Title	ON SOME IMPLICATIONS OF THE SEPARATING HYPERPLANE THEOREM
Sub Title	
Author	MUKHERJI, Anjan
Publisher	Keio Economic Society, Keio University
Publication year	1985
Jtitle	Keio economic studies Vol.22, No.2 (1985.) ,p.57- 71
JaLC DOI	
Abstract	The paper aims at unifying the various mathematical tools used in the analysis of linear economic models. Using the theorem of the Separating Hyperplane as stated in Gale's 'The Theory of Linear Economic Models', McGraw-Hill, 1960, a wide class of results are obtained: the Frobenius Theorem, the various alternative forms of the productivity conditions for a Leontief Matrix, the non-substitution theorem, and properties of the singular jacobian of excess demand functions under gross substitution. The concept of a dominant diagonal matrix plays an important role in the entire analysis.
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19850002-0057

ON SOME IMPLICATIONS OF THE SEPARATING HYPERPLANE THEOREM*

Anjan MUKHERJI

Abstract. The paper aims at unifying the various mathematical tools used in the analysis of linear economic models. Using the theorem of the Separating Hyperplane as stated in Gale's 'The Theory of Linear Economic Models', McGraw-Hill, 1960, a wide class of results are obtained: the Frobenius Theorem, the various alternative forms of the productivity conditions for a Leontief Matrix, the non-substitution theorem, and properties of the singular jacobian of excess demand functions under gross substitution. The concept of a dominant diagonal matrix plays an important role in the entire analysis.

INTRODUCTION

In economic theory, one often encounters non-negative matrices or matrices with all off diagonal elements non positive (B-matrices) and usually, one enquires into the nature of characteristic roots and associated characteristic vectors or whether certain linear equations involving these matrices have non-negative solutions. The primary source of results in this connection appears to be the Frobenius Theorem; that this result is considered to be of central importance, is perhaps best reflected by the fact that almost every text book dealing with mathematical economics contains a section or an appendix dealing with the topic, e.g. [1], [8], [12], [13]. Besides, there have been several papers, e.g. [3], [10], [11], each providing an alternative proof of the Frobenius Theorem.

An alternate source of similar results, according to Mckenzie [9] is Hadamard's Theorem which states that a matrix with a dominant diagonal must be non singular. Then there is the indigenous source—which uses the Frobenius Theorem and/or Hadamard's Theorem and/or various tedious properties of determinants and cofactors to establish some of these results. There is also the work of Gale [5] which begins with the Separating Hyperplane Theorem (SHT) and establishes some results in this connection; but the properties of characteristic roots and vectors find no place in Gale's analysis.

We shall show that the entire analysis of these class of problems may be based on the Separating Hyperplane Theorem alone. The importance of this result to economic theory cannot be overstressed and with our analysis, it occupies a central role in this area as well. Specifically, the route which we traverse may be seen as

* Helpful comments from the referee are gratefully acknowledged.

follows:

$$
\begin{array}{lll}
 & \quad \text{III} & \\
 & \Rightarrow \text{Hardamard's Theorem} & \Rightarrow \text{Properties of the Gross} \\
\text{SHT} & & \quad \text{Substitute System VIII} \\
 & \Rightarrow \text{Properties of } B\text{-matrices} & \Rightarrow \text{Frobenius Theorem IV, V} \\
 & & \text{Productivity Conditions} \\
 & & \quad \text{in the Leontief Model VI} \\
 & & \text{Non Substitution Theorem VII}
\end{array}
$$

To put our analysis in the proper perspective, let us compare the above scheme to the analysis in [3], [5] and [9]. In [3], IV is first established; then the relevant properties of B-matrices in III are demonstrated. In [5], beginning with SHT, the author straightway arrives at VI and goes on to VII for the case when each sector has a finite number of activities; but the conditions involving characteristic roots find no place in this analysis. In [9], the point of departure is III; these are used to demonstrate VI and in conjunction with IV, V, the stability of the Gross substitute system is demonstrated.

We show that B-matrices with dominant positive diagonal provide the common underlying structure and the SHT, the basic mathematical tool. Moreover, in obtaining an unified approach, the proofs of the various propositions become quite simple and elementary.

II. THE SEPARATING HYPERPLANE THEOREM

The starting point of this analysis is Gale's [5] Theorem 2.6:
Exactly one of the following alternatives hold:

$$
\begin{array}{ll}
\text{Either} & xA = b \text{ has a nonnegative solution} \\
\text{or} & Ay \geqq 0 \;\; by < 0 \text{ have a solution}\,.
\end{array}
\qquad \text{(SH)}
$$

It should be pointed out that the notation (and terms): $x > 0$ (positive), $x \geq 0$ (semi-positive) and $x \geqq 0$ (non-negative) are to be interpreted as in [5]. The following corollaries of (SH) would be frequently used; these are Theorems 2.9 and 2.10 [5, pages 48–49].

$$
\begin{array}{ll}
\text{Either} & xA = 0 \text{ has a semipositive solution} \\
\text{or} & Ay > 0 \text{ has a solution}
\end{array}
\qquad \text{(SH.1)}
$$

$$
\begin{array}{ll}
\text{Either} & xA \leqq 0 \text{ has a semipositive solution} \\
\text{or} & Ay > 0 \text{ has a nonnegative solution}
\end{array}
\qquad \text{(SH.2)}
$$

In each of the above cases, exactly one of the stated alternative hold.

