he has. Then we provide solution concepts for different types of information and evaluate the value of information. The paper illustrates some examples on a simple one-product oligopoly model.

1. INTRODUCTION

The purpose of this work is to investigate how given information about collusion affects the players' strategy selection in oligopoly models. This part of the paper is devoted to summarize the theoretical background to be used later.

An oligopoly model is an n-person game concerned with an economic situation where the players are producers, which sell their products in a homogenous market. The simplest version can be demonstrated as follows.

Assume that n different firms produce the same commodity and sell it on the same market. Let L_k denote the capacity of firm k and $x_k \in [0, L_k]$ the production level of firm k. Since the market is homogenous and no time lag is assumed for the firms in entering the market, the unit price function f depends only on the total production level $\sum_k x_k$.

Moreover assume that the production cost of each firm depends on only the volume produced.

If $x_1, x_2, \cdots, x_n$ denote the strategies selected by the players then the profit of player k is given as

$$\varphi_k(x_1, x_2, \cdots, x_n) = x_k f\left(\sum_{i=1}^{n} x\right) - K_k(x_k)\ , \qquad (1)$$

where the unit price function $f\colon R \to R$ and the cost function $K_k\colon R \to R$. The Nash equilibrium point is a strategy vector $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ characterized by the properties that

(a) $\qquad x_k^* \in [0, L_k]\ , \qquad k = 1, 2, \cdots, n;$

$$\text{(b)} \quad \varphi_k(x_1^*, x_2^*, \cdots, x_{k-1}^*, x_k^*, \cdots, x_n^*) \geqq \varphi_k(x_1^*, x_2^*, \cdots, x_{k-1}^*, x_k, x_{k-1}^*, \cdots, x_n^*), \tag{2}$$

$$\forall x_k \in [0, L_k], \qquad k=1, 2, \cdots, n.$$

The Nash equilibrium point of an oligopoly model is called also as the Cournot equilibrium.

The monograph by Okuguchi (1976) gives a detailed background on the oligopoly model. Some related results can be found in Szidarovszky (1978). Under certain continuity and concavity assumptions Szidarovszky and Yakowitz (1977; 1982) proved the existence and uniqueness of the equilibrium points and introduced a numerical method for their determination. Szidarovszky (1978) provided a generalization to multiproduct market cases.

Since each player intends to maximize his payoff some may collude if grouping results in higher profit. This question leads to the theory of cooperative games. There are many concepts for solving cooperative game (see for example Szep and Forgo (1985)) and many papers deal with efficient and stable coalitions and with questions how to distribute the group profit among the members, etc. These latter questions will not be concerns of our study.

This work focuses on a special cooperation case which is called the group-equilibrium problem in which some players form groups and the coalitions strive for equilibrium themselves. Each player is supposed to be a member of one or more group usually, and an individual player is considered as a one-person group. In classical oligopoly models group equilibrium can be formulated as follows.

Let $N_1, N_2, \cdots, N_m$ be any disjoint partition of the set $N=\{1, 2, \cdots, n\}$ of players, then for $k=1, 2, \cdots, m$, N_k is considered as the set of firms collaborating in group k.

The strategy set of group k is given by

$$S_k = \underset{i \in N_k}{\times} [0, L_i], \tag{3}$$

and the payoff of this group can be represented in the following form:

$$\Big(\sum_{i \in N_k} x_i\Big) f\Big(\sum_{l=1}^{n} x_l\Big) - \sum_{i \in N_k} K_i(x_i). \tag{4}$$

One can easily see that the group-equilibrium case can be transformed into a purely competitive m-person oligopoly game by solving the following nonlinear optimization problems:

$$\left.\begin{array}{rl} & \min \sum_{i \in N_k} K_i(x_i) \\ \text{s.t.} & \sum_{i \in N_k} x_i = s_k \\ & x_i \in [0, L_k] \quad (i \in N_k) \end{array}\right\} k=1, 2, \cdots, m, \tag{5}$$

where s_k is a parameter from the interval $[0, \sum_{i \in N_k} L_i]$.

The optimally allocated cost function for group k is equal to the optimum and will be denoted by $Q_k(s_k)$ in the paper. Moreover define the m-person purely competitive oligopoly model by the same unit price function f, strategy sets $[0, \sum_{i \in N_k} L_i]$ $(k=1, 2, \cdots, m)$ and cost functions Q_k.

