PARETO OPTIMALITY AND MONETARY COMPETITIVE EQUILIBRIUM IN THE OVERLAPPING GENERATIONS MODEL*

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Abstract: In this paper, the normative aspect of a monetary competitive equilibrium is studied in the overlapping-generations model. We will obtain the necessary and sufficient conditions under which (i) a monetary competitive equilibrium is Pareto optimal, and, (ii) a Pareto optimal allocation is a monetary competitive equilibrium.

1. INTRODUCTION

In the field of welfare economics, the following two 'Fundamental Theorems' are well known:

The First Theorem: Every Walrasian equilibrium is Pareto optimal.

The Second Theorem: Every Pareto optimal allocation is a Walrasian equilibrium.

These two theorems can be proved in a standard Arrow-Debreu model. This paper, however, attempts to study them in a different light. We will base our study on the overlapping-generations model and consider a monetary competitive equilibrium rather than a Walrasian equilibrium.

The first reason for making these modifications is that the First Theorem is known to fail in the overlapping-generations model, i.e. a Walrasian equilibrium does not necessarily yield a Pareto optimal allocation (Samuelson [7]). Although several studies have dealt with this subject (see, for example, Balasko and Shell [1] and Okuno and Zilcha [6]), it is still of interest to study an alternative set of conditions under which the First Theorem holds.

The second reason is that the overlapping-generations model is frequentry used as a model of a monetary economy, as it is the "only" natural, general-equilibrium type model in which money has a positive value without being an argument of the utility functions (see Kareken and Wallace [5]). It is thus important to know the properties of a monetary competitive equilibrium in the overlapping-generations model. Finally, because the overlapping-generations model is often used for evaluating monetary policy, it is also necessary to know the normative aspect of the model.

Thus, in this paper we will consider the necessary and sufficient conditions under which (i) a monetary competitive equilibrium is Pareto optimal, and, (ii) a Pareto optimal allocation is a monetary competitive equilibrium.

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This paper has several novel features. As for the First Theorem, although we follow the general practice of imposing conditions on the curvature of consumers' indifference surfaces, we use a new definition of curvature, i.e. a tangent hyperbola. This definition is a generalization of the one introduced by Borglin and Keiding [4]. In view of the hyperbola-like shape of indifference curves, this definition appears to be more natural than others. (A tangent circle and a tangent parabola are used in Balasko-Shell and Okuno-Zilcha respectively.) Moreover the characterization of a Pareto optimal allocation is given without assuming the boundedness of the allocation, which is assumed in both Balasko-Shell and Okuno-Zilcha. Our interpretation of the necessary and sufficient condition is also new.

As for the Second Theorem, it has already been proved that every Pareto optimal (or short-run Pareto optimal) allocation has a supporting price sequence (see [1]) under which the allocation can be attained as a Walrasian equilibrium. Whether or not this allocation can also be attained as a monetary competitive equilibrium (with a positive price of money) has, however, not yet been demonstrated. This paper, therefore, attempts to establish the conditions under which a Pareto optimal allocation is a *monetary* competitive equilibrium.

The model used here is the pure-exchange overlapping-generations model, in which agents possess perfect foresight. The amount of money is fixed in this study, but it is possible to modify the model so that the amount of money is variable.

The paper is organized as follows: Section 2 presents our model, notation, and assumptions. In Section 3, the definition of the curvature of an indifference surface using a tangent hyperbola is given. In Sections 4 and 5, the relationship between a monetary competitive equilibrium and a Pareto optimal allocation is studied. In Section 4, we focus on the conditions under which a monetary competitive equilibrium yields a Pareto optimal allocation, whereas, in Section 5, we study the conditions under which a Pareto optimal allocation is a monetary competitive equilibrium. Proofs are given in Section 6.

2. THE MODEL

The economy begins at period 1 and proceeds sequentially. Each period is denoted by t ($t=1, 2, \cdots$). Consumers are either present at the beginning of the economy and alive during period 1, or born at period t and alive during periods t and $t+1$ ($t=1, 2, \cdots$). For simplicity we assume that each generation consists of a single consumer, indexed by his birthdate, t ($t=0, 1, \cdots$). Therefore, in each period, the economy consists of two consumers, which we call "young" and "old" respectively.

There are L consumption goods available in each period. They are all perishable (i.e., they cannot be stored from one period to another) and no production takes place. There is also (fiat) money, which does not bring utility but which can be stored until the next period. The total amount of money is held constant over time.

The consumption set of consumer t is $X_0 \equiv R_+^L$ $(t=0)$ or $X_t \equiv R_+^{2L}$ $(t=1, 2, \cdots)$. Let $x_t^{s,i}$ be the consumption of commodity i $(i=1, 2, \cdots, L)$ by consumer t $(t=0, 1, \cdots)$ in period s $(s=1, 2, \cdots)$ and m_t be the purchase of money by consumer t $(t=0, 1, \cdots)$ in period t. We define

$$x_t^s=(x_t^{s,1}, x_t^{s,2}, \cdots, x_t^{s,L}),$$
$$x_0=x_0^1,$$
$$x_t=(x_t^t, x_t^{t+1}).$$

Consumer t $(t=0, 1, \cdots)$ has a preference relation on his consumption set, which is represented by the following utility function:

$$u_t: X_t \longrightarrow R.$$

We assume:

Assumption 1. The utility function, u_t $(t=0, 1, \cdots)$, is C^1, monotone and strictly quasi concave.

Assumption 2. Every indifference surface which passes through int X_t is contained in int X_t.

Assumption 2 excludes the possibility of a corner solution. Thus, in the following, we consider only strictly positive allocations, and we denote by X the set of all such allocations, i.e., $X \equiv \Pi_{t=0}^{\infty}$ (int X_t). Moreover, each consumer receives a strictly positive endowment of each commodity, which is denoted by

$$w_0=w_0^1=(w_0^{1,1}, w_0^{1,2}, \cdots, w_0^{1,L}) \in R_{++}^L,$$
$$w_t=(w_t^t, w_t^{t+1})=(w_t^{t,1}, w_t^{t,2}, \cdots, w_t^{t,L}, w_t^{t+1,1}, w_t^{t+1,2}, \cdots, w_t^{t+1,L}) \in R_{++}^{2L}$$
$$t=1, 2, \cdots.$$

Consumer 0 also has an endowment, $M>0$, of money.

