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# ON SOME IMPLICATIONS OF THE SEPARATING HYPERPLANE THEOREM* 

Anjan Mukherji


#### Abstract

The paper aims at unifying the various mathematical tools used in the analysis of linear economic models. Using the theorem of the Separating Hyperplane as stated in Gale's 'The Theory of Linear Economic Models', McGraw-Hill, 1960, a wide class of results are obtained: the Frobenius Theorem, the various alternative forms of the productivity conditions for a Leontief Matrix, the non-substitution theorem, and properties of the singular jacobian of excess demand functions under gross substitution. The concept of a dominant diagonal matrix plays an important role in the entire analysis.


## INTRODUCTION

In economic theory, one often encounters non-negative matrices or matrices with all off diagonal elements non positive ( $B$-matrices) and usually, one enquires into the nature of characteristic roots and associated characteristic vectors or whether certain linear equations involving these matrices have non-negative solutions. The primary source of results in this connection appears to be the Frobenius Theorem; that this result is considered to be of central importance, is perhaps best reflected by the fact that almost every text book dealing with mathematical economics contains a section or an appendix dealing with the topic, e.g. [1], [8], [12], [13]. Besides, there have been several papers, e.g. [3], [10], [11], each providing an alternative proof of the Frobenius Theorem.

An alternate source of similar results, according to Mckenzie [9] is Hadamard's Theorem which states that a matrix with a dominant diagonal must be non singular. Then there is the indigenous source-which uses the Frobenius Theorem and/or Hadamard's Theorem and/or various tedious properties of determinants and cofactors to establish some of these results. There is also the work of Gale [5] which begins with the Separating Hyperplane Theorem (SHT) and establishes some results in this connection; but the properties of characteristic roots and vectors find no place in Gale's analysis.

We shall show that the entire analysis of these class of problems may be based on the Separating Hyperplane Theorem alone. The importance of this result to economic theory cannot be overstressed and with our analysis, it occupies a central role in this area as well. Specifically, the route which we traverse may be seen as

* Helpful comments from the referee are gratefully acknowledged.
follows:

SHT \begin{tabular}{cc}
III \& <br>

$\Rightarrow$ Hardamard's Theorem \& $\Rightarrow$| Properties of the Gross |
| :---: |
| Substitute System VIII | <br>

\& $\Rightarrow$| Frobenius Theorem IV, V |
| :--- |
| Productivity Conditions |
| in the Leontief Model VI |
| Non Substitution Theorem VII |

\end{tabular}

To put our analysis in the proper perspective, let us compare the above scheme to the analysis in [3], [5] and [9]. In [3], IV is first established; then the relevant properties of $B$-matrices in III are demonstrated. In [5], beginning with SHT, the author straightway arrives at VI and goes on to VII for the case when each sector has a finite number of activities; but the conditions involving characteristic roots find no place in this analysis. In [9], the point of departure is III; these are used to demonstrate VI and in conjunction with IV, V, the stability of the Gross substitute system is demonstrated.

We show that $B$-matrices with dominant positive diagonal provide the common underlying structure and the SHT, the basic mathematical tool. Moreover, in obtaining an unified approach, the proofs of the various propositions become quite simple and elementary.

## II. the separating hyperplane theorem

The starting point of this analysis is Gale's [5] Theorem 2.6:
Exactly one of the following alternatives hold:
$\begin{array}{ll}\text { Either } & x A=b \text { has a nonnegative solution } \\ \text { or } & A y \geqq 0 \text { by }<0 \text { have a solution } .\end{array}$
It should be pointed out that the notation (and terms): $x>0$ (positive), $x \geq 0$ (semipositive) and $x \geqq 0$ (non-negative) are to be interpreted as in [5]. The following corollaries of ( SH ) would be frequently used; these are Theorems 2.9 and 2.10 [5, pages 48-49].

Either $\quad x A=0$ has a semipositive solution
or $\quad A y>0$ has a solution
Either $\quad x A \leqq 0$ has a semipositive solution
or $\quad A y>0$ has a nonnegative solution
In each of the above cases, exactly one of the stated alternative hold.

## III. MATRICES WITH DOMINANT DIAGONALS

A square matrix $A=\left(a_{i j}\right), i, j=1,2, \cdots, n$ is said to have a column dominant diagonal (c.d.d.) if there exist $d_{j}>0, j=1,2, \cdots, n$ such that

$$
\begin{equation*}
d_{j}\left|a_{j j}\right|>\sum_{i \neq j} d_{i}\left|a_{i j}\right|, \quad j=1,2, \cdots, n \tag{1}
\end{equation*}
$$

Let $B_{A}=\left(b_{i j}\right)$ be defined by

$$
\begin{aligned}
b_{i j} & =-\left|a_{i j}\right| & & i \neq j \\
& =\left|a_{j j}\right| & & \text { otherwise } .
\end{aligned}
$$

Then $A$ has a c.d.d. iff

$$
\begin{equation*}
d B_{A}>0 \quad \text { for some } \quad d \geqq 0 . \tag{la}
\end{equation*}
$$

$A$ is said to have a row dominant diagonal (r.d.d.) if and only if $A^{T}$ has a c.d.d. i.e., there exist $c_{j}>0, j=1,2, \cdots, n$ such that

$$
\begin{equation*}
c_{i}\left|a_{i i}\right|>\sum_{j \neq i}\left|a_{i j}\right| c_{j}, \quad i=1,2, \cdots, n \tag{2}
\end{equation*}
$$

or alternatively

$$
\begin{equation*}
B_{A} \cdot c>0 \quad \text { for some } \quad c \geqq 0 \tag{2a}
\end{equation*}
$$

We show first that
L1. A has a c.d.d. iff $A$ has a r.d.d.
Proof. If $A$ has a c.d.d. then (1) and (1a) hold; in particular,

$$
d B_{A}>0 \quad \text { for some } d^{*}>0
$$

Suppose to the contrary that $B_{A} \cdot c>0$ has a nonnegative solution. Then by (SH.2), $x B_{A} \leqq 0$ for some $x^{*} \geq 0$. Thus ( $\left.d^{*}-t x^{*}\right) B_{A}>0$ for any scalar $t>0$. In particular, consider $t^{*}=\min _{j \in x_{j}^{*}>0} d_{j}^{*} / x_{j}^{*}>0$. For such a choice, $\left(d^{*}-t^{*} x^{*}\right) B_{A}>0$ cannot hold. Hence no such $x^{*}$ exists so that (2a) holds and $A$ has a r.d.d. For the converse, apply the above to $A^{T}$.

