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EQUILIBRIUM IN A MARKET WITH DIVISIBLE AND INDIVISIBLE RISKY ASSETS

Rafael LAZIMY

Abstract. This study extends the classic Capital Asset Pricing Model (CAPM) by relaxing the assumptions that all assets are perfectly divisible and liquid, and that investors face the same set of investment opportunities. It is assumed that investors can invest in two types of assets: perfectly divisible, and perfectly indivisible (or discrete). Also, investors may face different sets of investment opportunities in risky indivisible assets. The paper focuses on the following subjects: single investor's equilibrium and his demand for risky assets; market equilibrium and risk-return relationships; and optimal decision rules for evaluating single risky projects. The modifications that the single investor and the market as a whole make in the equilibrium conditions due to the presence of investment opportunities in risky indivisible assets, are identified, studied, and interpreted. It is shown that the theoretical implications of the present extended model are more consistent with empirical findings, and that the relaxation of the assumption that all assets are perfectly divisible eliminates some of the most unattractive and disturbing implications of the classic CAPM, while preserving the more attractive properties of that model.

1. INTRODUCTION

On the basis of the pioneering work of Markowitz [17, 18] and Tobin [29] on portfolio selection, Sharpe [27], Lintner [14], Treynor [30], Mossin [24], and Fama [7] developed the Capital Asset Pricing Model (CAPM) for the equilibrium structure of risky asset prices. Two of the basic assumptions of the CAPM are: (1) All assets are perfectly divisible and perfectly liquid. (2) All investors face the same set of investment opportunities. One implication of the classic CAPM is the linear relationship in equilibrium between each asset's expected return and systematic risk. Another implication is that all investors hold identical portfolios of risky assets (the so-called "market portfolio," which consists of an investment in every available risky asset in proportion to its total value).

The empirical findings of studies that test the validity of the classic CAPM generally substantiate the linear relationship between each asset's expected return and risk. However, there are some discrepancies between other theoretical implications of this model and empirical findings. The most obvious (and perhaps most disturbing) one is that in the real world investors do not hold identical portfolios of risky assets: some hold extremely undiversified portfolios, whereas others' portfolios are well-diversified (see, e.g., Blume, Crockett, and Friend [2]; Blume and Friend [3]; and the Federal Reserve Board's 1967 survey of the Financial Characteristics of Consumers [33]). Another disturbing finding is that the market risk premium as suggested by the theoretical CAPM, is overstated (Black, Jensen, and Scholes [1]; Douglas [6]; Lintner [15]; Miller and Scholes [23]).

In the present study, we relax the assumptions that all assets are perfectly divisible and that all investors face the same set of investment opportunities. Thus, we assume that investors (individuals, firms, or institutions) can invest in two types of risky assets: perfectly divisible, and perfectly indivisible (or, discrete). We also assume that investors face the same set of investment opportunities in risky divisible assets, but not the same set of investment opportunities in risky indivisible assets. (The investor's wealth determines, to a large extent, the set of indivisible investment opportunities that are available to him.)

Risky stocks, shares of mutual funds, insurance contracts, and risky commodities, are only some examples of risky divisible assets. On the other hand, investment opportunities in risky indivisible assets may include physical assets (i.e., real estate, capital budgeting projects), and "human capital" assets. Divisible assets are usually marketable and liquid to a large extent. Some indivisible assets are marketable, whereas others are nonmarketable; the transaction costs of trading in the marketable indivisible assets are relatively large.

This study focuses on the task of deriving the equilibrium conditions for the individual investor and for the market as a whole. The equilibrium conditions in this extended model have the same basic structure as the equilibrium conditions in the CAPM, with one difference: they are modified in order to reflect the covariations between the returns on risky divisible assets and the return on risky indivisible assets. Furthermore, the theoretical implications of the extended model are more consistent with empirical findings. In this respect, the present model provides explanations for some of the major discrepancies between the theoretical CAPM and the real-world behavior of the market.

In Section 2 we derive the individual investor's equilibrium and his optimal demand for risky divisible assets. It is shown that each investor modifies his demand by adjusting his evaluation of the risk premium required on these assets. A major implication of this analysis is that investors may hold different portfolios of risky assets.

Market equilibrium and risk-return relationships are derived in Section 3. Similar to the classic CAPM, the extended model suggests linear relationship between each asset's expected return and systematic risk. However, the asset's systematic risk and the market price per unit of risk are modified, in order to reflect (1) the covariations between the returns on investors' portfolios of risky divisible assets and the returns on their portfolios of risky indivisible assets, and (2) the fact that investors may hold different portfolios of risky assets. It is also demonstrated that the market risk premium implied by the classic CAPM, is indeed overstated.