If $s_1^*, s_2^*, \cdots, s_m^*$ is the equilibrium point of the purely competitive m-person game and x_i^* $(i \in N_k)$ denotes the optimal solution of problem (5) at s_k^* $(k=1, 2, \cdots, m)$, then $(x_1^*, x_2^*, \cdots, x_n^*)$ is a group-equilibrium point. Furthermore any group-equilibrium point can be obtained this way.

In cooperative games the assumption that perfect information is available on players' collusion is generally in force. However, this assumption is rarely fulfilled in economic situations. One can easily ask the question how uncertainty or total lack of information about grouping affects the players' behaviour.

Our study intends to give an answer to this question. We shall represent how to formulate the strategy selection process for a certain player in case when the assumption of perfect information on collusion is disregarded. First we introduce some information concepts in order to present a clear formulation of the problem.

Let k be a certain player. Then k denotes either a group or an individual. Even if k is a group we many consider it as an individual, since our reduction theorem makes it possible to replace the group by an individual player. Thus the following consideration can be applied to both cases. If player k is not perfectly informed about the actual coalitions formed in the market, he can be either partially informed or totally uninformed. The case of no information can be regarded mathematically as a special case of partial information.

We distinguish two types of partial information: the partial deterministic and the partial stochastic ones. Deterministic information refers to an event that certainly occurs while stochastic information relates to one which is likely to happen. Both are partial in the sense that the available information concerns not with the whole market but only with some of the players.

In the case of partial deterministic information some subsets $M_1, M_2, \cdots, M_h \in N$ are given which are not to be further divided. In other words, we can say that for $j=1, 2, \cdots, h$, the players belonging to M_j certainly play together. It is also possible that the unions of certain sets M_j will give the actual collusions. Thus player k has to face with a great variety of possible coalition organisation.

The stochastic information case can be regarded as a generalization of the deterministic one.

In the case of stochastic information the players have some information about the probability of certain coalition formations. In other words, there is a given value p to subsets M_j $(j=1, 2, \cdots, h)$ which represents the probability that players belonging to M_j will play together. So we know some subsets M_j, but cannot be absolutely certain that the elements of a subset M_j will actually be the members of the same collusion, since the information is only probabilistic.

We assume that there is no further information for player k about the grouping in the market.

In the next sections we discuss above two cases of partial information in details.

2. DETERMINISTIC INFORMATION CASE

The analysis startes with the situation when a certain player does not know anything about the actual coalitions formed in the market. Then our player obtain information that some of the other players certainly play together and we formulate how this information affects his strategy selection.

Let $M_1, M_2, \cdots, M_h$ denote some subsets of N. As we previously stated the only thing that player k knows about other players' collusion is that subsets M_j ($j=1, 2, \cdots, h$) are not to be divided further during the game. In other words the actual collusions $N_1, N_2, \cdots, N_m$ will satisfy the following conditions:

(1) there exist a group i_0 such that $N_{i_0}=\{k\}$;

(2) for any $i\neq j$, $N_i \cap N_j=\phi$;

(3) $N_1 \cup N_2 \cup \cdots \cup N_m=N$;

(4) for any $i=1, 2, \cdots, m$ and $j=1, 2, \cdots, h$, $N_i \cap M_j$ is either the empty set or M_j

Let K be the set of all possible market situations $C=\{N_1, N_2, \cdots, N_m\}$ from the collusion's point of view satisfying the above conditions. Moreover let n-dimensional vector $x^*(C)$ denote a group-equilibrium point at market situation C and let $(n-1)$ dimensional vector $x^{*(k)}(C)$ denote the equilibrium strategies selected by the other players at group-equilibrium point $x^*(C)$. Then

$$x^*(C)=(x^{*(k)}(C), x_k^*(C)) . \tag{6}$$

We first assume that player k is the only one who does not know anything about what coalitions were actually formed in the market while the other players have perfect information on the subject. However none of them guesses that player k is an exception. Consequently player i, $i\neq k$ selects his equilibrium strategy of the perfect information case $x_i^{*(k)}(C)$. Player k has no doubt about the other players' behaviour but he can not make out in what market situation they play. Since he intends to select the strategy which provides him the highest profit even in the most pessimistic case, he maximizes his benefit as a minimum of his payoff function with respect to $x^{*(k)}(C)$, namely, he solves the optimization problem

$$\max \{\min_{C\in K} \varphi_k(x^{*(k)}(C), x_k) | x_k \in S_k\} \tag{7}$$

and selects the optimum solution as strategy. For the development to follow it is important to note that the optimum value in formula (7) is not equal to the characteristic function of player k. Even if no information is available and $h=n$ as well as $M_j=\{j\}$ ($j=1, 2, \cdots, h$) the two cases differ. The reason is that players belonging to $N-\{k\}$ do not select their strategies independently, but intend to reach the equilibrium situation.