Hence one need not distinguish between r.d.d. and c.d.d. and we shall refer to matrices as having a dominant diagonal (d.d.) if any one of (1), (1a), (2) or (2a) holds. L1 establishes that if any one of these holds, then so do the others. The fundamental property of such matrices is given by (Hardamard's Theorem):

L2. If $A$ has a d.d. then $A$ is non-singular.
Proof. By (1a) and (SH.2), $B_{A} \cdot y \leqq 0$ has no semipositive solution. Thus, if $A$ is singular, $A z=0$ for some $z \neq 0$; i.e.

$$
-a_{i i} \cdot z_{i}=\sum_{j \neq i} a_{i j} z_{i} \quad \text { for all } \quad i
$$

or

$$
\left|a_{i i}\right| \cdot\left|z_{i}\right|=\left|\sum_{j=i} a_{i j} z_{i}\right| \leqq \sum_{j \neq i}\left|a_{i j}\right|\left|z_{j}\right|
$$

or writting

$$
y_{i}=\left|z_{i}\right|, \quad y=\left(y_{i}\right), \quad y \geq 0 \quad \text { and } \quad B_{A} \cdot y \leqq 0: \text { a contradiction }
$$

Hence the claim.
Matrices such as $B_{A}$ would play an important role in our analysis. We shall refer to matrices such as these viz., with non-positive off diagonal entries, by the term $B$ matrices. For a $B$-matrix, $B=\left(b_{i j}\right)$, the above considerations imply: $B$ has a d.d. with $b_{i i}>0$ ( $+v e$ d.d.) iff

$$
\begin{equation*}
B d>0 \quad \text { for } \quad d \geqq 0 \tag{3}
\end{equation*}
$$

Moreover for any $B$-matrix, $B=\left(b_{i j}\right)$.
L3. $B x=c$ has an unique nonnegative solution for any $c \geqq 0$ iff $B$ has $a+$ ve d.d.
Proof. If $B$ has a $+v e$ d.d., then (3) implies $B^{J} \cdot d^{J}>0$ for any $J \subseteq\{1,2, \cdots, n\}$, $B^{J}=\left(b_{i j}\right), i, j \in J ; d^{J}=\left(d_{j}\right), j \in J$. Thus by (SH.2),

$$
\begin{equation*}
y^{J} \cdot B^{J} \leqq 0 \text { has a no semipositive solution. } \tag{4}
\end{equation*}
$$

Consider $y^{*}$ such that $y^{*} B \geqq 0$. If possible, let $J=\left\{j: y_{j}^{*}<0\right\} \neq \Phi$. Then $y^{* J} \cdot B^{J} \geqq 0$ and $y^{* J}<0$ violates (4) and hence $J=\Phi$. Thus $y^{*} B \geqq 0 \Rightarrow y^{*} \geqq 0$. Consequently $y B \geqq 0$ and $y c<0$ can have no solution whenever $c \geqq 0$. Hence by (SH) $B x=c$ has a nonegative solution for any $c \geqq 0$. The solution is unique by virtue of L2. The converse is trivial, given (3).

L4. For a $B$-matrix, $B^{-1} \geqq 0$ iff $B$ has $a+v e ~ d . d$.
The proof is trivial given L3 and hence omitted.
L5. For a B-matrix, B has $a+$ ve d.d. iff

Proof. The proof follows by virtue of L3 and the central result in Hawkins and Simon [6].

The above results will be shown to be central to not only the theory of linear economic models but to the Perron-Frobenius Theorem as well.

## IV. NONNEGATIVE MATRICES AND THE FROBENIUS THEOREM

Let $A \geqq 0$ be a square non-negative matrix of order $n$ and $F(A)=[\pi: \pi I-A$ has a
$+v e$ d.d.]. For any $\pi, \pi I-A$ is a $B$-matrix and hence the results of the last section apply. Moreover, for any $A \geqq 0, F(A) \neq \Phi$ : since any $\pi$ larger than the maximum column sum of $A$ must be an element of $F(A)$. Aslo $\pi \in F(A) \Rightarrow \pi \geqq 0$. Hence:

$$
\pi_{A}^{*}=\operatorname{lnf}_{\pi \in F(A)} \pi
$$

exists. We shall drop the subscript and write $\pi^{*}$ whenever the context makes it clear. First of all,

L6. $\pi^{*} \notin F(A), \pi^{*} \geqq 0 ; \pi>\pi^{*} \Rightarrow \pi \in F(A)$. Thus $F(A)=\left(\pi^{*},+\infty\right)$. The claim follows from (3) and the properties of the infimum.