At each date, only spot markets for current consumption goods and money are organized. Since each spot market is independent through time, we can normalize the prices in each period, and so we assume that the price of money equals one in each period. Let $p_{t,i}$ be the price of good i $(i=1, 2, \cdots, L)$ in period t $(t=1, 2, \cdots)$. We define p_t by

$$p_t=(p_{t,1}, p_{t,2} \cdots, p_{t,L}) \in R_{++}^L.$$

We denote by P the set of all price sequences, i.e., $P \equiv \Pi_{t=1}^{\infty} R_{++}^L$. Furthermore, we assume that all agents have perfect foresight.

We next consider the decision-making problem of the consumers who are present at period t $(t=1, 2, \cdots)$. For the "young" consumer, t, the problem is:

$$\begin{aligned} &\text{given} \quad (p_t, p_{t+1}) \in R_{++}^{2L} \\ &\text{maximize} \quad u_t(x_t^t, x_t^{t+1}) \\ &\text{subject to} \quad p_t x_t^t + m_t \leqq p_t w_t^t \\ &\qquad\qquad\quad p_{t+1} x_t^{t+1} \leqq p_{t+1} w_t^{t+1} + m_t. \end{aligned}$$

By Assumptions 1 and 2, this problem has a well-defined solution, which is denoted by the demand functions f_t^t, for current goods, and f_t^m for money;

$$f_t^t: R_{++}^{2L} \longrightarrow R_{++}^L$$
$$f_t^m: R_{++}^{2L} \longrightarrow R$$

For the "old" consumer, $t-1$, the problem is:

given $p_t \in R_{++}^L$, $x_{t-1}^{t-1} \in R_+^L$ and $m_{t-1}^{t-1} \in R_+$
maximize $u_{t-1}(x_{t-1}^{t-1}, x_{t-1}^t)$
subject to $p_t x_{t-1}^t \leqq p_t w_{t-1}^t + m_{t-1}^{t-1}$.

This problem also has a well-defined solution, which is denoted by the demand function, f_{t-1}^t, for current goods;

$$f_{t-1}^t: R_{++}^L \longrightarrow R_{++}^L.$$

DEFINITION 2.1. The set of *attainable allocations* is

$$A \equiv \{(x_0, x_1, \cdots) \in X \mid x_{t-1}^t + x_t^t = w_{t-1}^t + w_t^t,\ t=1, 2, \cdots\}$$

DEFINITION 2.2. The sequence of commodity prices $p=(p_1, p_2, \cdots) \in P$ and the allocations $x=(x_0, x_1, \cdots) \in X$ are called a *monetary competitive equilibrium*, (MCE), if each p_t, x_{t-1}^t and x_t^t $(t=1, 2, \cdots)$ satisfy the following conditions.

(2.3) $$x_{t-1}^t = f_{t-1}^t(p_t)\,,$$

(2.4) $$x_t^t = f_t^t(p_t, p_{t+1})\,,$$

(2.5) $$f_{t-1}^t(p_t) + f_t^t(p_t, p_{t+1}) = w_{t-1}^t + w_t^t\,,$$

(2.6) $$f_t^m(p_t, p_{t+1}) = M\,.$$

(2.3) and (2.4) represent the utility maximizing behavior of consumers $t-1$ and t respectively. (2.5) and (2.6) are the market clearing conditions for the goods markets and the money market.

DEFINITION 2.7. The allocation $x=(x_0, x_1, \cdots) \in A$ is *Pareto optimal* (PO), if there is no $y=(y_0, y_1, \cdots) \in A$ such that

$u_t(y_t) \geqq u_t(x_t)$, with at least one strict inequality, for $t=0, 1, \cdots$.

DEFINITION 2.8. The allocation $x=(x_0, x_1, \cdots) \in A$ is *short-run Pareto optimal* (SRPO), if there is no $y=(y_0, y_1, \cdots) \in A$ and $t' \geqq 0$ such that

$y_t = x_t$ for all $t \geqq t'$,

and if

$u_t(y_t) \geqq u_t(x_t)$, with at least one strict inequality for $t=0, 1, \cdots$.

LEMMA 2.9. If $x \in A$ is PO, then x is SRPO.

Proof. Obvious from the definition of PO and of SRPO.

Q.E.D.

EXAMPLE 2.10. Let $L=1$. Define the utility function of consumer t ($t=0, 1, \cdots$) by

$$u_0(x_0^1)=\log x_0^1 \qquad t=0\,,$$
$$u_t(x_t^t, x_t^{t+1})=\log x_t^t+\log x_t^{t+1} \qquad t=1, 2, \cdots$$

and let his endowment be

$$w_0^1=2 \quad \text{and} \quad M=1 \qquad t=0\,,$$
$$(w_t^t, w_t^{t+1})=(4, 2) \qquad t=1, 2, \cdots.$$

Then utility maximization implies

$$f_0{}^1(p_1)=\frac{2p_1+1}{p_1}\,,$$
$$f_t{}^t(p_t, p_{t+1})=\frac{2p_t+p_{t+1}}{p_t}\,,$$
$$f_t{}^{t+1}(p_{t+1})=\frac{1+2p_{t+1}}{p_{t+1}}\,,$$

and

$$f_t{}^m(p_t, p_{t+1})=2p_t-p_{t+1} \qquad t=1, 2, \cdots.$$

Therefore,

$$p=(p_1, p_2, \cdots, p_t, \cdots)$$
$$=(2, 3, \cdots, 2^{t-1}+1, \cdots)$$

and

$$x=(x_0^1, (x_1^1, x_1^2), \cdots, (x_t^t, x_t^{t+1}), \cdots)$$
$$=\left(\frac{5}{2}, \left(\frac{7}{2}, \frac{7}{3}\right), \cdots, \left(\frac{2^{t+1}+3}{2^{t-1}+1}, \frac{2^{t+1}+3}{2^t+1}\right), \cdots\right)$$

are a MCE.

This equilibrium, however, is not PO, because every consumer prefer y_t to x_t, where y_t is the t-th component of the attainable allocations,

$$y=(3, (3, 3), \cdots, (3, 3), \cdots)\,.$$

In section 4, we will consider this example in more detail.