Optimal decision rules for the evaluation of single risky projects (i.e., risky indivisible opportunities), are obtained in Section 4. It is argued that the "cost of capital" should be adjusted to reflect covariations of the returns on the project under consideration with the returns on the "market portfolio" of all existing risky projects.

This paper does not consider the issues of equilibrium prices and trading schemes in markets with indivisible risky assets (or commodities), beyond the analysis and derivation of optimal decision rules for evaluating risky indivisible opportunities as presented in Section 4. The important issues of equilibrium and trading schemes in markets with indivisible assets have been studied in the following three interrelated strands of economic research: equilibrium in exchange economies with non-convex consumption sets; the theory of monoply pricing schemes; and in auction and competitive bidding. In the first area of research, Broome [4] and Dierker [5] showed the existence of equilibrium in economies with indivisible commodities and a finite number of traders. In his model, Broome [4] also assumes the existence of a perfectly divisible commodity. Khan and Rashid [11] have extended this line of research and showed the existence of approximate equilibria in large but finite exchange economies in which all traders have neither convex preferences nor convex consumption sets: their model assumes markets with indivisible commodities and a perfectly divisible commodity. Other basic studies in this area of research are those of Mas-Colell [19], Yamazaki [32], and Khan and Yamazaki [10].

For many types of indivisible commodities, the seller of the commodity is monopolistic, and there may be many potential buyers for this commodity. The theory of monopoly pricing addresses the issues of equilibrium and trading schemes in this type of markets. Several monopolistic pricing schemes are commonly used in marketing the indivisible commodity: single-price strategy, auction, and "priority pricing" are the most common ones (see Harris and Raviv [9]). In [9], Harris and Raviv develop a model consisting of a single, monopolistic seller and N potential buyers; the seller produces the product with constant marginal production costs up to a capacity limit (which may or may not be binding); each buyer is assumed to demand up to one unit of the product at any price at or below his reservation price. They then proceed to derive results regarding the optimal marketing (trading) scheme, and the optimal (equilibrium) price under each trading scheme.

As mentioned above, auction and competitive bidding is one of the most common trading and pricing schemes used in markets with indivisible assets. A basic reference in this line of research is the study of Milgrom and Weber [22], in which a general auction model is presented, and results regarding expected prices of the objects being auctioned are obtained. Many other well-known auction models are shown to be special cases of this general model. An important and interesting result regarding winning bids in the context of auction was derived by

Milgrom [21]: In a sealed bid tender auction where each bidder has private information, the winning bid will converge in probability to the true value of the object at auction, even though no bidder knows the true value.

2. INVESTOR'S EQUILIBRIUM AND THE DEMAND FUNCTION FOR RISKY DIVISIBLE ASSETS

We will use the following notation:

x:	The n-vector of the investor's investment proportions in risky divisible assets.
y:	The m-vector of 0-1 variables such that $y_j=1$ if and only if risky indivisible asset j is undertaken by the investor, and $y_j=0$ otherwise.
μ^d, μ^i:	The n- and m-vectors of the expected "end-of-the-period" returns on the risky divisible and on the risky indivisible assets, respectively.
Σ^d, Σ^i:	The $n \times n$ and $m \times m$ variance-covariance matrices of the random returns $\tilde{\mu}^d$ and $\tilde{\mu}^i$ on risky divisible and on risky indivisible assets, respectively.
Σ^{di}:	The $n \times m$-covariance matrix of $(\tilde{\mu}^d, \tilde{\mu}^i)$.
v^i:	The m-vector of the current market values of the indivisible assets available to the investor.
W:	The investor's initial wealth.
Z:	An $m \times m$-matrix such that $Z_{ll}=v_l^i/W$ $l=1, \cdots, m$, but $Z_{lj}=0$ for all $l \neq j$.
$e=(1, \cdots, 1)$:	Vector of 1's of an appropriate order.

All investors are assumed to be risk-averse single-period expected utility maximizers, and that they all have homogeneous expectations. It is also assumed that all investors face the *same* set of investment opportunities in risky divisible assets, but that they face *different* sets of investment opportunities in risky indivisible assets. A risk-free divisible asset with return r is also assumed.

Let $U(\mu, \sigma^2)$ be the investor's utility function: it is assumed to possess the usual properties:

$$\frac{\partial U}{\partial \mu}>0, \qquad \frac{\partial U}{\partial \sigma^2}<0, \tag{1}$$

where μ and σ^2 are the expected return and variance, respectively, of a given portfolio (x, y) of divisible and indivisible assets, per \$ 1 of investment:

$$\mu=\mu^d x+\mu^i Zy+(1-eZy-ex)r, \tag{2}$$

$$\sigma^2=x\Sigma^d x+yZ\Sigma^i Zy+2x\Sigma^{di}Zy. \tag{3}$$

Then, the optimal portfolio (x, y) of investing simultaneously in divisible and indivisible assets is determined by solving the following mixed-integer nonlinear program P:

$$\text{(P)} \qquad \underset{x,y}{\text{maximize}}\, U(\mu, \sigma^2), \qquad \text{subject to} \quad y \in (0, 1),$$

where $y \in (0, 1)$ means that $y_j = 0$ or 1, all j.