III. MATRICES WITH DOMINANT DIAGONALS

A square matrix $A=(a_{ij})$, $i,j=1,2,\cdots,n$ is said to have a *column dominant diagonal* (c.d.d.) if there exist $d_j>0, j=1,2,\cdots,n$ such that

$$d_j|a_{jj}|>\sum_{i\neq j} d_i|a_{ij}|, \qquad j=1,2,\cdots,n. \tag{1}$$

Let $B_A=(b_{ij})$ be defined by

$$\begin{aligned} b_{ij}&=-|a_{ij}| && i\neq j \\ &=|a_{jj}| && \text{otherwise}. \end{aligned}$$

Then A has a c.d.d. iff

$$dB_A>0 \qquad \text{for some} \quad d\geqq 0. \tag{1a}$$

A is said to have a *row dominant diagonal* (r.d.d.) if and only if A^T has a c.d.d. i.e., there exist $c_j>0, j=1,2,\cdots,n$ such that

$$c_i|a_{ii}|>\sum_{j\neq i}|a_{ij}|c_j, \qquad i=1,2,\cdots,n \tag{2}$$

or alternatively

$$B_A\cdot c>0 \qquad \text{for some} \quad c\geqq 0 \tag{2a}$$

We show first that

L1. *A has a c.d.d. iff A has a r.d.d.*

Proof. If A has a c.d.d. then (1) and (1a) hold; in particular,

$$dB_A>0 \qquad \text{for some} \quad d^*>0.$$

Suppose to the contrary that $B_A\cdot c>0$ has a nonnegative solution. Then by (SH.2), $xB_A\leqq 0$ for some $x^*\geq 0$. Thus $(d^*-tx^*)B_A>0$ for any scalar $t>0$. In particular, consider $t^*=\min_{j\in x_j^*>0} d_j^*/x_j^*>0$. For such a choice, $(d^*-t^*x^*)B_A>0$ cannot hold. Hence no such x^* exists so that (2a) holds and A has a r.d.d. For the converse, apply the above to A^T.

Hence one need not distinguish between r.d.d. and c.d.d. and we shall refer to matrices as having a *dominant diagonal* (d.d.) if any one of (1), (1a), (2) or (2a) holds. L1 establishes that if any one of these holds, then so do the others. The fundamental property of such matrices is given by (Hardamard's Theorem):

L2. *If A has a d.d. then A is non-singular.*

Proof. By (1a) and (SH.2), $B_A\cdot y\leqq 0$ has no semipositive solution. Thus, if A is singular, $Az=0$ for some $z\neq 0$; i.e.

$$-a_{ii}\cdot z_i=\sum_{j\neq i} a_{ij}z_i \qquad \text{for all} \quad i$$

or

$$|a_{ii}| \cdot |z_i| = \left| \sum_{j=i} a_{ij} z_i \right| \leqq \sum_{j \neq i} |a_{ij}| |z_j|$$

or writting

$$y_i = |z_i|, \quad y = (y_i), \quad y \geq 0 \quad \text{and} \quad B_A \cdot y \leqq 0: \quad \text{a contradiction}.$$

Hence the claim.

Matrices such as B_A would play an important role in our analysis. We shall refer to matrices such as these viz., with non-positive off diagonal entries, by the term *B-matrices*. For a B-matrix, $B=(b_{ij})$, the above considerations imply: B has a d.d. with $b_{ii}>0$ ($+ve$ d.d.) iff

$$Bd > 0 \qquad \text{for} \quad d \geqq 0. \tag{3}$$

Moreover for any B-matrix, $B=(b_{ij})$.

L3. *$Bx=c$ has an unique nonnegative solution for any $c \geqq 0$ iff B has a $+ve$ d.d.*

Proof. If B has a $+ve$ d.d., then (3) implies $B^J \cdot d^J > 0$ for any $J \subseteq \{1, 2, \cdots, n\}$, $B^J = (b_{ij})$, $i, j \in J$; $d^J = (d_j)$, $j \in J$. Thus by (SH.2),

$$y^J \cdot B^J \leqq 0 \quad \text{has a no semipositive solution}. \tag{4}$$

Consider y^* such that $y^* B \geqq 0$. If possible, let $J = \{j : y_j^* < 0\} \neq \Phi$. Then $y^{*J} \cdot B^J \geqq 0$ and $y^{*J} < 0$ violates (4) and hence $J = \Phi$. Thus $y^* B \geqq 0 \Rightarrow y^* \geqq 0$. Consequently $yB \geqq 0$ and $yc < 0$ can have no solution whenever $c \geqq 0$. Hence by (SH) $Bx = c$ has a nonegative solution for any $c \geqq 0$. The solution is unique by virtue of L2. The converse is trivial, given (3).

L4. *For a B-matrix, $B^{-1} \geqq 0$ iff B has a $+ve$ d.d.*

The proof is trivial given L3 and hence omitted.

L5. *For a B-matrix, B has a $+ve$ d.d. iff*

$$\det B_k = \det \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ b_{k1} & \cdots & b_{kk} \end{pmatrix} > 0, \qquad k = 1, 2, \cdots, n \tag{HS}$$

Proof. The proof follows by virtue of L3 and the central result in Hawkins and Simon [6].

The above results will be shown to be central to not only the theory of linear economic models but to the Perron-Frobenius Theorem as well.

IV. NONNEGATIVE MATRICES AND THE FROBENIUS THEOREM

Let $A \geqq 0$ be a square non-negative matrix of order n and $F(A) = [\pi : \pi I - A$ has a

$+ve$ d.d.]. For any π, $\pi I - A$ is a B-matrix and hence the results of the last section apply. Moreover, for any $A \geqq 0$, $F(A) \neq \Phi$: since any π larger than the maximum column sum of A must be an element of $F(A)$. Aslo $\pi \in F(A) \Rightarrow \pi \geqq 0$. Hence:

$$\pi_A^* = \operatorname*{Inf}_{\pi \in F(A)} \pi$$

exists. We shall drop the subscript and write π^* whenever the context makes it clear. First of all,

L6. $\pi^* \notin F(A)$, $\pi^* \geqq 0$; $\pi > \pi^* \Rightarrow \pi \in F(A)$. *Thus* $F(A) = (\pi^*, +\infty)$. *The claim follows from* (3) *and the properties of the infimum.*

Moreover,

L7. π^* *is a characteristic root* (*c.r.*) *of* A. $|\alpha| \leqq \pi^*$ *for any other c.r. of* A.