3. THE CURVATURE OF THE INDIFFERENCE SURFACE

When we study the relationship between a MCE and a PO allocation, some conditions must be imposed on the curvature of the consumers' indifference surfaces, in order to obtain clear results. In this section, we will define the curvature of a consumer's indifference surface, and provide a geometric interpretation of it.

DEFINITION 3.1. Let $\bar{x}=(\bar{x}_1, \bar{x}_2)\in R_{++}^{2L}$, $\bar{q}=(\bar{q}_1, \bar{q}_2)\in R_{++}^{2L}$ and $a\geqq 0$. Define $C(\bar{x}, \bar{q}, a)$ by

$$C(\bar{x}, \bar{q}, a) \equiv \{(x_1, x_2) \in R_+^{2L} \mid \bar{q}_1(x_1-\bar{x}_1)+\bar{q}_2(x_2-\bar{x}_2) \geqq -\frac{a}{\bar{q}_1\bar{x}_1}(\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2)) \text{ and } \bar{q}_1(x_1-\bar{x}_1)+\bar{q}_2(x_2-\bar{x}_2) \geqq 0\}\,.$$

If $L=1$ and $a>0$, then, since the expression

$$\bar{q}_1(x_1-\bar{x}_1)+\bar{q}_2(x_2-\bar{x}_2) \geqq -\frac{a}{\bar{q}_1\bar{x}_1}(\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2))$$

is equivalent to

$$\left(x_1-\left(\bar{x}_1-\frac{\bar{x}_1}{a}\right)\right)\left(x_2-\left(\bar{x}_2-\frac{\bar{x}_1\bar{q}_1}{a\bar{q}_2}\right)\right) \geqq \frac{\bar{x}_1^2\bar{q}_1}{a^2\bar{q}_2}\,,$$

$C(\bar{x}, \bar{q}, a)$ can be illustrated as follows.

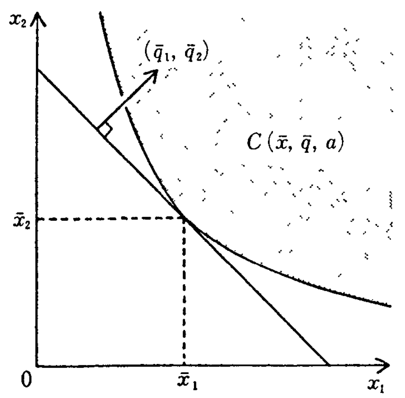

Fig. 1. Illustration of $C(\bar{x}, \bar{q}, a)$ when $L=1$ and $a>0$.

In other words, $C(\bar{x}, \bar{q}, a)$ is the convex hull of a hyperbola which passes through $(\bar{x}_1, \bar{x}_2)$ and whose tangent line at $(\bar{x}_1, \bar{x}_2)$ is perpendicular to $(\bar{q}_1, \bar{q}_2)$.

First, we note some basic properties of $C(\bar{x}, \bar{q}, a)$.

LEMMA 3.2. *For any* $\lambda>0$

$$C(\bar{x}, \lambda\bar{q}, a)=C(\bar{x}, \bar{q}, a).$$

Proof. Obvious from the definition of $C(\bar{x}, \bar{q}, a)$.

Q.E.D.

LEMMA 3.3. (a) *Let* $a_1 \geqq a_2 \geqq 0$. *Then* $C(\bar{x}, \bar{q}, a_1) \subset C(\bar{x}, \bar{q}, a_2)$.

(b) $C(\bar{x}, \bar{q}, a)$ *converges*[1] *to the set* $\{(x_1, x_2) \in R_+^{2L} \mid \bar{q}_1(x_1-\bar{x}_1)+\bar{q}_2(x_2-\bar{x}_2) \geqq 0\}$ *as* $a \to 0$.

(c) *When* $L=1$, $C(\bar{x}, \bar{q}, a)$ *converges to the set* $\{(x_1, x_2) \in R_+^2 \mid x_1 \geqq \bar{x}_1$ *and* $x_2 \geqq \bar{x}_2\}$ *as* $a \to \infty$.

Proof. See Section 6.

[1] See Section 6.

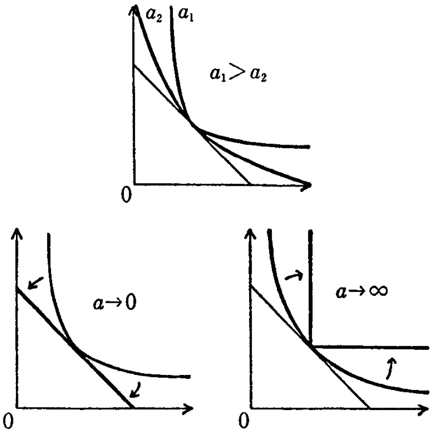

Fig. 2. The behavior of the set $C(\bar{x}, \bar{q}, a)$ when a is variable. (cf. Lemma 3.4)

We now work towards defining the curvature of the indifference surface. Let $t \in \{1, 2, \cdots\}$.

DEFINITION 3.4. The *upper contour set* of an allocation $\bar{x}_t \in X_t$ is

$$U(\bar{x}_t) \equiv \{x_t \in X_t \mid u_t(x_t) \geqq u_t(\bar{x}_t)\}$$

By assumption 1, $U(\bar{x}_t)$ is convex.

DEFINITION 3.5. Let $\bar{x}_t \in R^{2L}_{++}$. Then $q \in R^{2L}_{++}$ is called a *support* for $U(\bar{x}_t)$ at $\bar{x}_t$ if, for all $x_t \in U(\bar{x}_t)$, $q(x_t - \bar{x}_t) \geqq 0$.

By assumption 1, a support for $U(\bar{x}_t)$ at $\bar{x}_t$ exists and is uniquely determined up to scalar multiplications.