Remark 2.1. A mathematical programming algorithm is reported in Lazimy [12] for solving the mixed-integer nonlinear program P, along with a numerical example. A mean-variance analysis is also used in [12] to derive the investment frontier facing the investor who invests in both divisible and indivisible risky assets.

In order to examine the effects of investing in risky indivisible assets on the risk-return relationship and on the equilibrium structure of risky divisible assets, we assume that J is the set of risky indivisible assets in investor k's optimal portfolio (x^k, y^k), where $y_j^k = 1$ if and only if $j \in J$. (Recall, however, that x^k and y^k are determined simultaneously, rather than in stages.) Also, define $z^k \equiv y^k Z^k$ to be the vector of the proportions of the investor's wealth invested in the risky indivisible assets. (Investors have different matrices Z, since they face different sets of investment opportunities in indivisible assets, and since they differ in their wealth.)

Given y^k and z^k, the optimal investment strategy x^k in risky divisible assets is determined by maximizing the Lagrange function L_k:

$$L_k = x\Sigma^d x + 2x\Sigma^{di} z^k + z^k \Sigma^i z^k + 2\lambda[\mu - \mu^d x - \mu^i z^k - (1 - ex - ez^k)r]. \tag{4}$$

Let $\sigma_{j.}^d \equiv (\sigma_{j1}^d, \cdots, \sigma_{jn}^d)$ and $\sigma_{j.}^{di} \equiv (\sigma_{j1}^{di}, \cdots, \sigma_{jm}^{di})$ be the j^{th} row of Σ^d and Σ^{di}, respectively. Differentiating L_k with respect to x_j, equating the resulting equation to 0, and solving for λ, yields:

$$\frac{1}{\lambda_k} = \frac{\mu_j^d - r}{\sigma_{j.}^d x^k + \sigma_{j.}^{di} z^k}. \tag{5}$$

Investor k will be in equilibrium if and only if

$$\frac{\mu_j^d - r}{\sigma_{j.}^d x^k + \sigma_{j.}^{di} z^k} = \frac{\mu_l^d - r}{\sigma_{l.}^d x^k - \sigma_{l.}^{di} z^k} \tag{6}$$

for all pairs (j, l) of risky divisible assets. Define:

$$\tilde{\mu}_k^d \equiv \tilde{\mu}^d x^k + (1 - ex^k - ez^k) r; \qquad \tilde{\mu}_k^i \equiv \tilde{\mu}^i z^k. \tag{7}$$

$\tilde{\mu}_k^d$ is the random return on the *divisible assets* in investor k's portfolio, and $\tilde{\mu}_k^i$ is the random return on the *indivisible assets* in his portfolio.

The common ratio in equation (6) is $(1/\lambda_k)$. By multiplying the numerator and denominator of this ratio by x_j^k and summing up over j, we obtain:

$$\frac{1}{\lambda_k} = \frac{\mu^d x^k - rex^k}{x^k \Sigma^d x^k + x^k \Sigma^{di} z^k} \equiv \frac{E(\tilde{\mu}_k^d) - r(1 - ez^k)}{\sigma^2(\tilde{\mu}_k^d) + \operatorname{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^i)}. \tag{8}$$

By substituting (8) in (6) and observing that

$$\sigma_{j.}^{d} x^{k} = \mathrm{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d) \tag{9.a}$$

$$\sigma_{j.}^{di} z^{k} = \mathrm{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^i)\,, \tag{9.b}$$

we obtain the necessary and sufficient condition for investor k to be in equilibrium:

$$E(\tilde{\mu}_j^d) = r + \frac{E(\tilde{\mu}_k^d) - r(1 - ez^k)}{\sigma^2(\tilde{\mu}_k^d) + \mathrm{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^i)} \cdot [\mathrm{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d) + \mathrm{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^i)]\,. \tag{10}$$

or

$$E(\tilde{\mu}_j^d) = r + \frac{E(\tilde{\mu}_k^d) - r(1 - ez^k)}{\mathrm{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)} \cdot \mathrm{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)\,. \tag{11}$$

Let β_{kj}^* be asset j's systematic risk in investor k's portfolio:

$$\beta_{kj}^* = \mathrm{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)/\mathrm{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)\,. \tag{12}$$

Then, the necessary and sufficient condition for investor k to be in equilibrium (equation (11)) can be written as

$$E(\tilde{\mu}_j^d) = r + [E(\tilde{\mu}_k^d) + r(1 - ez^k)]\beta_{kj}^*\,. \tag{13}$$

On the individual level, it is apparent that in his evaluation of the asset's total risk, the investor takes into consideration the covariation of the asset's return with the returns on his overall portfolio, which includes both divisible and indivisible assets.