Moreover,
L7. $\pi^{*}$ is a characteristic root (c.r.) of $A .|\alpha| \leqq \pi^{*}$ for any other c.r. of $A$.
Proof. Let $c$ be a positive column vector $(n \times 1)$. Construct an $n \times(n+1)$ matrix whose first $n$ columns are the columns of $\pi^{*} I-A$ and the last column is $-c$. Denote this matrix by

$$
\left[\pi^{*} I-A,-c\right]
$$

Note that $y\left[\pi^{*} I-A,-c\right]>0$ can have no solution; for if a solution $\bar{y}$ exists, then

$$
\bar{y}\left(\pi^{*} I-A\right)>0 \quad \text { and } \quad \bar{y} \cdot c<0
$$

so that for some $\sigma>0$

$$
\bar{y}\left(\left(\pi^{*}+\sigma\right) I-A\right) \geqq 0 \quad \text { and } \quad \bar{y} \cdot c<0 \Rightarrow \text { by SH that } \quad\left(\left(\pi^{*}+\sigma\right) I-A\right) x=c
$$

can have no nonnegative solution $\Rightarrow$ by L 3 that $\pi^{*}+\sigma \notin F(A)$ : contrary to L6. Thus no such $\bar{y}$ exists. Hence by (SH.1) there is $\left[\begin{array}{l}\left.x_{\alpha^{*}}^{*}\right]\end{array}\right.$ semipositive where $x^{*}$ is $n \times 1$ and $\alpha^{*}$ a scalar such that

$$
\left[\pi^{*} I-A,-c\right]\left[\alpha_{\alpha^{*}}^{x^{*}}\right]=0
$$

or

$$
\begin{aligned}
& \left(\pi^{*} I-A\right) x^{*}=\alpha^{*} c \\
\because & \alpha^{*} \neq 0 \Rightarrow \pi^{*} I-A
\end{aligned}
$$

has a $+v e$ d.d. and $\pi^{*} \in F(A)$ : contrary to L6, we conclude $\alpha^{*}=0$. Hence $x^{*} \geq 0$ and $\left(\pi^{*} I-A\right) x^{*}=0$ so that $\pi^{*}$ is a c.r. of $A$.

For any other c.r. $\alpha$ let $z$ be its associated characteristic vector (c.v.). Then $A z=$ $\alpha z$ and in particular

$$
|\alpha| \cdot\left|z_{i}\right|=\left|\sum_{j} a_{i j} z_{j}\right| \leqq \sum_{j} a_{i j}\left|z_{i}\right| \quad \text { for all } \quad i
$$

or

$$
(|\alpha| I-A) y \leqq 0 \quad \text { where } \quad y_{i}=\left|z_{i}\right|, \quad \text { and so } \quad y \geq 0 .
$$

In case $|\alpha|>\pi^{*},|\alpha| \in F(A)$ whence by L4,

$$
(|\alpha| I-A)^{-1} \geqq 0 \text { so that } y \leqq 0: \text { a contradiction }
$$

Thus $|\alpha| \leqq \pi^{*}$.
The following fact, obtained in the above proof, is noted separately:
L8. There is $x^{*} \geq 0$ such that $A x^{*}=\pi^{*} x^{*}$.
L9. $(\pi I-A)^{-1} \geqq 0$ iff $\pi>\pi^{*}$. Further $L_{i j}(\pi) \geqq 0$ for $\pi \geqq \pi^{*}$ where $L_{i j}(\pi)$ is the cofactor of the $i, j$-th element in $(\pi I-A)$.

Proof. Follows from L4 by virtue of L6. Writing $L(\pi)$ for $\operatorname{det}(\pi I-A)$; the $j$, $i$-th element of $(\pi I-A)^{-1}$ is $L_{i j}(\pi) / L(\pi) . L(\pi)>0$ whenever $\pi>\pi^{*}$ (L5); and so $L_{i j}(\pi) \geqq 0$ for $\pi>\pi^{*}$ and hence $L_{i j}\left(\pi^{*}\right) \geqq 0$.

L10. If $A \geqq C \geqq 0$ then $\pi_{A}^{*} \geqq \pi_{C}^{*} \geqq 0$. Moreover if $C$ is any principal minor of $A$, then $\pi_{A}^{*} \geqq \pi_{C}^{*}$.

Proof. Follows from noting that $F(A) \subseteq F(C)$, and from the definition of $\pi^{*}$. If $C$ is a principal minor of $A$ of order $r$; fill out the remaining ( $n-r$ ) rows and columns to obtain $D$, a matrix of order $n$. Then

$$
D \leqq A \quad \text { and } \quad \pi_{D}^{*} \leqq \pi_{A}^{*} \quad \text { and } \quad \pi_{D}^{*}=\pi_{C}^{*} .
$$

Collecting the above results, we have shown that for any non-negative matrix $A$
(i) there is $\pi^{*} \geqq 0$ such that $\pi^{*}$ is a c.r. of $A$ and for any other c.r. $\alpha$, $|\alpha| \leqq \pi^{*} \quad$ (L7)
(ii) $(\pi I-A)^{-1} \geqq 0$ iff $\pi>\pi^{*} \quad$ (L9)
(iii) there are $x^{*} \geq 0$ and $p^{*} \geq 0$ such that

$$
\begin{aligned}
& A x^{*}=\pi^{*} x^{*} \\
& p^{*} A=\pi^{*} p^{*} \quad\left(\text { L8 and application to } A^{T}\right) .
\end{aligned}
$$

(iv) $A y \geqq \alpha y, y \geq 0 \Rightarrow \pi^{*} \geqq \alpha \quad$ (Proof of L7).

It may be of some interest to note that some authors e.g. [8, page 247] use (iv) to define $\pi^{*}$. The above set of results is referred to as the Perron-Frobenius Theorem e.g. [13, page 102].

## V. INDECOMPOSABILITY

If in addition to $A$ being nonnegative, one requires that $A$ be indecomposable, then the results of the last section are considerably strengthened. Let $I=$ $[1,2, \cdots, n]$. If there is some non-empty proper subset $J$ of $I$ such that

$$
a_{i j}=0, \quad i \notin J \quad \text { and } \quad j \in J
$$

then $A$ is defined to be decomposable. If no such $J$ exists, then $A$ is said to be indecomposable.