Proof. Let c be a positive column vector ($n \times 1$). Construct an $n \times (n+1)$ matrix whose first n columns are the columns of $\pi^* I - A$ and the last column is $-c$. Denote this matrix by

$$[\pi^* I - A, \ -c]\,.$$

Note that $y[\pi^* I - A, \ -c] > 0$ can have no solution; for if a solution $\bar{y}$ exists, then

$$\bar{y}(\pi^* I - A) > 0 \quad \text{and} \quad \bar{y} \cdot c < 0$$

so that for some $\sigma > 0$

$$\bar{y}((\pi^* + \sigma)I - A) \geqq 0 \quad \text{and} \quad \bar{y} \cdot c < 0 \ \Rightarrow \ \text{by SH that} \quad ((\pi^* + \sigma)I - A)x = c$$

can have no nonnegative solution $\Rightarrow$ by L3 that $\pi^* + \sigma \notin F(A)$: contrary to L6. Thus no such $\bar{y}$ exists. Hence by (SH.1) there is $\left[\begin{smallmatrix} x^* \\ \alpha^* \end{smallmatrix}\right]$ semipositive where x^* is $n \times 1$ and α^* a scalar such that

$$[\pi^* I - A, \ -c]\left[\begin{smallmatrix} x^* \\ \alpha^* \end{smallmatrix}\right] = 0$$

or

$$(\pi^* I - A)x^* = \alpha^* c$$

$$\because \ \alpha^* \neq 0 \Rightarrow \pi^* I - A$$

has a $+ve$ d.d. and $\pi^* \in F(A)$: contrary to L6, we conclude $\alpha^* = 0$. Hence $x^* \geq 0$ and $(\pi^* I - A)x^* = 0$ so that π^* is a c.r. of A.

For any other c.r. α let z be its associated characteristic vector (c.v.). Then $Az = \alpha z$ and in particular

$$|\alpha| \cdot |z_i| = \left|\sum_j a_{ij} z_j\right| \leqq \sum_j a_{ij} |z_i| \qquad \text{for all} \quad i$$

or

$$(|\alpha|I-A)y \leqq 0 \quad \text{where} \quad y_i=|z_i|, \quad \text{and so} \quad y \geq 0 .$$

In case $|\alpha|>\pi^*$, $|\alpha| \in F(A)$ whence by L4,

$$(|\alpha|I-A)^{-1} \geqq 0 \quad \text{so that} \quad y \leqq 0: \quad \text{a contradiction} .$$

Thus $|\alpha| \leqq \pi^*$.

The following fact, obtained in the above proof, is noted separately:

L8. *There is $x^* \geq 0$ such that $Ax^*=\pi^* x^*$.*

L9. *$(\pi I-A)^{-1} \geqq 0$ iff $\pi>\pi^*$. Further $L_{ij}(\pi) \geqq 0$ for $\pi \geqq \pi^*$ where $L_{ij}(\pi)$ is the cofactor of the i, j-th element in $(\pi I-A)$.*

Proof. Follows from L4 by virtue of L6. Writing $L(\pi)$ for $\det(\pi I-A)$; the j, i-th element of $(\pi I-A)^{-1}$ is $L_{ij}(\pi)/L(\pi)$. $L(\pi)>0$ whenever $\pi>\pi^*$ (L5); and so $L_{ij}(\pi) \geqq 0$ for $\pi>\pi^*$ and hence $L_{ij}(\pi^*) \geqq 0$.

L10. *If $A \geqq C \geqq 0$ then $\pi_A^* \geqq \pi_C^* \geqq 0$. Moreover if C is any principal minor of A, then $\pi_A^* \geqq \pi_C^*$.*

Proof. Follows from noting that $F(A) \subseteq F(C)$, and from the definition of π^*. If C is a principal minor of A of order r; fill out the remaining $(n-r)$ rows and columns to obtain D, a matrix of order n. Then

$$D \leqq A \quad \text{and} \quad \pi_D^* \leqq \pi_A^* \quad \text{and} \quad \pi_D^*=\pi_C^* .$$

Collecting the above results, we have shown that for any non-negative matrix A

(i) there is $\pi^* \geqq 0$ such that π^* is a c.r. of A and for any other c.r. α, $|\alpha| \leqq \pi^*$ (L7)

(ii) $(\pi I-A)^{-1} \geqq 0$ iff $\pi>\pi^*$ (L9)

(iii) there are $x^* \geq 0$ and $p^* \geq 0$ such that

$$Ax^*=\pi^* x^*$$

$$p^*A=\pi^* p^* \quad \text{(L8 and application to } A^T) .$$

(iv) $Ay \geqq \alpha y$, $y \geq 0 \Rightarrow \pi^* \geqq \alpha$ (Proof of L7).

It may be of some interest to note that some authors e.g. [8, page 247] use (iv) to define π^*. The above set of results is referred to as the Perron-Frobenius Theorem e.g. [13, page 102].