The investor's optimal demand function for risky divisible assets is derived and interpreted next. Upon differentiating the Lagrange function L_k (equation (4)) with respect to x and λ and solving the first order conditions for x and λ, we obtain:

$$x^k = \lambda V(\mu^d - re) - V\Sigma^{di} z^k\,, \tag{14}$$

$$\lambda_k = \frac{\mu - (r + E_k + rF_k)}{B - 2rA + r^2 C}\,, \tag{15}$$

where

$$V \equiv (\Sigma^d)^{-1}\,, \quad A \equiv \mu^d V e\,, \quad B \equiv \mu^d V \mu^d\,, \quad C \equiv eVe\,, \tag{16}$$
$$E_k \equiv (\mu^i - \mu^d V \Sigma^{di}) z^k\,, \quad F_k \equiv (e - eV\Sigma^{di}) z^k\,.$$

A portfolio (μ, σ^2) is optimal if and only if (μ, σ^2) is the tangency point between the investor's indifference curve and the investment frontier facing the investor, or if and only if

$$\lambda_k = \frac{1}{2} (\partial\sigma^2/\partial\mu)_k\,,$$

where $(\partial\sigma^2/\partial\mu)_k$ is the investor's marginal substitution rate between risk and return. Also, note that (see equation (9.b))

$$\Sigma^{di}z^k=\begin{pmatrix}\operatorname{cov}(\tilde{\mu}_1^d,\tilde{\mu}_k^i)\\ \vdots\\ \operatorname{cov}(\tilde{\mu}_n^d,\tilde{\mu}_k^i)\end{pmatrix}\equiv\operatorname{cov}(\tilde{\mu}^d,\tilde{\mu}_k^i)\,.$$

Therefore, investor k's optimal demand for risky divisible assets is:

$$x^k=\frac{1}{2}\left(\frac{\partial\sigma^2}{\partial\mu}\right)_k V\left[\mu^d-2\left(\frac{\partial\mu}{\partial\sigma^2}\right)_k\operatorname{cov}(\tilde{\mu}^d,\tilde{\mu}_k^i)-re\right]. \tag{17}$$

Expression (17) is identical to the expression of the classic CAPM for the demand for risky assets, except for the term $2(\partial\mu/\partial\sigma^2)_k\operatorname{cov}(\tilde{\mu}^d,\tilde{\mu}_k^i)$. Under the present extended model, the investor modifies his demand for risky divisible assets by making adjustments in his evaluation of all risk premiums required on risky divisible assets. These adjustments reflect the fact that his optimal portfolio includes both divisible and indivisible risky assets, and they are function of: (1) $\operatorname{cov}(\tilde{\mu}^d,\tilde{\mu}_k^i)$, which is the vector of covariations of the returns on risky divisible assets with the returns on the portfolio y^k of risky indivisible assets. (2) $(\partial\mu/\partial\sigma^2)_k$, which is the inverse of the investor risk-aversion measure. Furthermore:

(a) The adjustments are proportional to the vector of covariations $\operatorname{cov}(\tilde{\mu}^d,\tilde{\mu}_k^i)$: the investor will attach relatively more weight to risky divisible assets that have relatively low covariation with the returns on risky indivisible assets.

(b) The more risk-averse the investor, the greater are the adjustments that he will make in evaluating the risk premiums required on risky divisible assets and, consequently, in his demand for these assets.

It should be emphasized that the investor's overall demand (x^k, y^k) for divisible and indivisible risky assets is determined *simultaneously* by solving the mixed-integer program P, rather than in stages. Thus, the practical importance of expression (17) for the individual investor is limited, since the vector of covariations $\operatorname{cov}(\tilde{\mu}^d,\tilde{\mu}_k^i)$ is unknown prior to solving program P. However, the optimal solution (x^k, y^k) of program P is such that the adjustments that he makes in evaluating all risk premiums (as determined by the vector $2(\partial\mu/\partial\sigma^2)_k\operatorname{cov}(\tilde{\mu}^d,\tilde{\mu}_k^i)$), fit his preferences and attitudes towards risk-return.

Remark 2.2. Mayers [20] extended the CAPM to include claims on risky nonmarketable assets. His expression for the demand for marketable assets [20, equation (62)] is similar to expression (17). However, in Mayers' model the claims on nonmarketable assets are known, so that the vector of covariations Σ_i^H [20, equation (62)] is known and the investor can make the indicated modifications in his demand for marketable assets.