L11. If $A \geqq 0$ is indecomposable, then
(i) $\pi^{*}>0$.
(ii) $x^{*}\left(\right.$ and $\left.p^{*}\right)>0$.
(iii) Any other c.v. corresponding to $\pi^{*}$ is a scalar multiple of $x^{*}$ (or $p^{*}$ ).
(iv) No other c.r. has an associated c.v. which is non-negative.
(v) If $C$ is such that $A \geqq C \geqq 0$ then $\pi_{A}^{*}>\pi_{C}^{*}$ if $C \neq A$.
(vi) If $C$ is a principal minor of $A, C \neq A$ then $\pi_{A}^{*}>\pi_{C}^{*}$.
(vii) $(\pi I-A) x \geq 0$ for $x \geq 0 \Rightarrow(\pi I-A)^{-1}>0$,
and
(viii) $\pi^{*}$ is a simple root of the characteristic equation of $A$.

Proof. See Nikaido [13, page 107]. A simpler demonstration of (viii) is as follows: $\alpha$ is a c.r. of $B=\pi^{*} I-A$ iff $\left(\pi^{*}-\alpha\right)$ is a c.r. of $A$. The characteristic equation of $B$ is $\pi^{n}-\pi^{n-1} \cdot\left(\sum_{j} b_{j j}\right)+\cdots+(-1)^{n-1}$ (sum of det of all p.r. minors of $B$ of order $n-1$ ) $\pi+(-1)^{n} \operatorname{det} B=0$. By virtue of (vi) and (L9), $L_{j j}\left(\pi^{*}\right)>0$ for all $j$; so that the last but one term cannot vanish and hence 0 is a simple root of the above equation.

## VI. A PRODUCTIVE LEONTIEF MODEL

Assume that there are $n$-sectors of production; each sector $j$ produces only an output $j$ by using as inputs the outputs of other sectors and a single nonproduced factor traditionally identified with labour. For each sector, there is an unique method of producing its output and suppose ( $a_{j}, b_{j}$ ) is required to produce one unit of $j$; here, $a_{j}$ stands for the vector of produced commodities required as inputs and $b_{j}$, a scalar, represents the amount of labour required. Let $A$ denote the matrix whose $j$-th column is $a_{j}$.

The viability of the above system of production lies in its ability to produce a surplus (over and above its input requirements) of each commodity. Thus $A$ is said to be productive [5, page 296] if

$$
\begin{equation*}
(I-A) x>0 \quad \text { for some } \quad x \geqq 0 \tag{P1}
\end{equation*}
$$

$x_{j} \geqq 0$, is the activity level or equivalently, the gross output for sector $j . A x$ then stands for the intersectoral input requirements to produce $x$ and so $x-A x$ is the surplus. Note that $(I-A)$ is a $B$-matrix and hence the results of Section III are applicable. We use these results to restate ( Pl ) in several equivalent forms.

A is productive
iff $(I-A)$ has a + ve d.d. [from (P1) and (3)]
iff for any $c \geqq 0,(I-A) x=c$ has an unique nonnegative solution
iff $(I-A)^{-1} \geqq 0 \quad$ [follows from (P3) and L4]
iff $\pi_{A}^{*}<1$ [follows from (P4) and L9]
iff the Hawkins-Simon Conditions hold [follows from (P2) and L5] (P6)
iff for some $c>0,(I-A) x=c$ has an unique nonnegative solution
iff all principal minors of $(I-A)$ have positive determinants
Notice that $(\mathrm{P} 3) \Rightarrow(\mathrm{P} 7) \Rightarrow(\mathrm{P} 2) \Rightarrow(\mathrm{P} 3)$. Also $(\mathrm{P} 8) \Rightarrow(\mathrm{P} 6)$; whereas $(\mathrm{P} 6) \Leftrightarrow(\mathrm{P} 2) \Rightarrow B^{J}$ has a $+v e$ d.d. for every $J$, where $B=(I-A), J \subseteq[1,2, \cdots, n]$, $B^{J}=\left(b_{i j}\right), i, j \in J$ and hence $\operatorname{det} B^{J}>0$ by (L5) $\Rightarrow(\mathrm{P} 8)$.

One may also note the following conditions as well:

$$
(\mathrm{P} 1) \Rightarrow \sum_{i} a_{i j}<1 \quad \text { for some } \quad j
$$

and

$$
\begin{equation*}
\sum_{i} a_{i j}<1 \quad \text { for every } \quad j \Rightarrow(\mathrm{P} 1) \tag{P9}
\end{equation*}
$$

which is a trivial consequence of (P2).

## VII. NON-SUBSTITUTION THEOREM

Consider the production model of the last section; except instead of the unique configuration of inputs $\left(a_{j}, b_{j}\right)$ required to produce one unit of $j$, we now assume that there is a collection $T_{j}$ of such processes for each $j$. We shall insist that

$$
\begin{equation*}
T_{j} \text { is a closed subset of } R_{n+1}^{+} \text {and }\left(a_{j}, b_{j}\right) \in T_{j} \Rightarrow b_{j}>0 \tag{I}
\end{equation*}
$$

When each sector chooses a process, say $j$ chooses $\left(\bar{a}_{j}, \bar{b}_{j}\right)$ from $T_{j}$, the $\bar{a}_{j}$ 's form a Leontief matrix $\bar{A}$ whose $j$-th column is $\bar{a}_{j}$. For each configuration of choice, a separate Leontief matrix becomes applicable. For the model to be viable, we shall also insist that
(II) $\quad$ There is $\left(a_{j}^{*}, b_{j}^{*}\right) \in T_{j}$ such that $A^{*}=\left(a_{j}^{*}\right)$ is productive .