DEFINITION 3.6. Let $i, j \in \{1, 2, \cdots, L\}$.

$$C_{ij}(\bar{x}, \bar{q}, a) \equiv C(\bar{x}, \bar{q}, a) \cap \{(x_1, x_2) \in R^{2L}_{+} \mid x_{1k} = \bar{x}_{1k} \text{ for all } k \neq i \text{ and } x_{2n} = \bar{x}_{2n} \text{ for all } n \neq j\}$$

$$U_{ij}(\bar{x}_t) \equiv U(\bar{x}_t) \cap \{(x_1, x_2) \in R^{2L}_{+} \mid x_{1k} = \bar{x}_{1k} \text{ for all } k \neq i \text{ and } x_{2n} = \bar{x}_{2n} \text{ for all } n \neq j\}.$$

DEFINITION 3.7. Let $\bar{x}_t \in R^{2L}_{++}$ and let $\bar{q} \in R^{2L}_{++}$ be a support of $U(\bar{x}_t)$ at $\bar{x}_t$. Then the *outer curvature*, $a_t(\bar{x}_t)$, of $U(\bar{x}_t)$ at $\bar{x}_t$, is

$$a_t(\bar{x}_t) \equiv \sup \{a \mid U(\bar{x}_t) \subset C(\bar{x}_t, \bar{q}, a)\},$$

and the *inner curvature*, $b_t(\bar{x}_t)$, of $U(\bar{x}_t)$ at $\bar{x}_t$, is

$$b_t(\bar{x}_t) \equiv \min_{i,j \in \{1, \cdots, L\}} \inf \{a \mid C_{ij}(\bar{x}_t, \bar{q}, a) \subset U_{ij}(\bar{x}_t)\}.$$

The outer curvature at $\bar{x}_t$ is measured by the smallest tangent hyperbola (hyperboloid) at $\bar{x}_t$ which contains $U(\bar{x}_t)$, and the inner curvature at $\bar{x}_t$ is measured by the largest hyperbola at $\bar{x}_t$ which is contained in $U_{ij}(\bar{x}_t)$, for all $i, j \in \{1, 2, \cdots, L\}$.

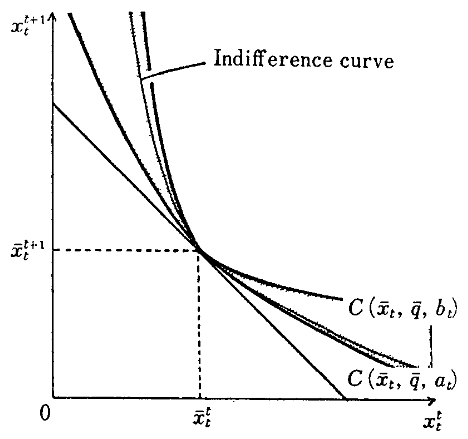

Fig. 3. Illustration of the outer and inner curvatures

This definition of the outer and inner curvatures is a generalization of the one introduced by Borglin and Keiding [4].

By Lemma 3.2 and the uniqueness of the support, q, up to scalar multiplications, a_t and b_t are independent of q. Lemma 3.3 ensures that these curvatures are well-defined. Moreover, by Assumption 1, a rather large class of economies satisfies $a_t>0$ and $b_t<\infty$.

As is seen in Fig. 3, this definition of the curvatures of the indifference surface is more natural than the one which uses a tangent circle (Balasko and Shell [1]) or a tangent parabola (Okuno and Zilcha [6]).

4. MONETARY COMPETITIVE EQUILIBRIUM AND PARETO OPTIMALITY: (THE FIRST THEOREM OF WELFARE ECONOMICS)

In this section, we will characterize the PO allocation with its support p, and prove the First Theorem of Welfare Economics in the overlapping-generations model.

DEFINITION 4.1. Let $x=(x_0, x_1, \cdots)\in X$ be an allocation. The price sequence $p=(p_1, p_2, \cdots)\in P$ is said to be a *support of* x if, for every t ($t=1, 2, \cdots$), (p_t, p_{t+1}) is a support for $U(x_t)$ at x_t.

LEMMA 4.2. *The following two propositions are equivalent*:
(i) x *is* SRPO.
(ii) *There exists a support*, $p\in P$, *of* x.

Proof. See Balasko and Shell [1], p. 292, Lemma 4.3.

By assumption 1, this support is uniquely determined up to scalar multiplications.

LEMMA 4.3. *Let* p, x *be a* MCE. *Then*, (i) x *is* SRPO *and*, (ii) p *is its support*.

Proof. See Section 6.

DEFINITION 4.4. Let $x=(x_0, x_1, \cdots)\in X$ be an allocation.
Condition A. There exists $\bar{a}>0$ such that $a_t(x_t)>\bar{a}$ for all $t=1, 2, \cdots$.
Condition B. There exists $\bar{b}<\infty$ such that $b_t(x_t)<\bar{b}$ for all $t=1, 2, \cdots$.

Condition A (resp. B) says that the curvature of every consumer's indifference surface at x is uniformly bounded from below (resp. above).

Let $\bar{x}=(\bar{x}_0, \bar{x}_1, \cdots)\in X$ be a SRPO allocation and $\bar{p}=(\bar{p}_1, \bar{p}_2, \cdots)\in P$ be its support. The existence of the support is ensured by Lemma 4.2.

PROPOSITION 4.5. *Assume that Condition A holds at $\bar{x}$. If*

$$\sum_{t=1}^{\infty}\frac{1}{\bar{p}_t\bar{x}_t^t}=\infty$$

then $\bar{x}$ is PO.

Proof. See Section 6.

PROPOSITION 4.6. *Assume that Condition B holds at $\bar{x}$. If $\bar{x}$ is* PO, *then*

$$\sum_{t=1}^{\infty}\frac{1}{\bar{p}_t\bar{x}_t^t}=\infty\,.$$

Proof. See Section 6.

THEOREM 4.7. *Assume that Conditions A and B hold at $\bar{x}$. Then $\bar{x}$ is* PO *if and only if*

$$\sum_{t=1}^{\infty}\frac{1}{\bar{p}_t\bar{x}_t^t}=\infty\,.$$

Proof. Follows from Proposition 4.5 and 4.6.

Q.E.D.

COROLLARY 4.8 (The First Theorem of Welfare Economics). *Let $\bar{p}$, $\bar{x}$ be a* MCE. *Assume that Conditions A and B hold at $\bar{x}$. Then $\bar{x}$ is* PO *if and only if*

$$\sum_{t=1}^{\infty}\frac{1}{\bar{p}_t\bar{x}_t^t}=\infty\,.$$

Proof. Follows from Lemma 4.3 and Theorem 4.7.

Q.E.D.