Observe that the elements $\operatorname{cov}(\tilde{\mu}_j^d,\tilde{\mu}_k^i)$ $j=1,\cdots,n$ of the covariations vector $\operatorname{cov}(\tilde{\mu}^d,\tilde{\mu}_k^i)$ are function of the particular set J of indivisible risky assets undertaken by the investor, and of the proportions z_l^k, $l\in J$, of the investor's wealth W_k

invested in the indivisible assets. Furthermore, $\mathrm{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^i) \to 0$ as $z^k \to 0$; in other words, investors who invest smaller proportions of their wealth in indivisible assets will make smaller adjustments in their evaluation of the risk premiums required on risky divisible assets and, consequently, the impact on their demand for divisible assets will be smaller.

The demand function (17) can be written in the form:

$$x^k + V \cdot \mathrm{cov}(\tilde{\mu}^d, \tilde{\mu}_k^i) = \frac{1}{2}\left(\frac{\partial \sigma^2}{\partial \mu}\right)_k V \cdot (\mu^d - re). \tag{18}$$

The right-hand-side of (18) is the well-known expression for the demand for risky assets under the classic CAPM. Therefore, *after* each investor makes the adjustments $V \cdot \mathrm{cov}(\tilde{\mu}^d, \tilde{\mu}_k^i)$ due to his holdings of risky indivisible assets, then *all investors hold a linear combination of the same two portfolios*, one consists of the risk-free asset only, and the other is the so-called "market portfolio" of risky assets. However, unlike the classic CAPM where all frontier portfolios have the *same* composition of risky assets (except for a scaling constant), in the present extended model the composition of risky portfolios of divisible assets along the investment frontier, changes, even in the presence of a riskless asset. Consequently, separation (at least as it exists in the classic CAPM) does not hold in the present model.

Finally, it is clear that investors differ from one another with respect to the adjustments that they make in their evaluation of the risk premiums, as these adjustments are determined by $2(\partial \mu / \partial \sigma^2)_k \mathrm{cov}(\tilde{\mu}^d, \tilde{\mu}_k^i)$. The sources of these differences are two-fold: (1) Investors differ in their wealth and in the investment opportunities in risky indivisible assets that are available to them. (2) Investors differ in their preferences and in the degree of their risk-aversion. Consequently, *investors are likely to hold different portfolios of risky divisible assets*. This result is clearly consistent with the real-world behavior of investors, who differ widely in their investment strategy: some hold well-diversified portfolios, but other hold only few assets.

3. MARKET EQUILIBRIUM AND RISK-RETURN RELATIONSHIP

In this section we derive the equilibrium structure of risky divisible asset prices and the risk-return relationship, and examine the structure of the systematic risk of risky assets, the market price per unit of risk, and the market risk premium,.

3.1 *The Equilibrium Structure of Risky Asset Prices*

Equation (17) can be written in the form

$$[E(\tilde{\mu}_j^d) - r]\,\mathrm{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i) = [E(\tilde{\mu}_k^d) - r(1 - ez^k)]\,\mathrm{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i). \tag{19}$$

Multiplying equation (19) by W_k and summing up *only for investors k who hold risky divisible asset j*, yields:

$$[E(\tilde{\mu}_j^d)-r]\sum_k W_k \operatorname{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d+\tilde{\mu}_k^i)=\sum_k W_k[E(\tilde{\mu}_k^d)-r(1-ez^k)]\operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d+\tilde{\mu}_k^i)\,, \quad (20)$$

or

$$E(\tilde{\mu}_j^d)=r+\frac{\sum_k \gamma_k}{\sum_k W_k \operatorname{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d+\tilde{\mu}_k^i)}\cdot\frac{\sum_k \gamma_k \operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d+\tilde{\mu}_k^i)}{\sum_k \gamma_k}\,, \quad (21)$$

where

$$\gamma_k \equiv W_k[E(\tilde{\mu}_k^d)-r(1-ez^k)]\,. \quad (22)$$

Define:

V_j^d = the total market value of risky divisible asset j at the beginning of the period.

N_j = the number of outstanding shares of risky divisible asset j.

v_j^d = the current equilibrium price of one share of risky divisible asset j.