V. INDECOMPOSABILITY

If in addition to A being nonnegative, one requires that A be indecomposable, then the results of the last section are considerably strengthened. Let $I=[1, 2, \cdots, n]$. If there is some non-empty proper subset J of I such that

$$a_{ij}=0 , \qquad i \notin J \quad \text{and} \quad j \in J$$

then A is defined to be *decomposable*. If no such J exists, then A is said to be *indecomposable*.

L11. *If $A \geqq 0$ is indecomposable, then*
(i) $\pi^* > 0$.
(ii) x^* *(and p^*)* > 0.
(iii) *Any other c.v. corresponding to π^* is a scalar multiple of x^* (or p^*).*
(iv) *No other c.r. has an associated c.v. which is non-negative.*
(v) *If C is such that $A \geqq C \geqq 0$ then $\pi_A^* > \pi_C^*$ if $C \neq A$.*
(vi) *If C is a principal minor of A, $C \neq A$ then $\pi_A^* > \pi_C^*$.*
(vii) $(\pi I - A)x \geq 0$ *for* $x \geq 0 \Rightarrow (\pi I - A)^{-1} > 0$,
and
(viii) *π^* is a simple root of the characteristic equation of A.*

Proof. See Nikaido [13, page 107]. A simpler demonstration of (viii) is as follows: α is a c.r. of $B = \pi^* I - A$ iff $(\pi^* - \alpha)$ is a c.r. of A. The characteristic equation of B is $\pi^n - \pi^{n-1} \cdot (\sum_j b_{jj}) + \cdots + (-1)^{n-1}$ (sum of det of all p.r. minors of B of order $n-1$) $\pi + (-1)^n \det B = 0$. By virtue of (vi) and (L9), $L_{jj}(\pi^*) > 0$ for all j; so that the last but one term cannot vanish and hence 0 is a simple root of the above equation.

VI. A PRODUCTIVE LEONTIEF MODEL

Assume that there are n-sectors of production; each sector j produces only an output j by using as inputs the outputs of other sectors and a single nonproduced factor traditionally identified with labour. For each sector, there is an unique method of producing its output and suppose (a_j, b_j) is required to produce one unit of j; here, a_j stands for the vector of produced commodities required as inputs and b_j, a scalar, represents the amount of labour required. Let A denote the matrix whose j-th column is a_j.

The viability of the above system of production lies in its ability to produce a surplus (over and above its input requirements) of each commodity. Thus A is said to be *productive* [5, page 296] if

$$(I - A)x > 0 \qquad \text{for some} \quad x \geqq 0 \tag{P1}$$

$x_j \geqq 0$, is the activity level or equivalently, the gross output for sector j. Ax then stands for the intersectoral input requirements to produce x and so $x - Ax$ is the surplus. Note that $(I - A)$ is a B-matrix and hence the results of Section III are applicable. We use these results to restate (P1) in several equivalent forms.

A is productive

iff $(I - A)$ has a $+ve$ d.d. [from (P1) and (3)] (P2)

iff for any $c \geqq 0$, $(I - A)x = c$ has an unique nonnegative solution

[follows from (P2) and L3] (P3)

iff $(I-A)^{-1} \geqq 0$ [follows from (P3) and L4] (P4)

iff $\pi_A^* < 1$ [follows from (P4) and L9] (P5)

iff the Hawkins-Simon Conditions hold [follows from (P2) and L5] (P6)

iff for *some* $c > 0$, $(I-A)x = c$ has an unique nonnegative solution (P7)

iff *all* principal minors of $(I-A)$ have positive determinants (P8)

Notice that (P3) $\Rightarrow$ (P7) $\Rightarrow$ (P2) $\Rightarrow$ (P3). Also (P8) $\Rightarrow$ (P6); whereas (P6) $\Leftrightarrow$ (P2) $\Rightarrow B^J$ has a $+ve$ d.d. for every J, where $B = (I-A)$, $J \subseteq [1, 2, \cdots, n]$, $B^J = (b_{ij})$, $i, j \in J$ and hence $\det B^J > 0$ by (L5) $\Rightarrow$ (P8).

One may also note the following conditions as well:

$$(\text{P1}) \Rightarrow \sum_i a_{ij} < 1 \qquad \text{for some} \quad j$$

and

$$\sum_i a_{ij} < 1 \qquad \text{for every} \quad j \Rightarrow (\text{P1}) \tag{P9}$$

which is a trivial consequence of (P2).

VII. NON-SUBSTITUTION THEOREM

Consider the production model of the last section; except instead of the unique configuration of inputs (a_j, b_j) required to produce one unit of j, we now assume that there is a collection T_j of such processes for each j. We shall insist that

(I) T_j is a closed subset of R_{n+1}^+ and $(a_j, b_j) \in T_j \Rightarrow b_j > 0$.

When each sector chooses a process, say j chooses $(\bar{a}_j, \bar{b}_j)$ from T_j, the $\bar{a}_j$'s form a Leontief matrix $\bar{A}$ whose j-th column is $\bar{a}_j$. For each configuration of choice, a separate Leontief matrix becomes applicable. For the model to be viable, we shall also insist that

(II) There is $(a_j^*, b_j^*) \in T_j$ such that $A^* = (a_j^*)$ is productive.

Let

$$p^*(I-A^*) = b^* \qquad \text{where} \quad b^* = (b_j^*) \tag{4}$$

Thus, we have normalised prices by taking the wage rate as unit. In this set up, it is meaningful to enquire into the question of choice of processes by each sector. Given that labour is the only primary factor it is reasonable to expect each sector to minimize its labour costs; and if there is a set of processes one for each sector, which minimises these costs, then this set of processes may be chosen regardless of what surplus has to be generated. Thus even though substitution possibilities exist, no substitution may occur: the non-substitution theorem.