Our next step is to analyze the players' behaviour in an oligopoly model under the information discussed above. Assume that the price function f is monotonically decreasing. Then

$$\varphi_k(x^{*(k)}(C), x_k)=x_k f(s^{*(k)}(C)+x_k)-K_k(x_k)$$

where (8)

$$s^{*(k)}(C)=\sum_{1\neq k} x_1^{*(k)}(C) \quad \text{and} \quad x^{*(k)}(C)=(x_1^{*(k)}(C))_{1\neq k}\ .$$

Introduce the notation $s_{\max}^{*(k)}$ according to the formula

$$s_{\max}^{*(k)}=\max_{C\in K}\{s^{*(k)}(C)\}\ . \tag{9}$$

Then (7) can be rewritten as follows:

$$\begin{aligned}&\max\ \{x_k f(s_{\max}^{*(k)}+x_k)-K_k(x_k)\}\\ &\text{s.t.}\quad x_k\in[0, L_k]\ .\end{aligned} \tag{10}$$

We close this section with an illustrative example.

Example 1. Consider a five-person oligopoly model defined by price function $f(s)=5-s$, strategy sets $L_k=5$ and cost functions $K_k(x_k)=kx_k/5+0.005$, for $k=1, 2, \cdots, 5$. Then the profit of player k is given by

$$\varphi_k(x_1, x_2, \cdots, x_5)=x_k(5-s)-kx_k/5-0.005\ , \tag{11}$$

where $s=\sum_{i=1}^{5} x_i$.

We shall study the model from the point of view of player 3 who is assumed to be an individual player and we shall calculate the group-equilibrium points in all possible market situations. There are fifteen different situations listed below:

1, 2, 3, 4, 5;	(1, 2), 3, 4, 5;	(1, 4), 2, 3, 5;	(1, 5), 2, 3, 4;	(2, 4), 1, 3, 5;
(2, 5), 1, 3, 4;	(4, 5), 1, 2, 3;	(1, 2), 3, (4, 5);	(1, 4), 3, (2, 5);	(1, 5), 3, (2, 4);
(1, 2, 4), 3, 5;	(1, 2, 5), 3, 4;	(1, 4, 5), 2, 3;	(2, 4, 5), 1, 3;	(1, 2, 4, 5), 3;

where numbers in parentheses indicate players belonging to the same groups. The equilibrium strategies and payoffs are shown in Table 1, respectively. The results were calculated by reducing the group equilibrium problem to a purely competitive oligopoly case and by using the numerical method introduced by Szidarovszky (1978).

First we assume that no information is available for player 3 concerning the other players' collusions. Then the total production level provided in the most pessimistic case by all other players is $s_{\max}^{*(k)}=2.9332$. Consequently, problem (10) can be rewritten in the following form:

$$\begin{aligned}&\max\ x_3(5-2.9332-x_3)-(0.6x_3+0.005)\\ &\text{s.t.}\quad 0\leq x_3\leq 5\ .\end{aligned} \tag{12}$$

TABLE 1. GROUP-EQUILIBRIUM POINTS

1; 2; 3; 4; 5				$s^{*(3)}=2.9332$	
x	1.1333	0.9333	0.7333	0.5333	0.3333
φ	1.2945	0.8785	0.5425	0.2865	0.1105
(1; 2); 3; 4; 5				$s^{*(3)}=2.56$	
x	1.32	0	0.92	0.72	0.52
φ	(1,2) 1.7324	—	0.8414	0.5134	0.2654
(1,4); 2; 3; 5				$s^{*(3)}=2.72$	
x	1.24	1.04	0.84	0	0.44
φ	(1,4) 1.5276	1.0766	0.7006	—	0.1886
(1,5); 2; 3; 4				$s^{*(3)}=2.8$	
x	1.2	1.0	0.8	0.6	0
φ	(1,5) 1.430	0.995	0.635	0.355	—
(2,4); 1; 3; 5				$s^{*(3)}=2.72$	
x	1.24	1.04	0.84	0	0.44
φ	1.5326	(2,4) 1.0716	0.7006	—	0.1886
(2,5); 3; 1; 4				$s^{*(3)}=2.8$	
x	1.2	1	0.8	0.6	0
φ	1.435	(2,5) 0.99	0.635	0.355	—
(4,5); 1; 2; 3				$s^{*(3)}=2.8$	
x	1.2	1	0.8	0.6	0
φ	1.435	0.995	0.635	(4,5) 0.350	—
(1,2); 3; (3,5)				$s^{*(3)}=2.3$	
x	1.45	0	1.05	0.85	0
φ	(1,2) 2.0925	—	1.0975	0.7125	—
(1,4); 3; (2,5)				$s^{*(3)}=2.5$	
x	1.35	1.15	0.95	0	0
φ	(1,4) 1.8125	(2,5) 1.3125	0.8975	—	—
(1,5); 3; (2,4)				$s^{*(3)}=2.5$	
x	1.35	1.15	0.95	0	0
φ	(1,5) 1.8125	(2,4) 1.3125	0.8975	—	—