Let

$$
\begin{equation*}
p^{*}\left(I-A^{*}\right)=b^{*} \quad \text { where } \quad b^{*}=\left(b_{j}^{*}\right) \tag{4}
\end{equation*}
$$

Thus, we have normalised prices by taking the wage rate as unit. In this set up, it is meaningful to enquire into the question of choice of processes by each sector. Given that labour is the only primary factor it is reasonable to expect each sector to minimize its labour costs; and if there is a set of processes one for each sector, which minimises these costs, then this set of processes may be chosen regardless of what surplus has to be generated. Thus even though substitution possibilities exist,
no substitution may occur: the non-substitution theorem.
Let $L$ denote the quantity of labour available; then given

$$
\begin{aligned}
& \left(a_{j}, b_{j}\right) \in T_{j}, \quad j=1,2, \cdots, n, \quad A=\left(a_{j}\right), \quad b=\left(b_{j}\right), \\
& U(A, b)=[y \geqq 0: y \leqq(I-A) x, b x \leqq L, x \geqq 0] .
\end{aligned}
$$

We shall write $(A, b) \in T$ whenever $A=\left(a_{j}\right), b=\left(b_{j}\right)$ and $\left(a_{j}, b_{j}\right) \in T_{j}$ for each $j$. If these is $(\hat{A}, \hat{b}) \in T$ such that for all $(A, b) \in T, U(A, b) \subseteq U(\hat{A}, \hat{b})$, then $(\hat{A}, \hat{b})$ has the non-substitution property [7]. What Johansen [7] did not clinch was the existence of such an $(\hat{A}, \hat{b})$; see, for example, Dasgupta [2], who presented an argument to establish the existence. A more direct demonstration is presented below; in the process, the crucial role of the productivity conditions ( P 1$)-(\mathrm{P} 9)$ stand revealed.

We begin by observing that
A. If $\left(a_{j}^{s}, b_{j}^{s}\right) \in T_{j}, s=1,2, \cdots$ and $b_{j}^{s} \rightarrow 0$ then $a_{k j}^{s} \rightarrow+\infty$ for some $k$.

Proof. For, if not, then ( $a_{j}^{s}, b_{j}^{s}$ ) forms a bounded sequence and must have a limit pt. $\left(\bar{a}_{j}, 0\right)$ and which must be contained in $T_{j}$ : a contradiction.

Let $P=\left[p: 0 \leqq p \leqq p^{*}\right.$ and for each $p$ there is some $(A, b) \in T$ such that $p(I-A)=b]$.

In the definition of $P, p^{*}$ is as in (4) above. Hence $p^{*} \in P$. Thus $P$ is nonempty and bounded. Let $p^{s}, s=1,2, \cdots$ be a sequence in $P$; without any loss of generality, assume $p^{s} \rightarrow p^{0}$ as $s \rightarrow \infty$. Since $p^{s} \in P$, for all $s$, there exist $\left(A^{s}, b^{s}\right) \in T$ such that

$$
p^{s}\left(I-A^{s}\right)=b^{s}
$$

Thus by virtue of (P2), $A^{s}$ is productive for all $s$. Moreover,
B. $\quad a_{i j}^{s} \rightarrow+\infty$ for some $i, j \Rightarrow p_{i}^{s} \rightarrow 0$. This is immediate since

$$
\begin{equation*}
p_{j}^{*} \geqq p_{j}^{s}=\sum_{k=1}^{n} p_{k}^{s} a_{k j}^{s}+b_{j}^{s} \geqq b_{j}^{s}>0 \tag{5}
\end{equation*}
$$

C. Pis compact.

Proof. This would be established by showing that $p^{0} \in P$. If possible let $J=$ $\left[j: p_{j}^{0}=0\right] \neq \Phi$. Then by virtue of (5), B, and A ,

$$
\begin{aligned}
j \in J & \Rightarrow b_{j}^{s} \rightarrow 0 \\
& \Rightarrow a_{k j}^{s} \rightarrow+\infty \quad \text { for some } \quad k \\
& \Rightarrow k \in J \quad\left(\because p_{k}^{s} \rightarrow 0\right) .
\end{aligned}
$$

or

$$
j \in J \Rightarrow \sum_{i \in J} a_{i j}^{s} \rightarrow+\infty
$$

$\therefore$ For $s$ sufficiently large, $A_{j}^{s}=\left(a_{i j}^{s}\right), i, j \in J$
cannot be productive (by (P9)): which would contradict the fact that $A^{s}$ is productive for all $s$. Hence $J=\Phi$ or $p^{0}>0$. Thus by $\mathrm{B}, a_{j}^{s}$ is bounded for all $j ; b_{j}^{s}$ is bounded by (5) and hence ( $a_{j}^{s}, b_{j}^{s}$ ) has a limit $p t$. $\left(a_{j}^{0}, b_{j}^{0}\right) \in T_{j}$. Since $p^{s}\left(I-A^{s}\right)=b^{s}$, $p^{0}\left(I-A^{0}\right)=b^{0}$; moreover $0 \leqq p^{s} \leqq p^{*}$ implies $0 \leqq p^{0} \leqq p^{*}$ and so $p^{0} \in P$. Hence the claim.

By virtue of C , one may now make the following assertion:
D. There exists $\hat{p} \in P$ such that $\hat{p}$ solves $\min \sum_{i} p_{i} s \cdot t, p \in P$. Since $\hat{p} \in P$, there is $(\hat{A}, \hat{b}) \in T$ such that

$$
\hat{p}(I-\hat{A})=\hat{b}
$$

since $\hat{b}>0$, the above implies that $\hat{A}$ is productive ( P 2 ).
For this $(\hat{A}, \hat{b})$, we have the following claim:
E. $U(\hat{A}, \hat{b})$ has the nonsubstitution property.