Then:[1]

$$\begin{aligned}\sum_k \gamma_k &= \sum_k W_k \sum_j x_j^k E(\tilde{\mu}_j^d) - r\sum_k W_k \sum_j x_j^k \\ &= \sum_j E(\tilde{\mu}_j^d)\sum_k x_j^k W_k - r\sum_j\sum_k x_j^k W_k \\ &= \sum_j E(\tilde{\mu}_j^d) v_j^d \sum_k N_j^k - r\sum_j v_j^d \sum_k N_j^k\,. \end{aligned} \quad (23)$$

Market equilibrium requires that all assets be held, i.e., $\sum_k N_j^k = N_j$. Therefore, we have in equilibrium (see (23))

$$\sum_k \gamma_k = \sum_j E(\tilde{\mu}_j^d) v_j^d N_j - r\sum_j v_j^d N_j = \sum_j E(\tilde{\mu}_j^d) V_j^d - r\sum_j V_j^d \equiv E(\tilde{R}_M^d) - rV_M^d\,, \quad (24)$$

since $V_j^d = v_j^d N_j$. The term

$$\tilde{R}_M^d = \sum_j \tilde{\mu}_j^d V_j^d \quad (25)$$

is the random total cash flow paid on the "market portfolio" of all risky divisible

[1] Since

$$\tilde{\mu}_k^d = \sum x_j^k \tilde{\mu}_j^k + r(1-ez^k-ex^k)\,,$$

therefore

$$E(\tilde{\mu}_k^d) - r(1-ez^k) = \sum x_j^k E(\tilde{\mu}_j^d) - r\sum x_j^k\,.$$

Also, we substitute $x_j^k = (v_j^d N_j^k)/W_k$ for the proportion of investor k's wealth invested in risky asset j, where N_j^k is the number of shares of asset j held by investor k.

assets.[2] (Recall that $\tilde{\mu}_j^d$ is the random return on asset j per \$ 1 of investment.) The term

$$V_M^d \equiv \sum_j V_j^d \tag{26}$$

is the beginning-of-the-period total market value of all risky divisible assets.

By substituting (24) in (21), we obtain the expression for the equilibrium structure of risky divisible asset prices under the extended model:

$$E(\tilde{\mu}_j^d) = r + \left[\frac{E(\tilde{R}_M^d) - rV_M^d}{\sum_k W_k \operatorname{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)}\right]\left[\frac{\sum_k \gamma_k \operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)}{\sum_k \gamma_k}\right]. \tag{27}$$

Define:

$\tilde{\mu}_M^d \equiv \tilde{R}_M^d / V_M^d$: the random return on the "market portfolio" of risky divisible assets.[3]

Therefore, $[E(\tilde{R}_M^d) - rV_M^d] = [V_M^d(E(\tilde{\mu}_M^d) - r)]$, and expression (27) can be written as:

$$E(\tilde{\mu}_j^d) = r + \pi^* \operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_M^*), \tag{28}$$

where

$$\pi^* \equiv \frac{E(\tilde{\mu}_M^d) - r}{\sum_k (W_k/V_M^d) \operatorname{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)} \tag{29}$$

[2] It is also clear from equation (24) that $\tilde{R}_M^d$ can be expressed as $\tilde{R}_M^d = \sum_k W_k \tilde{\mu}_k^d$, where $\tilde{\mu}_k^d$ is the random return on investor k's portfolio of risky divisible assets. Using definition (7) of $\tilde{\mu}_k^d$, we obtain:

$$\tilde{R}_M^d = \sum_k W_k \tilde{\mu}_k^d = \sum_k W_k \left[\sum_j \tilde{\mu}_j^d x_j^k + r\left(1 - \sum_{j \in J} z_j^k - \sum_j x_j^k\right)\right].$$

Since $x_j^k = (v_j^d N_j^k)/W_k$ and $\sum_k N_j^k = N_j$, it follows that

$$\tilde{R}_M^d = \sum_j \tilde{\mu}_j^d \sum_k v_j^d N_j^k + r \sum_k W_k \left(1 - \sum_j z_j^k - \sum_j x_j^k\right) = \sum_j \tilde{\mu}_j^d v_j^d N_j + r \sum_k \left(1 - \sum_j z_j^k - \sum_j x_j^k\right).$$

But

$$\sum_j \tilde{\mu}_j^d v_j^d N_j = \sum_j \tilde{\mu}_j^d V_j^d \equiv \tilde{R}_M^d.$$

Therefore:

$$\sum_k W_k \left(1 - \sum_{j \in J} z_j^k - \sum_j x_j^k\right) = 0,$$

which means that in equilibrium, the total market new borrowing or lending is zero. (This, however, does not imply that investor k's borrowing or lending is zero, since it is still possible that $(1 - ez^k - ex^k) \neq 0$ for the individual investor k.)