Let L denote the quantity of labour available; then given

$$(a_j, b_j) \in T_j\,, \quad j=1,2,\cdots,n\,, \quad A=(a_j)\,, \quad b=(b_j)\,,$$
$$U(A,b)=[y \geqq 0 : y \leqq (I-A)x,\ bx \leqq L,\ x \geqq 0]\,.$$

We shall write $(A, b) \in T$ whenever $A=(a_j)$, $b=(b_j)$ and $(a_j, b_j) \in T_j$ for each j. If these is $(\hat{A}, \hat{b}) \in T$ such that for all $(A, b) \in T$, $U(A, b) \subseteq U(\hat{A}, \hat{b})$, then $(\hat{A}, \hat{b})$ has the non-substitution property [7]. What Johansen [7] did not clinch was the existence of such an $(\hat{A}, \hat{b})$; see, for example, Dasgupta [2], who presented an argument to establish the existence. A more direct demonstration is presented below; in the process, the crucial role of the productivity conditions (P1)–(P9) stand revealed.

We begin by observing that

A. *If* $(a_j^s, b_j^s) \in T_j$, $s=1,2,\cdots$ *and* $b_j^s \to 0$ *then* $a_{kj}^s \to +\infty$ *for some* k.

Proof. For, if not, then (a_j^s, b_j^s) forms a bounded sequence and must have a limit *pt.* $(\bar{a}_j, 0)$ and which must be contained in T_j: a contradiction.

Let $P=[p:\ 0 \leqq p \leqq p^*$ and for each p there is some $(A, b) \in T$ such that $p(I-A)=b]$.

In the definition of P, p^* is as in (4) above. Hence $p^* \in P$. Thus P is nonempty and bounded. Let p^s, $s=1,2,\cdots$ be a sequence in P; without any loss of generality, assume $p^s \to p^0$ as $s \to \infty$. Since $p^s \in P$, for all s, there exist $(A^s, b^s) \in T$ such that

$$p^s(I-A^s)=b^s\,.$$

Thus by virtue of (P2), A^s is productive for all s. Moreover,

B. $a_{ij}^s \to +\infty$ *for some* $i, j \Rightarrow p_i^s \to 0$. *This is immediate since*

$$p_j^* \geqq p_j^s = \sum_{k=1}^{n} p_k^s a_{kj}^s + b_j^s \geqq b_j^s > 0 \tag{5}$$

C. *P is compact.*

Proof. This would be established by showing that $p^0 \in P$. If possible let $J=[j: p_j^0=0] \neq \Phi$. Then by virtue of (5), B, and A,

$$\begin{aligned} j \in J &\Rightarrow b_j^s \to 0 \\ &\Rightarrow a_{kj}^s \to +\infty \qquad \text{for some} \quad k \\ &\Rightarrow k \in J \quad (\because\ p_k^s \to 0)\,. \end{aligned}$$

or

$$j \in J \Rightarrow \sum_{i \in J} a_{ij}^s \to +\infty\,.$$

$\therefore$ For s sufficiently large, $A_j^s=(a_{ij}^s)$, $i, j \in J$

cannot be productive (by (P9)): which would contradict the fact that A^s is productive for all s. Hence $J=\Phi$ or $p^0>0$. Thus by B, a_j^s is bounded for all j; b_j^s is bounded by (5) and hence (a_j^s, b_j^s) has a limit *pt.* $(a_j^0, b_j^0)\in T_j$. Since $p^s(I-A^s)=b^s$, $p^0(I-A^0)=b^0$; moreover $0\leqq p^s\leqq p^*$ implies $0\leqq p^0\leqq p^*$ and so $p^0\in P$. Hence the claim.

By virtue of C, one may now make the following assertion:

D. *There exists $\hat{p}\in P$ such that $\hat{p}$ solves* $\min\sum_i p_i s\cdot t$, $p\in P$. *Since $\hat{p}\in P$, there is $(\hat{A}, \hat{b})\in T$ such that*

$$\hat{p}(I-\hat{A})=\hat{b}$$

since $\hat{b}>0$, the above implies that $\hat{A}$ is productive (P2).

For this $(\hat{A}, \hat{b})$, we have the following claim:

E. *$U(\hat{A}, \hat{b})$ has the nonsubstitution property.*

Proof. For if not, there is $(A, b)\in T$ and $y\in U(A, b)$ such that $y\notin U(\hat{A}, \hat{b})$. Hence $\hat{b}(I-\hat{A})^{-1}y>L$, as $\hat{A}$ is productive. Moreover, there is $x\geqq 0$ such that

$$(I-A)x\geqq y\,, \qquad b\cdot x\leqq L\,.$$

$$\therefore\ \hat{p}(I-A)x\geqq\hat{p}y=\hat{b}(I-\hat{A})^{-1}y>L\geqq bx$$

so that $[\hat{p}(I-A)-b]x>0$ and hence, there exists j_1 such that

$$\hat{p}_{j_1}-\hat{p}a_{j_1}-b_{j_1}>0\,.$$

Define next $(\bar{A}, \bar{b})\in T$ such that $\bar{A}=(\bar{a}_j)$, $\bar{b}=(\bar{b}_j)$ and

$$\begin{aligned}(\bar{a}_j, \bar{b}_j)&=(\hat{a}_j, \hat{b}_j)\,, && j\neq j_1\\ &=(a_{j_1}, b_{j_1})\,, && \text{otherwise}\,.\end{aligned}$$

Note that

$$\begin{aligned}\hat{p}\bar{a}_j+\bar{b}_j&=\hat{p}\hat{a}_j+\hat{b}_j=\hat{p}_j\,, && j\neq j_1\\ &=\hat{p}a_{j_1}+b_{j_1}<\hat{p}_{j_1}\,, && \text{otherwise}\end{aligned}$$

so that $\hat{p}(I-\bar{A})\geq\bar{b}>0$; hence $\bar{A}$ is productive (P2). Hence $\hat{p}\geq\bar{b}(I-\bar{A})^{-1}=\bar{p}$ say.