Proof. For if not, there is $(A, b) \in T$ and $y \in U(A, b)$ such that $y \notin U(\hat{A}, \hat{b})$. Hence $\hat{b}(I-\hat{A})^{-1} y>L$, as $\hat{A}$ is productive. Moreover, there is $x \geqq 0$ such that

$$
\begin{aligned}
&(I-A) x \geqq y, \quad b \cdot x \leqq L . \\
& \therefore \hat{p}(I-A) x \geqq \hat{p} y=\hat{b}(I-\hat{A})^{-1} y>L \geqq b x
\end{aligned}
$$

so that $[\hat{p}(I-A)-b] x>0$ and hence, there exists $j_{1}$ such that

$$
\hat{p}_{j_{1}}-\hat{p} a_{j_{1}}-b_{j_{1}}>0 .
$$

Define next $(\bar{A}, \bar{b}) \in T$ such that $\bar{A}=\left(\bar{a}_{j}\right), \bar{b}=\left(\bar{b}_{j}\right)$ and

$$
\begin{aligned}
\left(\bar{a}_{j}, \bar{b}_{j}\right) & =\left(\hat{a}_{j}, \hat{b}_{j}\right), & & j \neq j_{1} \\
& =\left(a_{j_{1}}, b_{j_{1}}\right), & & \text { otherwise } .
\end{aligned}
$$

Note that

$$
\begin{aligned}
\hat{p} \bar{a}_{j}+\bar{b}_{j} & =\hat{p} \hat{a}_{j}+\hat{b}_{j}=\hat{p}_{j}, & & j \neq j_{1} \\
& =\hat{p} a_{j_{1}}+b_{j_{1}}<\hat{p}_{j_{1}}, & & \text { otherwise }
\end{aligned}
$$

so that $\hat{p}(I-\bar{A}) \geq \bar{b}>0$; hence $\bar{A}$ is productive (P2). Hence $\hat{p} \geq \bar{b}(I-\bar{A})^{-1}=\bar{p}$ say.
The inequality must remain strict for at least one component since $(I-\bar{A})^{-1} \geqq 0$ and $(I-\bar{A})^{-1}$ cannot have a row of zeros.

Hence $\bar{p} \in P$ : a contradiction to the definition of $\hat{p}$. Thus no such $y,(A, b)$ can exist and E is established.

We may conclude with two related observations:
F. $\hat{p}$ whose existence is asserted in D , is unique.

Proof. For suppose, to the contrary, that $\hat{p}, \bar{p}$ with $\hat{p} \neq \bar{p}$ both solve the problem in D . Then there exist $(\hat{A}, \hat{b}),(\bar{A}, \bar{b}) \in T$ such that

$$
\begin{aligned}
& \hat{p}(I-\hat{A})=\hat{b} \\
& \bar{p}(I-\bar{A})=\bar{b}
\end{aligned}
$$

Moreover, as

$$
\sum_{j} \hat{p}_{j}=\sum_{j} \bar{p}_{j}, \quad \bar{p} \neq \hat{p} \Rightarrow J=\left\{j: p_{j}>\bar{p}_{j}\right\} \neq \Phi
$$

consider $\tilde{p},\left(a_{j}, b_{j}\right) \in T_{j}$ defined by

$$
\begin{aligned}
\tilde{p}_{j} & =\bar{p}_{j}, & j \in J ; & \left(a_{j}, b_{j}\right)
\end{aligned}=\left(\bar{a}_{j} . \bar{b}_{j}\right), \quad j \in J ; ~ 子, ~\left(\hat{a}_{j}, \hat{b}_{j}\right), \quad j \notin J .
$$

Let $A=\left(a_{j}\right), b=\left(b_{j}\right)$. Then $(A, b) \in T$ and for

$$
\begin{array}{ll}
j \in J, & \tilde{p} a_{j}+b_{j}=\sum_{i \in J} \bar{p}_{i} \bar{a}_{i j}+\sum_{i \notin J} \hat{p}_{i} \bar{a}_{i j}+\bar{b}_{j} \leqq \sum_{i} \bar{p}_{i} \bar{a}_{i j}+\bar{b}_{j}=\bar{p}_{j}=\tilde{p}_{j} ; \\
j \notin J, \quad \tilde{p} a_{j}+b_{j}=\sum_{i \in J} \bar{p}_{i} \hat{a}_{i j}+\sum_{i \notin J} \hat{p}_{i} \hat{a}_{i j}+\hat{b}_{j}<\sum_{i} \hat{p}_{i} \hat{a}_{i j}+\hat{b}_{j}=\hat{p}_{j}=\tilde{p}_{j} .
\end{array}
$$

Hence $\tilde{p} A+b \leq \tilde{p}$ so that $A$ is productive and $\tilde{p} \geq b(I-A)^{-1}=p$ say, as noted in the proof of E since $\hat{p} \geq \tilde{p} \geq p, p \in P$ : a contradiction. Hence the claim.

Finally,
G. If $(A, b) \in T$ has the non-substitution property, then $b(I-A)^{-1}=\hat{p}$.

Proof. First of all, since $(A, b)$ has the non-substitution property, $A$ must be productive. Now if $b(I-A)^{-1}=p \neq \hat{p}$ then $F \Rightarrow \sum_{i} p_{i}>\sum_{i} \hat{p}_{i}$, provided $p \in P$. So consider $c>0$ such that $\sum_{i=1}^{n} p_{i} c_{i}=L>\sum_{i} \hat{p}_{i} c_{i}$ where $c_{i}=L / \sum p_{i}$ for each $i$. Hence there is $\hat{c}>c$ where $\hat{c}_{i}=L / \sum \hat{p}_{i}>c_{i}$ such that $\sum p_{i} \hat{c}_{i}>L=\sum \hat{p}_{i} \hat{c}_{i}$ so that $\hat{c} \in U(\hat{A}, \hat{b})$ but $c \notin U(A, b)$ : a contradiction. So $p \notin P$ i.e., $p_{j}>p_{j}^{*}$ for some $j_{1}$. Now define, $c_{j_{1}}=$ $L / p_{j_{1}}^{*}, c_{j}=0, j \neq j_{1}$. Clearly $p \cdot c=p_{j_{1}} / p_{j_{1}}^{*} \cdot L>L$ but $p^{*} c=L$ and so $c \in U\left(A^{*}, b^{*}\right)$ but $c \notin U(A, b)$ : again a contraction. Hence $p=\hat{p}$.