(1,2,4); 3; 5				$s^{*(3)}=2.2$	
x	1.5	0	1.1	0	0.7
φ	(1,2,4) 2.2350	—	1.205	—	0.485

(1,2,5); 3; 4				$s^{*(3)}=2.3$	
x	1.45	0	1.05	0.85	0
φ	(1,2,5) 2.0875	—	1.0975	0.7175	—

(1; 4; 5); 2; 3				$s^{*(3)}=2.5$	
x	1.35	1.15	0.95	0	0
φ	(1,4,5) 1.8075	1.3175	0.8975	—	—

(2,4,5); 1; 3				$s^{*(3)}=2.5$	
x	1.35	1.15	0.95	0	0
φ	1.8175	(2,4,5) 1.3075	0.8975	—	—

(1,2,4,5); 3				$s^{*(3)}=1.7333$	
x	1.7333	0	1.3333	0	0
φ	(1,2,4,5) 2.9843	—	1.7726	—	—

The reader can easily confirm that the optimal solution is $x_3=0.7333$ and the optimum equals 0.5425.

Our next step is to demonstrate the case when the only information which player 3 obtained about the other players' collusion is the fact that player 1 and 5 certainly play together. This information implies that the following five of the fifteen possible market situations can actually happen:

(1, 5), 2, 3, 4; (1, 5), 3, (2, 4); (1, 2, 5), 3, 4; (1, 4, 5), 2, 3; (1, 2, 4, 5), 3.

Thus set K has now only five elements. On the basis of this information $s^*=2.8$, and now problem (10) is as follows:

$$\max x_3(5-2.8-x_3)-(0.6x_3+0.005)$$
$$\text{st.}\quad 0\leq x_3\leq 5\,.$$

The optimum is 0.635 and the optimal solution is $x_3=0.8$.

One can easily pose the further problem to evaluate the value of an adequate deterministic information. We shall answer the question in Section 3 after investigating the case of stochastic information.

3. STOCHASTIC INFORMATION CASE

In this paragraph we demonstrate how given stochastic information concerning other players' collusions affects the strategy selection of a certain player. At the end of this section we provide formula for evaluating the value of such information.

First we mention that all assumptions given in Section 2 are also in force during this section.

Moreover assume that player k received the information about subsets $M_1, M_2, \cdots, M_h$ and a value p expressing that players belonging to $M_j\,(j=1, 2, \cdots, h)$ play together at probability p.

Let K denote the set of possible market situations when these sets will not be divided into further parts, and let $\bar{K}$ denote the complement of K. Thus $P(K)=p$ and $P(\bar{K})=1-p$.

Since no further information is given to player k, he considers market situations in sets K and $\bar{K}$ to occur in uniform distribution.

Let $|K|$ and $|\bar{K}|$ denote the number of elements in sets K and $\bar{K}$, respectively. Then the probability that any market situation C occurs equals

$$P(C)=\begin{cases} \dfrac{p}{|K|} & \text{if } C\in K \\ \dfrac{1-p}{|\bar{K}|} & \text{if } C\in \bar{K}\,. \end{cases} \tag{13}$$

Thus a discrete probability measure is defined over the set of all possible market situations.

Assume that player k is controlled by the "Bayesian principle" (see Harsanyi (1967)), that is, he intends to select the strategy which provides the highest payoff as expected value. The "Bayesian behaviour" is represented by the following optimization problem:

$$\max_{C} \{E[\varphi_k(x^{*(k)}(C), x_k)]\}$$
$$\text{s.t.} \qquad x_k\in S_k \tag{14}$$

where $E[\varphi_k]$ is the expectation of the profit of player k with respect to the discrete probability distribution (13).

If absolutely no information is given to player k about other players' collusion, then he solves this problem under the assumption of uniform distribution over the set of all possible market situation.