The condition

$$\sum_{t=1}^{\infty}\frac{1}{\bar{p}_t\bar{x}_t^t}=\infty$$

means that $\{\bar{p}_t\bar{x}_t^t\}$ does not diverge rapidly. Economically, this condition excludes the following two situations.

(i) $\{\bar{p}_t\}$ diverges rapidly, i.e., a spiral inflation goes on throughout the period.

(ii) $\{\bar{x}_t\}$ diverges rapidly, i.e., the economy grows rapidly.

In both situations, the existing money stock becomes less and less useful as a means of storing value, because, in situation (i), the price of money (fixed at one) becomes relatively lower and lower as the prices of goods keep on rising and thus the purchasing power of money decreases, and, in situation (ii), the amount of goods purchased keeps on rising while the amount of money is held constant (fixed at M) throughout the period, and thus the purchasing power of money decreases relatively.

Therefore, Corollary 4.8 shows that when the existing money stock fully carries out its function as a means of storing value, every MCE is PO.

EXAMPLE 4.9. Take the same economy as in Example 2.10. Considering the market clearing conditions, we obtain multiple equilibria, which are represented by the price sequences satisfying

$$p_{t+1}=2p_t-1$$
$$p_t>0 \quad \text{for all} \quad t=1, 2, \cdots .$$

For example, if $p_1=1$, it follows that $p_t=1$ for all $t=1, 2, \cdots, x_0^1=3$ and $x_t=(3, 3)$ for all $t=1, 2, \cdots$, are a MCE. Or, if $p_1=2$, then

$$p=(p_1, p_2, \cdots, p_t, \cdots)$$
$$=(2, 3, \cdots, 2^{t-1}+1, \cdots)$$

and

$$x=(x_0^1, (x_1^1, x_1^2), \cdots, (x_t^t, x_t^{t+1}), \cdots)$$
$$=\left(\frac{5}{2}, \left(\frac{7}{2}, \frac{7}{3}\right), \cdots, \left(\frac{2^{t+1}+3}{2^{t-1}+1}, \frac{2^{t+1}+3}{2^t+1}\right), \cdots\right)$$

are also a MCE. (This is the MCE of Example 2.10)

As for efficiency, the former allocation is PO, while the latter is not. In fact, if we choose $p_1>1$, no MCE is PO.

We can check this last claim as follows.

$$p_{t+1}=2p_t-1$$

means

$$p_{t+1}=2^{t-1}(p_1-1)+1 .$$

Since $p_1>1, p_t>1$ for all $t=1, 2, \cdots$, and thus $x_t^t \geqq 3$ for all $t=1, 2, \cdots$. Therefore

$$\sum_{t=1}^{\infty} \frac{1}{p_t x_t^t}=\sum_{t=1}^{\infty} \frac{1}{(2^{t-1}(p_1-1)+1)x_t^t}$$
$$<\sum_{t=1}^{\infty} \frac{1}{2^{t-1}(p_1-1)\cdot 3}$$
$$=\frac{1}{3(p_1-1)} \sum_{t=1}^{\infty} \frac{1}{2^{t-1}}$$
$$=\frac{2}{3(p_1-1)}<\infty .$$

Hence, by Corollary 4.8, no MCE is PO when $p_1>1$.

EXAMPLE 4.10. When the curvature of the indifference surface is not bounded, our characterization of PO allocations breaks down. Suppose $L=1$, $M=1$ and

$$\begin{aligned} u_0(x_0)&=x_0^1 && t=0\,,\\ u_t(x_t)&=x_t^t+x_t^{t+1} && t=1, 2, \cdots,\\ w_0&=2 && t=0\,,\\ w_t&=(2, 2) && t=1, 2, \cdots. \end{aligned}$$

Then

$$\begin{aligned} p_t&=1 \qquad \text{for all} \quad t=1, 2, \cdots,\\ x_0&=3\,, \end{aligned}$$

and

$$x_t=(1, 3) \qquad \text{for all} \quad t=1, 2, \cdots$$

are a MCE, and this allocation is not PO. (Consider $y=(4, (0, 4), \cdots, (0, 4), \cdots)$.) However

$$\sum_{t=1}^{\infty}\frac{1}{p_t x_t^t}=\infty\,.$$

5. MONETARY COMPETITIVE EQUILIBRIUM AND PARETO OPTIMALITY: (THE SECOND THEOREM OF WELFARE ECONOMICS)

In this section, we will state the Second Theorem of Welfare Economics in the overlapping-generations model. In contrast to the previous section, we do not need to assume that the curvature conditions hold at all. By Lemma 4.2 (and Lemma 2.9), we know that there exists a supporting price for any PO allocation. Thus our main concern is to find a way of redistributing the initial endowments (and introducing money at period 1) so that any PO allocation can be attained as a MCE using a supporting price.

THEOREM 5.1 (The Second Theorem of Welfare Economics). *Let $\bar{x}\in X$ be a* PO *allocation and $\bar{p}\in P$ be its support. Then $\bar{x}$ can be attained as a* MCE *if and only if*

$$\inf_t \bar{p}_t\bar{x}_{t-1}^t=\varepsilon>0\,.$$

Proof. See Section 6.

When $\bar{x}$ is attained as a MCE, $\bar{p}$ becomes an equilibrium price and $\bar{p}_t\bar{x}_{t-1}^t$ indicates the nominal income which consumer $t-1$ has at period t. Therefore the condition

$$\inf_t \bar{p}_t\bar{x}_{t-1}^t=\varepsilon>0$$

means that the "old" consumer has at least ε units of income at each period.

EXAMPLE 5.2. Consider the same utility functions as in Example 2.10. Let

$$\begin{aligned} x_0^1&=2\,,\\ x_t&=(1, 2) \qquad t=1, 2, \cdots. \end{aligned}$$

Then it can easily be shown that x is SRPO and that

$$p=(p_1, p_2, \cdots, p_t, \cdots)$$
$$=\left(1, \frac{1}{2}, \cdots, \frac{1}{2^{t-1}}, \cdots\right)$$

is its support price. Since

$$\sum_{t=1}^{\infty} \frac{1}{p_t x_t^t} = \sum_{t=1}^{\infty} 2^{t-1} = \infty ,$$

x is PO (by Theorem 4.7). However, since

$$\inf_t p_t x_{t-1}^t = 0 ,$$

x cannot be attained as a MCE.