[3] Since $\tilde{R}_M^d = \sum_k W_k \tilde{\mu}_k^d$ (footonte 2), it follows that

$$\tilde{\mu}_M^d = \sum_k (W_k/V_M^d) \tilde{\mu}_k^d.$$

Therefore, $\tilde{\mu}_M^d$ is a weighted average of $\tilde{\mu}_k^d$, the random return on investor k's portfolio of risky divisible assets.

is the market price per unit of risk, and

$$\tilde{\mu}_M^* \equiv \frac{\sum_k \gamma_k(\tilde{\mu}_k^d + \tilde{\mu}_k^i)}{\sum_k \gamma_k} \tag{30}$$

is the random return on the "overall market portfolio" which is a weighted average of all investors' portfolios, and includes all risky assets, divisible as well as indivisible.

The extended model preserves the linear relationship between the asset's expected return and risk, as can be seen in equation (28). The implications of the extended model for the structure of the market systematic risk of risky assets and for the market price per unit of risk, are summarized below.

(a) The market systematic risky of risky asset j can be interpreted in two ways. If we write it as (see equation (27))

$$\left[\sum_k \gamma_k \operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)\right] \Big/ \sum_k \gamma_k \tag{31}$$

and recall that $\operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)$ is asset j's systematic risk in investor k's portfolio (equation (11)), then asset j's market systematic risk is a weighted average of that asset's systematic risk in all investors' portfolios. If, however, we write it as (see equations (28) and (30))

$$\operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_M^*) \equiv \operatorname{cov}\left(\tilde{\mu}_j^d, \left[\sum_k \gamma_k(\tilde{\mu}_k^d + \tilde{\mu}_k^i)\right] \Big/ \sum_k \gamma_k\right), \tag{32}$$

then asset j's risk that cannot be diversified away is measured by the covariation of the asset's returns with the returns $\tilde{\mu}_M^*$ on the "overall market portfolio," which is a weighted average of all investors' portfolios and includes all risky assets, divisible as well as indivisible.

Furthermore, observe that the asset's systematic risk in investor k's portfolio and the asset's market systematic risk can be written, respectively, as

$$\operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d) + \operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^i),$$

and

$$\operatorname{cov}(\tilde{\mu}_j^d, (\textstyle\sum \gamma_k \tilde{\mu}_k^d)/\sum \gamma_k) + \operatorname{cov}(\tilde{\mu}_j^d, (\sum \gamma_k \tilde{\mu}_k^i)/\sum \gamma_k).$$

Therefore, on the individual level, each investor modifies his evaluation of the asset's systematic risk by adding the covariation of the asset's returns with the returns on the portfolio y^k of risky indivisible assets held by him. On the aggregate level, on the other hand, the market as a whole modifies the asset's systematic risk by adding the covariation of the asset's returns with the returns on the "market portfolio" of risky indivisible assets, which is a weighted average of all investors' portfolios of risky indivisible assets.

(b) The market price per unit of risk π^* is measured by the risk premium required on the market portfolio of risky divisible assets per unit of *total risk* on this portfolio. Since

$$\sum_k (W_k/V_M^d)\operatorname{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d+\tilde{\mu}_k^i)=\sum_k (W_k/V_M^d)\sigma^2(\tilde{\mu}_k^d)+\sum_k (W_k/V_M^d)\operatorname{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^i)\,,$$

it is clear that the total risk on the market portfolio of risky divisible assets is a weighted sum of the risks on all investors' portfolios of divisible assets, and it is composed of two parts: a weighted sum of the variances of all investors' portfolios of risky divisible assets, and a weighted sum of the covariances of the returns on the investors' portfolios of divisible assets with the returns on their portfolios of indivisible assets.

3.2 *Risk-Return Relationships and Market Risk Premium*

One of the most disturbing findings of empirical studies that test the validity of the theoretical CAPM is that the market risk premium $(E(\tilde{\mu}_M)-r)$ as suggested by the theoretical CAPM, is overstated. In these studies, the risk-return relationship $(E(\tilde{\mu}_j)-r)=(E(\tilde{\mu}_M)-r)\beta_j$ where $\beta_j=\operatorname{cov}(\tilde{\mu}_j, \tilde{\mu}_M)/\sigma^2(\tilde{\mu}_M)$ is tested by running the cross-section regression $\bar{\mu}_j-r=\gamma_0+\gamma_j\hat{\beta}_j+e_j$, where $\bar{\mu}_j$ is the average return on asset j, e_j is a residual term, and $\hat{\beta}_j$ is the estimate for the systematic risk β_j. ($\hat{\beta}_j$ is obtained from the time-series regression $\mu_{jt}=\alpha_j+\beta_j\mu_{mt}+e_{jt}$, where μ_{jt} and μ_{mt}, respectively, are the returns on asset j and on the market portfolio at time period t.) Most empirical findings show that the estimate $\hat{\gamma}_j$ of the market risk premium as obtained from the cross-section regression is significantly smaller than $(\bar{\mu}_m-r)$, where $\bar{\mu}_m$ is the average return on the market portfolio as observed in the market. (See, e.g., Black, Jensen, and Scholes [1]; Douglas [6]; Lintner [15]; Miller and Scholes [23].) We will show that the extended model provides an explanation for this discrepancy between the theoretical CAPM and the empirical findings.