The inequality must remain strict for at least one component since $(I-\bar{A})^{-1}\geqq 0$ *and* $(I-\bar{A})^{-1}$ cannot have a row of zeros.

Hence $\bar{p}\in P$: a contradiction to the definition of $\hat{p}$. Thus no such y, (A, b) can exist and E is established.

We may conclude with two related observations:

F. *$\hat{p}$ whose existence is asserted in* D, *is unique.*

Proof. For suppose, to the contrary, that $\hat{p}, \bar{p}$ with $\hat{p}\neq\bar{p}$ both solve the problem in D. Then there exist $(\hat{A}, \hat{b})$, $(\bar{A}, \bar{b})\in T$ such that

$$\hat{p}(I-\hat{A})=\hat{b}$$
$$\bar{p}(I-\bar{A})=\bar{b}$$

Moreover, as

$$\sum_j \hat{p}_j=\sum_j \bar{p}_j\,, \quad \bar{p}\neq\hat{p}\Rightarrow J=\{j:\, p_j>\bar{p}_j\}\neq\Phi$$

consider $\tilde{p}$, $(a_j, b_j)\in T_j$ defined by

$$\begin{aligned}\tilde{p}_j&=\bar{p}_j\,, \quad j\in J\,; \qquad & (a_j, b_j)&=(\bar{a}_j.\, \bar{b}_j)\,, \quad j\in J\,;\\ &=\hat{p}_j\,, \quad j\notin J\,; & &=(\hat{a}_j, \hat{b}_j)\,, \quad j\notin J\,.\end{aligned}$$

Let $A=(a_j)$, $b=(b_j)$. Then $(A, b)\in T$ and for

$$j\in J\,, \quad \tilde{p}a_j+b_j=\sum_{i\in J}\bar{p}_i\bar{a}_{ij}+\sum_{i\notin J}\hat{p}_i\bar{a}_{ij}+\bar{b}_j\leqq\sum_i \bar{p}_i\bar{a}_{ij}+\bar{b}_j=\bar{p}_j=\tilde{p}_j\,;$$
$$j\notin J\,, \quad \tilde{p}a_j+b_j=\sum_{i\in J}\bar{p}_i\hat{a}_{ij}+\sum_{i\notin J}\hat{p}_i\hat{a}_{ij}+\hat{b}_j<\sum_i \hat{p}_i\hat{a}_{ij}+\hat{b}_j=\hat{p}_j=\tilde{p}_j\,.$$

Hence $\tilde{p}A+b\leq\tilde{p}$ so that A is productive and $\tilde{p}\geq b(I-A)^{-1}=p$ say, as noted in the proof of E since $\hat{p}\geq\tilde{p}\geq p$, $p\in P$: a contradiction. Hence the claim.

Finally,

G. *If $(A, b)\in T$ has the non-substitution property, then $b(I-A)^{-1}=\hat{p}$.*

Proof. First of all, since (A, b) has the non-substitution property, A must be productive. Now if $b(I-A)^{-1}=p\neq\hat{p}$ then $F\Rightarrow\sum_i p_i>\sum_i \hat{p}_i$, provided $p\in P$. So consider $c>0$ such that $\sum_{i=1}^n p_ic_i=L>\sum_i \hat{p}_ic_i$ where $c_i=L/\sum p_i$ for each i. Hence there is $\hat{c}>c$ where $\hat{c}_i=L/\sum\hat{p}_i>c_i$ such that $\sum p_i\hat{c}_i>L=\sum\hat{p}_i\hat{c}_i$ so that $\hat{c}\in U(\hat{A}, \hat{b})$ but $c\notin U(A, b)$: a contradiction. So $p\notin P$ i.e., $p_j>p_j^*$ for some j_1. Now define, $c_{j_1}=L/p_{j_1}^*$, $c_j=0, j\neq j_1$. Clearly $p\cdot c=p_{j_1}/p_{j_1}^*\cdot L>L$ but $p^*c=L$ and so $c\in U(A^*, b^*)$ but $c\notin U(A, b)$: again a contraction. Hence $p=\hat{p}$.

VIII. SINGULAR B-MATRICES: THE GROSS SUBSTITUTE SYSTEM

For the theory of linear economic models, we have seen that B-matrices and their nonnegative inverses played a major role. The theory of stability of competitive equilibrium also entails the investigation of properties of a matrix which is often assumed to be a B-matrix; the only difference being that the B-matrix is known to be singular. This problem has led to the exclusion of one row and column of the B-matrix by the device of choosing a numeraire so that once again, one has a non-singular B-matrix. Since the original singular B-matrix is often not considered, there does not exist, to the best of our knowledge, an analysis of the properties of such matrices. We show below that the tools developed above may be applied to such a problem.

Let $Z_i(p_1, \cdots, p_n)$ denote the excess demand function for the i-th good, $i=1,2,\cdots,n$; $Z_i(\ )$ is assumed to be differentiable in the interior of the nonnegative orthant; it is homogeneous of degree zero in the prices i.e.,

$$Z_i(p)=Z_i(\lambda p)\,, \qquad \forall\,\lambda>0 \tag{1}$$

and satisfies Walras law i.e.,

$$\sum_{i=1}^{n} p_i Z_i(p)=0 \qquad \text{for all} \quad p>0\,. \tag{2}$$

Let $E=[p>0: Z_i(p)=0, \forall\, i]$: the set of equilibrium prices. Consider

$$J(p)=\left(\frac{\partial Z_i(p)}{\partial P_j}\right), \qquad i,j=1,2,\cdots,n$$

which is defined for all $p>0$. Then (1) implies

$$J(p)\cdot p=0 \tag{3}$$

and (2) implies

$$p\cdot J(p)=-Z(p) \tag{4}$$

We begin by observing that

L12. *$J(p)$ is symmetric $\Rightarrow p\in E$.*

This is immediate from (3) and (4).