6. PROOFS

Proof of Lemma 3.3. (a) Let $(x_1, x_2) \in C(\bar{x}, \bar{q}, a_1)$. By the definition of $C(\bar{x}, \bar{q}, a_1)$, we obtain

$$\bar{q}_1(x_1-\bar{x}_1)+\bar{q}_2(x_2-\bar{x}_2) \geqq -\frac{a_1}{\bar{q}_1\bar{x}_1}(\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2)) \tag{6.1}$$

and

$$\bar{q}_1(x_1-\bar{x}_1)+\bar{q}_2(x_2-\bar{x}_2) \geqq 0 . \tag{6.2}$$

Case 1. $\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2)<0$
Since $a_1 \geqq a_2$,

$$-\frac{a_1}{\bar{q}_1\bar{x}_1}(\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2)) \geqq -\frac{a_2}{\bar{q}_1\bar{x}_1}(\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2)) .$$

Hence (by (6.1))

$$\bar{q}_1(x_1-\bar{x}_1)+\bar{q}_2(x_2-\bar{x}_2) \geqq -\frac{a_2}{\bar{q}_1\bar{x}_1}(\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2)) .$$

Thus

$$(x_1, x_2) \in C(\bar{x}, \bar{q}, a_2) .$$

Case 2. $\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2) \geqq 0$
Since

$$0 \geqq -\frac{a_2}{\bar{q}_1\bar{x}_1}(\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2)) ,$$

we obtain (by (6.2))

$$\bar{q}_1(x_1-\bar{x}_1)+\bar{q}_2(x_2-\bar{x}_2) \geqq -\frac{a_2}{\bar{q}_1\bar{x}_1}(\bar{q}_1(x_1-\bar{x}_1)\cdot\bar{q}_2(x_2-\bar{x}_2)) .$$

Thus

$$(x_1, x_2) \in C(\bar{x}, \bar{q}, a_2).$$

Q.E.D.

(b) and (c): At first we define some notation. Let $d(x, y)$ be the Euclidian metric on R^{2L} and A and B be non empty sets in R^{2L}. Define $d(x, B)$, $\rho(A, B)$ and $h(A, B)$ by

$$d(x, B) \equiv \inf_{y \in B} d(x, y)$$
$$\rho(A, B) \equiv \sup_{x \in A} d(x, B)$$
$$h(A, B) \equiv \max(\rho(A, B), \rho(B, A)).$$

Let

$$C_0 \equiv \{(x_1, x_2) \in R_+^{2L} \mid \bar{q}_1(x_1 - \bar{x}_1) + \bar{q}_2(x_2 - \bar{x}_2) \geqq 0\}$$

and

$$C_\infty \equiv \{(x_1, x_2) \in R_+^2 \mid x_1 \geqq \bar{x}_1 \text{ and } x_2 \geqq \bar{x}_2\}.$$

Then we can state the exact meaning of this lemma:

(b) $h(C(\bar{x}, \bar{q}, a), C_0) \to 0$ as $a \to 0$,

(c) $h(C(\bar{x}, \bar{q}, a), C_\infty) \to 0$ as $a \to \infty$.

Proof of (b): Since $C(\bar{x}, \bar{q}, 0) = C_0$ and $C(\bar{x}, \bar{q}, a)$ is a continuous correspondence of a, $h(C(\bar{x}, \bar{q}, a), C_0) \to 0$ as $a \to 0$.

Q.E.D.

Proof of (c): Since $L=1$, the boundary of $C(\bar{x}, \bar{q}, a)$ is a hyperbola whose asymptotes are

$$x_1 = \bar{x}_1 - \frac{\bar{x}_1}{a},$$

and

$$x_2 = \bar{x}_2 - \frac{\bar{x}_1 \bar{q}_1}{a \bar{q}_2}.$$

Therefore, as $a \to \infty$, these asymptotes approach $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$, and $h(C(\bar{x}, \bar{q}, a), C_\infty) \to 0$.

Q.E.D.

Proof of Lemma 4.3. (i) Suppose x is not SRPO. Then by the definition of SRPO, there exists $t' \geqq 0$ and $y = (y_0, y_1, \cdots)$ such that

(6.3) $\quad y_{t-1}^t + y_t^t = x_{t-1}^t + x_t^t \quad$ for all $t = 1, 2, \cdots$,

(6.4) $\quad y_t = x_t \quad$ for all $t \geqq t'$,

(6.5) $\quad u_t(y_t) \geqq u_t(x_t) \quad$ with at least one strict inequality for $t = 0, 1, \cdots$.

Thus, from (6.5),

$$p_1 y_0^1 \geqq p_1 x_0^1\,,$$
$$p_t y_t^t + p_{t+1} y_t^{t+1} \geqq p_t x_t^t + p_{t+1} x_t^{t+1} \qquad \text{for all} \quad t=1, 2, \cdots,$$

and there is at least one strict inequality. Therefore, considering (6.4), we obtain

$$p_1 y_0^1 + p_1 y_1^1 + \cdots + p_{t'-1} y_{t'-1}^{t'-1} + p_{t'} y_{t'-1}^{t'}$$
$$> p_1 x_0^1 + p_1 x_1^1 + \cdots + p_{t'-1} x_{t'-1}^{t'-1} + p_{t'} x_{t'-1}^{t'}\,.$$

This contradicts (6.3).

(ii) Since x_t maximizes $u_t(x_t)$ under the constraints

$$p_t x_t^t + m_t \leqq p_t w_t^t \qquad \text{and} \qquad p_{t+1} x_t^{t+1} \leqq p_{t+1} w_t^{t+1} + m_t\,,$$

(p_t, p_{t+1}) is a support of $U(x_t)$ at x_t. Therefore p is a support of x.

Q.E.D.

Before proving Propositions 4.5 and 4.6, we have to prove two lemmas.

DEFINITION 6.6. Let $x \in X$ be an allocation. A sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots) \in \Pi_{t=1}^{\infty} R^L$ is called *Pareto-improving upon* x if

$$u_0(x_0^1 + \varepsilon_1) \geqq u_0(x_0^1)$$

and

$$u_t(x_t^t - \varepsilon_t,\ x_t^{t+1} + \varepsilon_{t+1}) \geqq u_t(x_t^t,\ x_t^{t+1})\,,$$
$$\text{with at least one strict inequality for} \quad t=0, 1, \cdots.$$

ε_t can be interpreted as the amount of commodities "transfered" by consumer t to consumer $t-1$. Clearly an allocation x is PO if and only if there is no Pareto-improving sequence upon x.