## VIII. singular $B$-matrices: the gross substitute system

For the theory of linear economic models, we have seen that $B$-matrices and their nonnegative inverses played a major role. The theory of stability of competitive equilibrium also entails the investigation of properties of a matrix which is often assumed to be a $B$-matrix; the only difference being that the $B$ matrix is known to be singular. This problem has led to the exclusion of one row and column of the $B$-matrix by the device of choosing a numeraire so that once again, one has a non-singular $B$-matrix. Since the original singular $B$-matrix is often not considered, there does not exist, to the best of our knowledge, an
analysis of the properties of such matrices. We show below that the tools developed above may be applied to such a problem.

Let $Z_{i}\left(p_{1}, \cdots, p_{n}\right)$ denote the excess demand function for the $i$-th good, $i=$ $1,2, \cdots, n ; Z_{i}(\quad)$ is assumed to be differentiable in the interior of the nonnegative orthant; it is homogeneous of degree zero in the prices i.e.,

$$
\begin{equation*}
Z_{i}(p)=Z_{i}(\lambda p), \quad \forall \lambda>0 \tag{1}
\end{equation*}
$$

and satisfies Walras law i.e.,

$$
\begin{equation*}
\sum_{i=1}^{n} p_{i} Z_{i}(p)=0 \quad \text { for all } \quad p>0 \tag{2}
\end{equation*}
$$

Let $E=\left[p>0: Z_{i}(p)=0, \forall i\right]$ : the set of equilibrium prices. Consider

$$
J(p)=\left(\frac{\partial Z_{i}(p)}{\partial P_{j}}\right), \quad i, j=1,2, \cdots, n
$$

which is defined for all $p>0$. Then (1) implies

$$
\begin{equation*}
J(p) \cdot p=0 \tag{3}
\end{equation*}
$$

and (2) implies

$$
\begin{equation*}
p \cdot J(p)=-Z(p) \tag{4}
\end{equation*}
$$

We begin by observing that
L12. $J(p)$ is symmetric $\Rightarrow p \in E$.
This is immediate from (3) and (4).
Since $J(p)$ is singular, we can at best seek to determine the properties of the adjoint. Let $A_{i j}(p)$ denote the cofactor of the $i, j$-th element in $J(p)$ and let $A(p)=$ $\left(A_{j i}(p)\right)$ : the adjoint. Then

$$
\begin{equation*}
A(p) \cdot J(p)=J(p) \cdot A(p)=0 \tag{5}
\end{equation*}
$$

where 0 denotes the $n \times n$ null matrix. We have then
L13. Either $A_{j i}(p)=0$ for all $i, j$ or $A_{j i}(p)=\lambda^{i} p_{j}$ for some scalar $\lambda^{i} \neq 0$.
Proof. Suppose $A_{r s}(p) \neq 0$ for some $r$ and $s$. Then rank $J(p)=n-1$. Consequently, $X=[x: J(p) \cdot x=0]$ is a subspace of rank unity; by virtue of (3), $p \in X$; by virtue of (5), $A^{i}(p)$ (the $i$-th column of $\left.A(p)\right) \in X$ for all $i \Rightarrow A^{i}(p)=\lambda^{i} p$ and the claim follows.

While investigating the stability of competitive equilibrium, a standard assumption has been to require that $J(p)$ satisfy

$$
\begin{equation*}
\frac{\partial Z_{i}(p)}{\partial p_{j}}>0, \quad i \neq j, \quad p>0 \tag{GS}
\end{equation*}
$$

Uder $G S, J(p)$ becomes the negative of a $B$-matrix; but under (3), $-J(p)$ is a
singular $B$-matrix. The usual practice in such situations has been to drop say, the first row and column from $J(p)$ and then note that

$$
\begin{equation*}
J_{11}(p) \cdot p_{\sim 1}<0 \tag{6}
\end{equation*}
$$

from (3) where

$$
J_{11}(p)=\left(\frac{\partial Z_{i}(p)}{\partial p_{j}}\right), \quad i, j \neq 1
$$

and

$$
p_{\sim 1}=\left(p_{2}, \cdots, p_{n}\right),
$$

so that $J_{11}(p)$ has a dominant diagonal which is negative; in other words $-J_{11}(p)$ is a $B$-matrix with a dominant positive diagonal and so earlier results are immediately applicable. Moreover, it has been standard practice to consider $J_{11}(p)$ for $p \in E$; with the result that properties of $J_{11}(p), p \notin E$ are not usually discussed. We shall, in the results below consider $J(p)$ and show that $J(p)$ for $p \in E$ is different from $J(p)$ for $p \notin E$ in an important manner. First of all,

L14. Under GS, there is $y>0$ such that $y J(p)=0$ and $y=\lambda p$ for $\lambda \neq 0$, if and only if $p \in E$.

Proof. Suppose to the contrary $y J(p)=0$ has no positive solution. Then under a corollary to (SH),

$$
\begin{aligned}
& J(p) x \geq 0 \quad \text { has a solution } \quad x^{*}, \\
\therefore & J(p)\left(-x^{*}\right) \leq 0 .
\end{aligned}
$$