Since $J(p)$ is singular, we can at best seek to determine the properties of the adjoint. Let $A_{ij}(p)$ denote the cofactor of the i,j-th element in $J(p)$ and let $A(p)=(A_{ji}(p))$: the adjoint. Then

$$A(p)\cdot J(p)=J(p)\cdot A(p)=0 \tag{5}$$

where 0 denotes the $n\times n$ null matrix. We have then

L13. *Either $A_{ji}(p)=0$ for all i, j or $A_{ji}(p)=\lambda^i p_j$ for some scalar $\lambda^i\neq 0$.*

Proof. Suppose $A_{rs}(p)\neq 0$ for some r and s. Then rank $J(p)=n-1$. Consequently, $X=[x: J(p)\cdot x=0]$ is a subspace of rank unity; by virtue of (3), $p\in X$; by virtue of (5), $A^i(p)$ (the i-th column of $A(p)$) $\in X$ for all $i \Rightarrow A^i(p)=\lambda^i p$ and the claim follows.

While investigating the stability of competitive equilibrium, a standard assumption has been to require that $J(p)$ satisfy

$$\frac{\partial Z_i(p)}{\partial p_j}>0\,, \qquad i\neq j\,, \quad p>0\,. \tag{GS}$$

Uder GS, $J(p)$ becomes the negative of a B-matrix; but under (3), $-J(p)$ is a singular B-matrix. The usual practice in such situations has been to drop say, the first row and column from $J(p)$ and then note that

$$J_{11}(p)\cdot p_{\sim 1}<0\,, \tag{6}$$

from (3) where

$$J_{11}(p)=\left(\frac{\partial Z_i(p)}{\partial p_j}\right), \qquad i,j\neq 1$$

and

$$p_{\sim 1}=(p_2,\cdots,p_n)\,,$$

so that $J_{11}(p)$ has a dominant diagonal which is negative; in other words $-J_{11}(p)$ is a B-matrix with a dominant positive diagonal and so earlier results are immediately applicable. Moreover, it has been standard practice to consider $J_{11}(p)$ for $p\in E$; with the result that properties of $J_{11}(p)$, $p\notin E$ are not usually discussed. We shall, in the results below consider $J(p)$ and show that $J(p)$ for $p\in E$ is different from $J(p)$ for $p\notin E$ in an important manner. First of all,

L14. *Under GS, there is $y>0$ such that $yJ(p)=0$ and $y=\lambda p$ for $\lambda\neq 0$, if and only if $p\in E$.*

Proof. Suppose to the contrary $yJ(p)=0$ has no positive solution. Then under a corollary to (SH),

$$J(p)x\geq 0 \qquad \text{has a solution} \quad x^*\,,$$

$$\therefore\; J(p)(-x^*)\leq 0\,.$$

By virtue of (3), $J(p)\cdot(x^*+\mu p)\leq 0$ for any scalar μ and hence $J(p)\cdot z^*\leq 0$ for $z^*>0$ i.e., $z^*=-x^*+\mu p$, μ chosen appropriately. But this implies that $J(p)\cdot z<0$ has a nonnegative solution; for if no such solution exists, $w\cdot J(p)\geqq 0$ has a semipositive solution, by (SH.2). Consequently if $I=[i\colon w_i>0]\neq\Phi$, writing $J_I(p)=(\partial Z_i(p)/\partial p_j)$, $i,j\in I$, $w_I=(w_i)$, $i\in I$, $z_I=(z_j^*)$, $j\in I$, we have:

$$w_I\cdot J_I(p)\geqq 0\,, \qquad J_I(p)\cdot z_I<0\,, \quad w_I>0\,, \quad z_I>0:$$

which is a contradiction. Thus $J(p)\cdot z<0$ has a nonnegative solution $z^*\Rightarrow z^*>0$ and hence $J(p)$ has a negative dominant diagonal and hence non-singular: which too, is a contradiction. Thus, the first part of the claim follows. For the remaining part, note

$$y=p\Leftrightarrow p\cdot J(p)=0\Leftrightarrow Z(p)=0 \text{ from (4)}$$

$$\Leftrightarrow p\in E\,.$$

L15. *Under GS, all elements of $A(p)$ are non zero and have the same sign; $A(p)>0$ if n is odd; $A(p)<0$ if n is even.*

Proof. As noted above, $J_{11}(p)$ has a dominant diagonal which is negative and hence non-singular. $\therefore A_{11}(p) \neq 0$. By L13, $A_{ji}(p) = \lambda^i p_j$. By L14, and the method of proof of L13,

$$A_{ji}(p) = \lambda_j y_i$$

$$\therefore \lambda^i p_j = \lambda_j y_i \Rightarrow \lambda^i \text{ and } \lambda_j$$

have the same sign for all λ^i and λ_j, $i, j = 1, 2, \cdots, n$. Also $A_{11}(p) = \lambda_1 y_1 = \lambda^1 p_1$. By (HS) conditions $\det(-J_{11}(p)) > 0 \Rightarrow A_{11}(p)$ has the sign of $(-1)^{n-1} \Rightarrow A_{11}(p) > 0$ iff n odd, $A_{11}(p) < 0$ if n even. Thus $\lambda_1, \lambda^1 > 0$ if n odd, $\lambda_1, \lambda^1 < 0$ if n even; and since λ^i, λ_j have the same signs, the claim follows..