LEMMA 6.7. *Let x be* SRPO *and ε be Pareto-improving sequence upon x. If t' denotes the smallest t $(t=1, 2, \cdots)$ such that $\varepsilon_t \neq 0$, then $\varepsilon_t \neq 0$ for $t=t', t'+1, \cdots$.*

Proof.[2] Assume that there is some $s > t'$ such that $\varepsilon_s = 0$. Then

$$u_0(x_0^1 + \varepsilon_1) \geqq u_0(x_0^1)\,,$$
$$\vdots$$
$$u_{s-1}(x_{s-1}^{s-1} - \varepsilon_{s-1},\ x_{s-1}^s) \geqq u_{s-1}(x_{s-1}^{s-1},\ x_{s-1}^s)$$
$$u_s(x_s^s,\ x_s^{s+1} + \varepsilon_{s+1}) \geqq u_s(x_s^s,\ x_s^{s+1})$$
$$\vdots\ .$$

Then the allocation $x' \equiv (x_0^1 + \varepsilon_1, (x_1^1 - \varepsilon_1, x_1^2 + \varepsilon_2), \cdots, (x_{s-1}^{s-1} - \varepsilon_{s-1}, x_{s-1}^s), (x_s^s, x_s^{s+1}), (x_{s+1}^{s+1}, x_{s+1}^{s+2}), \cdots)$ is attainable, $x' \neq x$ and

$$u_t(x_t') \geqq u_t(x_t) \qquad \text{for all} \quad t=0, 1, \cdots.$$

By the strict quasi concavity of u_t,

[2] See Balasko-Shell [1], p. 295, Lemma 5.4.

$$u_t\left(\frac{x_t'+x_t}{2}\right)\geqq u_t(x_t) \qquad \text{with at least one strict inequality for} \quad t=0, 1, \cdots .$$

Since

$$x_t'=x_t \qquad \text{for all} \quad t\geqq s\,,$$

this is a contradiction to the hypothesis that x is SRPO.

Q.E.D.

LEMMA 6.8. *Let $x\in X$ be* SRPO, *$p\in P$ be its support and ε be Pareto-improving upon x. Then we have the following inequalities* ($t=1, 2, \cdots$)

$$0\leqq p_1\varepsilon_1\leqq \cdots \leqq p_t\varepsilon_t\leqq \cdots ,$$

with the inequalities being strict for $t\geqq t'$, where t' is defined as in Lemma 6.7.

Proof.[3] Since $u_0(x_0^1+\varepsilon_1)\geqq u_0(x_0^1)$,

$$p_1(x_0^1+\varepsilon_1)\geqq p_1x_0^1 .$$

Thus

$$p_1\varepsilon_1\geqq 0 .$$

If $\varepsilon_1\neq 0$, by the strict quasi concavity of u_0,

$$p_1\varepsilon_1>0 .$$

For $t\geqq 1$, since

$$u_t(x_t^t-\varepsilon_t,\, x_t^{t+1}+\varepsilon_{t+1})\geqq u_t(x_t^t,\, x_t^{t+1})\,,$$
$$p_t(x_t^t-\varepsilon_t)+p_{t+1}(x_t^{t+1}+\varepsilon_{t+1})\geqq p_tx_t^t+p_{t+1}x_t^{t+1} .$$

Thus

$$p_t\varepsilon_t\leqq p_{t+1}\varepsilon_{t+1} .$$

If $\varepsilon_t\neq 0$, by the strict quasi concavity of u_t,

$$p_t\varepsilon_t<p_{t+1}\varepsilon_{t+1} .$$

By Lemma 6.7, the proof is complete.

Q.E.D.

Proof of Proposition 4.5. Assume that $\bar{x}$ is not PO. Then there exists a Pareto-improving sequence ε upon $\bar{x}$. By the definition of ε

$$(\bar{x}_t^t-\varepsilon_t,\, \bar{x}_t^{t+1}+\varepsilon_{t+1})\in U(\bar{x}_t) \qquad t=1, 2, \cdots$$

and from the assumption of this proposition (and Lemma 3.3), we have

$$(\bar{x}_t^t-\varepsilon_t,\, \bar{x}_t^{t+1}+\varepsilon_{t+1})\in C(\bar{x}_t, (\bar{p}_t, \bar{p}_{t+1}), \bar{a}) .$$

Hence

$$-\bar{p}_t\varepsilon_t+\bar{p}_{t+1}\varepsilon_{t+1}\geqq \frac{\bar{a}}{\bar{p}_t\bar{x}_t^t}\bar{p}_t\varepsilon_t\cdot\bar{p}_{t+1}\varepsilon_{t+1} \qquad t=1, 2, \cdots .$$

[3] See Balasko-Shell [1], p. 295, Lemma 5.5.

Letting $\delta_t=\bar{p}_t\varepsilon_t$, $t=1, 2, \cdots$, and choosing t' such that $\varepsilon_{t'}\neq 0$ (so that $\delta_t\neq 0$ for all $t\geqq t'$ (Lemma 6.8)), we have

$$\frac{1}{\delta_t}-\frac{1}{\delta_{t+1}}\geqq\frac{\bar{a}}{\bar{p}_t\bar{x}_t^t}\qquad t\geqq t'\,.$$

Summing this from $t=t'$ to T ($T\geqq t'$) and cross-cancelling,

$$\sum_{t=t'}^{T}\frac{\bar{a}}{\bar{p}_t\bar{x}_t^t}=\frac{1}{\delta_{t'}}-\frac{1}{\delta_{T+1}}<\frac{1}{\delta_{t'}}\,.$$

Hence

$$\sum_{t=1}^{\infty}\frac{1}{\bar{p}_t\bar{x}_t^t}<\infty\,,$$

which contradicts the hypothesis of this proposition.

Q.E.D.