In the case of an oligopoly model this problem can be rewritten as

$$\max_{C} \{x_k E[f(s^{*(k)}(C)+x_k)]-K_k(x_k)\}$$
$$\text{s.t.} \qquad x_k\in[0, L_k] \tag{15}$$

We present now an example for the stochastic information case.

Example 2. Consider the same situation as in Example 1 and study the behaviour of player k in the case of obtaining information that player 2 and 4 intend to join the same collusion with probability 0.2. Then grouping situations

$$(2, 4), 1, 3, 5; \quad (2, 4, 5), 1, 3; \quad (2, 4), 3, (1, 5); \quad (1, 2, 4, 5), 3; \quad (1, 2, 4), 3, 5;$$

belong to set K. The market situations not belonging to set K form the set $\bar{K}$. Thus $p=0.2$, $|K|=5$, $|\bar{K}|=10$. Consequently

$$P(C)=\begin{cases} 0.04 & \text{if} \quad C\in K \\ 0.08 & \text{if} \quad C\in\bar{K}\,. \end{cases}$$

Since f is a linear function, the expected value can be obtained in a simple manner. Thus the following optimization problem is derived:

$$\begin{aligned} &\max x_3(5-\bar{\mathfrak{z}}^{*(3)}-x_3)-(0.6x_3+0.005) \\ &\text{s.t.} \quad 0\leq x_3\leq 5\,, \end{aligned}$$

where expectation of the total production level $\bar{\mathfrak{z}}^{*(3)}$ of the other players equals 2.563188. We get the optimum at $x_3^*=0.918406$ and the optimum equals 0.8384696.

As a consequence of our investigation observe that concept (14) is applicable to all cases when discrete probability distributions can be defined over the set of all possible grouping situations according to the information available.

Now we represent how to determine the probability distribution in some cases, which are more complicated than the previous ones.

Consider first the case when both deterministic and stochastic informations are available. In other words, we assume that there are given subsets $M_1, M_2, \cdots, M_h$ and the probability p that players belonging to M_j $(j=1, 2, \cdots, h)$ play together. Furthermore, subsets $M_1', M_2', \cdots, M_{h'}'$ are given for which players belonging to M_j' $(j'=1, 2, \cdots, h')$ certainly collude.

Let K_0 denote the market situations consistent with the deterministic information while set K has the same meaning as before. Then the probability that any grouping situation C occurs is equal to

$$P(C)=\begin{cases} \dfrac{p}{|K\cap K_0|} & \text{if} \quad C\in K\cap K_0 \\ \dfrac{1-p}{|\bar{K}\cap K_0|} & \text{if} \quad C\in\bar{K}\cap K_0 \\ 0 & \text{if} \quad C\notin K_0\,. \end{cases}$$

Consider next the case when different type of stochastic information are available, and let p and p' be probabilities that players belonging to set M_j $(j=1, 2,$

$\cdots, h$) and to set $M'_{j'}$ ($j'=1, 2, \cdots, h'$), respectively, play together.

If we assume independence of the two types of information, then we conclude that the subset-system $M_1, M_2, \cdots, M_h, M'_1, M'_2, \cdots, M'_h$ occurs with probability $p \cdot p'$.

If independence does not hold, then *a* conditional probability $p'=P$ ($M'_1, M'_2, \cdots, M'_{h'} | M_1, M_2, \cdots, M_h$) is needed to take into account the combined information.

The final part of the paper is devoted to the determination of the value of stochastic and/or deterministic information.

Assume that the probability distribution of p over the set of all possible market situations is apriori known. Otherwise we can assume uniform distribution in [0, 1].

First we solve problem (15) with respect to p'. Let $\bar{\varphi}_k(p')$ denote the optimum. Moreover let $\bar{\bar{\varphi}}_k$ denote the optimum of problem (15) with respect to p only. Then the value of the stochastic information given by subsets $M'_1, M'_2, \cdots, M'_h$ and probability value p' is as follows:

$$E'_p[\bar{\varphi}_k(p')] - \bar{\bar{\varphi}}_k \tag{21}$$

Note that the stochastic result can be readily adapted to deterministic cases since all deterministic information formula can be rewritten as a stochatic one, where p or $p'=1$.

In the previous study all problems were discussed from the point of view of player k. However, several other interesting questions are important and to be investigated. The behaviour of a player, who intends to join a collusion, can be discussed or we can focus on information available either for the whole market or for some players and groups. It is also interesting to investigate how information about a group effect the profit of the group itself. If some players collude then they often intend to keep it in secret and gain a higher profit as a result. In addition, they may benefit by spreading out false information.

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