L16. *Under GS, zero is not a repeated characteristic root of $J(p)$; all other characteristic roots of $J(p)$ have negative real parts.*

Proof. By *GS*, there is $\gamma > 0$ such that $J(p) + \gamma I$ is a positive matrix. By the Frobenius Theorem, there is $\alpha^* > 0$ such that $(J(p) + \gamma I)y^* = \alpha^* y^*$ where $y^* > 0$. Moreover α^* is the only characteristic root of $J(p) + \gamma I$ which has an associated nonnegative characteristic vector. But by (3),

$$(J(p) + \gamma I)p = \gamma p ,$$

$$p > 0 \Rightarrow \gamma = \alpha^* .$$

Also, α^* is not a repeated characteristic root of $J(p) + \gamma I$ and $\alpha^* \geqq |\beta|$ for any other characteristic root β of $J(p) + \gamma I$; further β is a characteristic root of $J(p) + \gamma I \Rightarrow \beta - \gamma$ is a characteristic root of $J(p)$. Thus $\beta - \alpha^*$ is a root of $J(p)$ whenever β is root of $J(p) + \gamma I$ implies the claim, given the above properties of α^*.

L17. *$J(p)$ is quasi-negative semi-definite if and only if $p \in E$; (i.e., $xJ(p)x \leqq 0$ for all x iff $p \in E$).*

Proof. Note that

$$xJ(p)x = \frac{1}{2} x(J(p) + J^T(p))x .$$

First note that

$$xJ(p)x \leqq 0 , \ \forall x \quad \Rightarrow \quad x(J(p) + J^T(p))x \leqq 0 , \ \forall x$$

so that $J(p) + J^T(p)$ is negative semi-definite. Moreover from (3) and (4)

$$p(J(p) + J^T(p))p = 0 .$$

This must imply that $p(J(p) + J^T(p)) = 0$. For suppose $p(J(p) + J^T(p)) = y \neq 0$. Then consider $p + ty$ where t is a scalar:

$$0 \geqq (p + ty)'(J(p) + J^T(p))(p + ty), \text{ since } J(p) + J^T(p) \text{ is negative semi-definite}$$

$$= 2t[yy + t \cdot yJ(p)y] > 0 \text{ for } t > 0 \text{ and small} .$$

$$\therefore y = 0 .$$

Consequently, $p(J(p)+J^T(p))=0$ i.e. $Z(p)=0$ since $pJ^T(p)=0$ by (3) and (4) holds. Thus $p \in E$.

For the converse, $p \in E \Rightarrow p(J(p)+J^T(p)=0$ by virtue of (3) and (4) and definition of E. Thus $J(p)+J^T(p)$ satisfy the conditions used in the proof of L16; hence all characteristic roots, apart from zero, of $J(p)+J^T(p)$ must be negative. Since $J(p)+J^T(p)$ is symmetric, this means that $J(p)+J^T(p)$ is negative semi-definite, and hence

$$xJ(p)x \leqq 0\,, \qquad \forall\, x \neq 0\,.$$

We conclude by making the following

Remark. Note that the first part of the proof does not utilize the *GS* property. Thus one may state: Under (1) and (2)

$$J(p) \text{ is quasinegative semi def. } \Rightarrow p \in E\,.$$

Jawaharlal Nehru University

REFERENCES

[1] Arrow, K. J. and F. H. Hahn, *General Competitive Analysis*, Holden-Day Inc., San Francisco, 1971.

[2] Dasgupta, D., "A Note on Johansen's Non Substitution Theorem and Malinvaud's Decentralization Procedure," *Journal of Economic Theory*, Vol. 9, No. 3, 1974, pp. 340–349.

[3] Debreu, G. and I. N. Herstein, "Non negative Square Matrices," *Econometrica*, Vol. 21, 1953, pp. 597–607.

[4] Dorfman, R., P. A. Samuelson and R. Solow, *Linear Programming and Economic Analysis*, McGraw-Hill, New York, 1960.

[5] Gale, D., *The Theory of Linear Economic Models*, McGraw-Hill, New York, 1960.

[6] Hawkins, D. and H. A. Simon, "Note: Some Conditions of Macroeconomic Stability," *Econometrica*, **17**, 1949, pp. 245–248.

[7] Johansen, L., Simple and General Non Substitution Theorems for Input-Output Models, *Journal of Economic Theory*, Vol. 5, No. 3, 1972, pp. 383–394.

[8] Karlin, S., *Mathematical Methods in Games Programming and Economics, Volume I*, Addison-Wesley, Reading, Mass., 1959.

[9] McKenzie, L. W., "Matrices with dominant diagonals and economic theory" in *Mathematical Methods in the Social Sciences*, 1959, K. Arrow, S. Karlin and P. Suppes (editors), Stanford University Press, Stanford, 1960, pp. 47–62.

[10] Morishima, M. and T. Fujimoto, "The Frobenius Theorem, its Samuelson-Solow Extension and the Kuhn-Tucker Theorem," *Journal of Mathematical Economics*, Vol. 1, 1974, pp. 199–205.

[11] Murata, Y., "An Alternative Proof of the Frobenius Theorem," *Journal of Economic Theory*, Vol. 5, 1972, pp. 285–293.

[12] Murata, Y., *Mathematics for Stability and Optimization of Economic Systems*, Academic Press, New York, 1977.

[13] Nikaido, H., *Convex Structures and Economic Theory*, Academic Press, New York, 1968.