Proof of Proposition 4.6. Assume

$$B\equiv\sum_{t=1}^{\infty}\frac{1}{\bar{p}_t\bar{x}_t^t}<\infty\,.$$

Then

$$\lim_{t\to\infty}\bar{p}_t\bar{x}_t^t=\infty\,.$$

Hence we can find $t'>1$ such that, for all $t\geqq t'$, there exists $i(t)\in\{1, 2, \cdots, L\}$ such that

$$\bar{p}_{t,i(t)}\bar{x}_t^{t,i(t)}\geqq 1\,.$$

Define ε_t ($t=1, 2, \cdots$) by

$$\varepsilon_t=0\qquad\text{for}\quad t<t'$$

$$\varepsilon_{t'}=\frac{1}{\bar{p}_{t',i(t')}(1+\bar{b}B)}$$

and

$$\frac{1}{\bar{p}_{t+1,i(t+1)}\varepsilon_{t+1}}=\frac{1}{\bar{p}_{t,i(t)}\varepsilon_t}-\frac{\bar{b}}{\bar{p}_t\bar{x}_t^t}\qquad\text{for}\quad t\geqq t'$$

recursively. Since for any $t>t'$

$$\begin{aligned}\frac{1}{\bar{p}_{t+1,i(t+1)}\varepsilon_{t+1}}&=\frac{1}{\bar{p}_{t',i(t')}\varepsilon_{t'}}-\sum_{j=t'}^{t}\frac{\bar{b}}{\bar{p}_j\bar{x}_j^j}\\&=1+\bar{b}\sum_{j=1}^{\infty}\frac{1}{\bar{p}_j\bar{x}_j^j}-\sum_{j=t'}^{t}\frac{\bar{b}}{\bar{p}_j\bar{x}_j^j}\quad\text{(by the definition of }\varepsilon_{t'})\\&\geqq 1\,,\end{aligned}$$

ε_{t+1} is well-defined and

$$0\leqq\bar{p}_{t,i(t)}\varepsilon_t\leqq 1\qquad\text{for all}\quad t=1, 2, \cdots\,.$$

Define $\tilde{x}_t^t$ and $\tilde{x}_t^{t+1}$ by

$$\tilde{x}_t^t = \bar{x}_t^t - \varepsilon_t e_{i(t)}$$

and

$$\tilde{x}_t^{t+1} = \bar{x}_t^{t+1} + \varepsilon_{t+1} e_{i(t+1)}$$

where $e_{i(t)}$ is a $i(t)$-th unit vector. Then

$$\begin{aligned}
&\bar{p}_t(\tilde{x}_t^t - \bar{x}_t^t) + \bar{p}_{t+1}(\tilde{x}_t^{t+1} - \bar{x}_t^{t+1}) \\
&\quad = \bar{p}_t(-\varepsilon_t e_{i(t)}) + \bar{p}_{t+1}(\varepsilon_{t+1} e_{i(t+1)}) \\
&\quad = \bar{p}_{t+1,i(t+1)}\varepsilon_{t+1} - \bar{p}_{t,i(t)}\varepsilon_t \\
&\quad \geqq \frac{\bar{b}}{\bar{p}_t \bar{x}_t^t} \bar{p}_{t,i(t)}\varepsilon_t \cdot \bar{p}_{t+1,i(t+1)}\varepsilon_{t+1} \quad \text{(from the definition of } \varepsilon\text{)} \\
&\quad = -\frac{\bar{b}}{\bar{p}_t \bar{x}_t^t} \bar{p}_t(\tilde{x}_t^t - \bar{x}_t^t) \cdot \bar{p}_{t+1}(\tilde{x}_t^{t+1} - \bar{x}_t^{t+1})
\end{aligned}$$

(Notice that $\tilde{x}_t^{t,j} = \bar{x}_t^{t,j}$ for all $j \neq i(t)$ and $\tilde{x}_t^{t+1,j} = \bar{x}_t^{t+1,j}$ for all $j \neq i(t+1)$) .

Therefore

$$\tilde{x}_t \in C(\bar{x}_t, (\bar{p}_t, \bar{p}_{t+1}), \bar{b}) \subset U(\bar{x}_t) \quad \text{(by Condition B).}$$

Also, since

$$0 \leqq \bar{p}_{t,i(t)}\varepsilon_t \leqq 1 \leqq \bar{p}_{t,i(t)}\bar{x}_t^{t,i(t)} \qquad \text{for} \quad t \geqq t',$$

we have

$$0 \leqq \varepsilon_t \leqq \bar{x}_t^{t,i(t)} .$$

Thus $\tilde{x}$ is attainable. This contradicts the Pareto optimality of $\bar{x}$.

Q.E.D.

Proof of Theorem 5.1. (Sufficiency) Since, for any $\alpha > 0$, $\alpha\bar{p}$ is a support of $\bar{x}$, we can assume without loss of generality that

$$\inf_t \bar{p}_t \bar{x}_{t-1}^t = 1 .$$

Define the redistribution of initial endowments as follows. For any t ($t=1, 2, \cdots$) choose $w_{t-1}^t \geqq 0$ so that $\bar{x}_{t-1}^t \geqq w_{t-1}^t$ and $\bar{p}_t w_{t-1}^t = \bar{p}_t \bar{x}_{t-1}^t - 1 \geqq 0$, and define w_t^t by

$$w_t^t = \bar{x}_t^t + (\bar{x}_{t-1}^t - w_{t-1}^t) \qquad \text{for} \quad t=1, 2, \cdots .$$

Then

$$\bar{p}_1 w_0^1 + 1 = \bar{p}_1 \bar{x}_0^1 ,$$

and

$$\bar{p}_t w_t^t + \bar{p}_{t+1} w_t^{t+1} = \bar{p}_t \bar{x}_t^t + \bar{p}_{t+1} \bar{x}_t^{t+1} .$$

Hence $\bar{x}$ can be attained as a MCE.

(Necessity) Suppose

$$\inf_t \bar{p}_t \bar{x}^t_{t-1} = 0$$

and $\bar{x}$ can be attained as a MCE. Then, in any period t ($t=1, 2, \cdots$), the "old" consumer, $t-1$, has $M>0$ units of money and the price of money is assumed to be one. Hence consumer $t-1$ has at least M units of income in each period, i.e.,

$$\bar{p}_t \bar{x}^t_{t-1} \geqq M > 0 \qquad \text{for all} \quad t=1, 2, \cdots,$$

which contradicts $\inf_t \bar{p}_t \bar{x}^t_{t-1} = 0$.

Q.E.D.

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