By virtue of (3), $J(p) \cdot\left(x^{*}+\mu p\right) \leq 0$ for any scalar $\mu$ and hence $J(p) \cdot z^{*} \leq 0$ for $z^{*}>0$ i.e., $z^{*}=-x^{*}+\mu p, \mu$ chosen appropriately. But this implies that $J(p) \cdot z<0$ has a nonnegative solution; for if no such solution exists, $w \cdot J(p) \geqq 0$ has a semipositive solution, by (SH.2). Consequently if $I=\left[i: w_{i}>0\right] \neq \Phi$, writing $J_{I}(p)=\left(\partial Z_{i}(p) / \partial p_{j}\right)$, $i, j \in I, w_{I}=\left(w_{i}\right), i \in I, z_{I}=\left(z_{j}^{*}\right), j \in I$, we have:

$$
w_{I} \cdot J_{I}(p) \geqq 0, \quad J_{I}(p) \cdot z_{I}<0, \quad w_{I}>0, \quad z_{I}>0:
$$

which is a contradiction. Thus $J(p) \cdot z<0$ has a nonnegative solution $z^{*} \Rightarrow z^{*}>0$ and hence $J(p)$ has a negative dominant diagonal and hence non-singular: which too, is a contradiction. Thus, the first part of the claim follows. For the remaining part, note

$$
\begin{aligned}
y=p & \Leftrightarrow p \cdot J(p)=0 \Leftrightarrow Z(p)=0 \text { from }( \\
& \Leftrightarrow p \in E .
\end{aligned}
$$

L15. Under GS, all elements of $A(p)$ are non zero and have the same sign; $A(p)>0$ if $n$ is odd; $A(p)<0$ if $n$ is even.

Proof. As noted above, $J_{11}(p)$ has a dominant diagonal which is negative and
hence non-singular. $\therefore A_{11}(p) \neq 0$. By L13, $A_{j i}(p)=\lambda^{i} p_{j}$. By L14, and the method of proof of L13,

$$
\begin{aligned}
& A_{j i}(p)=\lambda_{j} y_{i} \\
\therefore & \lambda^{i} p_{j}=\lambda_{j} y_{i} \Rightarrow \lambda^{i} \text { and } \lambda_{j}
\end{aligned}
$$

have the same sign for all $\lambda^{i}$ and $\lambda_{j}, i, j=1,2, \cdots, n$. Also $A_{11}(p)=\lambda_{1} y_{1}=\lambda^{1} p_{1}$. By (HS) conditions $\operatorname{det}\left(-J_{11}(p)\right)>0 \Rightarrow A_{11}(p)$ has the sign of $(-1)^{n-1} \Rightarrow A_{11}(p)>0$ iff $n$ odd, $A_{11}(p)<0$ if $n$ even. Thus $\lambda_{1}, \lambda^{1}>0$ if $n$ odd, $\lambda_{1}, \lambda^{1}<0$ if $n$ even; and since $\lambda^{i}, \lambda_{j}$ have the same signs, the claim follows.

L16. Under GS, zero is not a repeated characteristic root of $J(p)$; all other characteristic roots of $J(p)$ have negative real parts.

Proof. By $G S$, there is $\gamma>0$ such that $J(p)+\gamma I$ is a positive matrix. By the Frobenius Theorem, there is $\alpha^{*}>0$ such that $(J(p)+\gamma I) y^{*}=\alpha^{*} y^{*}$ where $y^{*}>0$. Moreover $\alpha^{*}$ is the only characteristic root of $J(p)+\gamma I$ which has an associated nonnegative characteristic vector. But by (3),

$$
\begin{gathered}
(J(p)+\gamma I) p=\gamma p, \\
p>0 \Rightarrow \gamma=\alpha^{*} .
\end{gathered}
$$

Also, $\alpha^{*}$ is not a repeated characteristic root of $J(p)+\gamma I$ and $\alpha^{*} \geqq|\beta|$ for any other characteristic root $\beta$ of $J(p)+\gamma I$; further $\beta$ is a characteristic root of $J(p)+$ $\gamma I \Rightarrow \beta-\gamma$ is a characteristic root of $J(p)$. Thus $\beta-\alpha^{*}$ is a root of $J(p)$ whenever $\beta$ is root of $J(p)+\gamma I$ implies the claim, given the above properties of $\alpha^{*}$.

L17. $J(p)$ is quasi-negative semi-definite if and only if $p \in E$; (i.e., $x J(p) x \leqq 0$ for all $x$ iff $p \in E$ ).

Proof. Note that

$$
x J(p) x=\frac{1}{2} x\left(J(p)+J^{T}(p)\right) x .
$$

First note that

$$
x J(p) x \leqq 0, \forall x \quad \Rightarrow \quad x\left(J(p)+J^{T}(p)\right) x \leqq 0, \forall x
$$

so that $J(p)+J^{T}(p)$ is negative semi-definite. Moreover from (3) and (4)

$$
p\left(J(p)+J^{T}(p)\right) p=0
$$

This must imply that $p\left(J(p)+J^{T}(p)\right)=0$. For suppose $p\left(J(p)+J^{T}(p)\right)=y \neq 0$. Then consider $p+t y$ where $t$ is a scalar:

$$
\begin{aligned}
0 \geqq(p+t y)^{\prime}(J(p) & \left.+J^{T}(p)\right)(p+t y), \text { since } J(p)+J^{T}(p) \text { is negative semi-definite } \\
& =2 t[y y+t \cdot y J(p) y]>0 \text { for } t>0 \text { and small } . \\
\therefore y & =0 .
\end{aligned}
$$

Consequently, $p\left(J(p)+J^{T}(p)\right)=0$ i.e. $Z(p)=0$ since $p J^{T}(p)=0$ by (3) and (4) holds. Thus $p \in E$.

For the converse, $p \in E \Rightarrow p\left(J(p)+J^{T}(p)=0\right.$ by virtue of (3) and (4) and definition of $E$. Thus $J(p)+J^{T}(p)$ satisfy the conditions used in the proof of L16; hence all characteristic roots, apart from zero, of $J(p)+J^{T}(p)$ must be negative. Since $J(p)+J^{T}(p)$ is symmetric, this means that $J(p)+J^{T}(p)$ is negative semidefinite, and hence

$$
x J(p) x \leqq 0, \quad \forall x \neq 0 .
$$

We conclude by making the following
Remark. Note that the first part of the proof does not utilize the $G S$ property. Thus one may state: Under (1) and (2)
$J(p)$ is quasinegative semi def. $\Rightarrow p \in E$.

Jawaharlal Nehru University

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