Recall equation (13) for the necessary and sufficient condition for investor k to be in equilibrium. Multiplying this equation by W_k, summing up *only for investors k who hold risky divisible asset j*, and solving for $E(\tilde{\mu}_j^d)$, yields:

$$E(\tilde{\mu}_j^d)=r+\frac{\sum_k \gamma_k\beta_{kj}^*}{\sum_k W_k}\,, \tag{33}$$

where β_{kj}^* is defined by (12), and γ_k by (22). Clearly, (33) can be written as

$$E(\tilde{\mu}_j^d)=r+\frac{\sum_k \gamma_k}{\sum_k W_k}\cdot\frac{\sum_k \gamma_k\beta_{kj}^*}{\sum_k \gamma_k}\,. \tag{34}$$

Recalling that market equilibrium requires that $\sum_k \gamma_k=(E(\tilde{R}_M^d)-rV_M^d)$ (equation (24)), we obtain the following expression for the equilibrium structure of risky divisible asset prices:

$$E(\tilde{\mu}_j^d)=r+\frac{E(\tilde{R}_M^d)-rV_M^d}{\sum_k W_k}\cdot\frac{\sum_k \gamma_k\beta_{kj}^*}{\sum_k \gamma_k}. \tag{35}$$

$V_M^k\equiv\sum_j V_j^d\equiv\sum_j\sum_k v_j^d N_j^k$ is the beginning-of-period total market value of all risky divisible assets (see equations (26), (24), and (23)). Similarly, define:

$$V_M^i\equiv\sum_k\sum_{j\in J_k} v_j^i. \tag{36}$$

V_M^i is the beginning-of-period total market value of all risky indivisible assets. (J_k is the set of indivisible assets in investor k's portfolio, and recall that v_j^i is the current market value of indivisible asset j.) Market equilibrium implies that (see footnote 2):

$$\sum_k W_k\left(1-\sum_{j\in J_k} z_j^k-\sum_j x_j^k\right)=0, \quad \text{or}$$

$$\sum_k W_k=\sum_k W_k\sum_j x_j^k+\sum_k W_k\sum_{j\in J_k} z_j^k=\sum_k\sum_j v_j^d N_j^k+\sum_k\sum_{j\in J_k} v_j^i\equiv V_M^d+V_M^i, \tag{37}$$

since $z_j^k\equiv v_j^i/w_k$ and $x_j^k\equiv(v_j^d N_j^k)/W_k$. By substituting (37) in (35) and recalling that $(E(\tilde{R}_M^d)-rV_M^d)=V_M^d[E(\tilde{\mu}_M^d)-r]$, the expression for the equilibrium structure of risky divisible asset prices becomes:

$$E(\tilde{\mu}_j^d)=r+\frac{V_M^d}{V_M^d+V_M^i}[E(\tilde{\mu}_M^d)-r]\beta_j^*, \tag{38}$$

where the modified market systematic risk

$$\beta_j^*=\left(\sum_k \gamma_k\beta_{kj}^*\right)\Big/\sum_k\gamma_k \tag{39}$$

is a weighted average of the asset's systematic risk

$$\beta_{kj}^*=\operatorname{cov}(\tilde{\mu}_j^d,\ \tilde{\mu}_k^d+\tilde{\mu}_k^i)/\operatorname{cov}(\tilde{\mu}_k^d,\ \tilde{\mu}_k^d+\tilde{\mu}_k^i)$$

in each investor's portfolio.

The linear relationship between the asset's expected return $E(\tilde{\mu}_j^d)$ and risk β_j^* is clearly seen in equation (39). Furthermore,

$$\frac{V_M^d}{V_M^d+V_M^i}[E(\tilde{\mu}_M^d)-r]<[E(\tilde{\mu}_M)-r], \tag{40}$$

from which it follows that the market risk premium implied by the classic CAPM is indeed overstated.

An interesting question is whether the market systematic risk $\beta_j^*=(\sum\gamma_k\beta_{kj}^*)/\sum\gamma_k$ in the extended model is greater or smaller than the systematic risk $\beta_j=\operatorname{cov}(\tilde{\mu}_j,\tilde{\mu}_M)/\sigma^2(\tilde{\mu}_M)$ in the classic CAPM. However, nothing can be said a priori regarding the relationship between β_j^* and β_j, since it depends on the relationships between β_j and $\beta_{kj}^*=\operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_k^d+\tilde{\mu}_k^i)/\operatorname{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d+\tilde{\mu}_k^i)$. If, for instance, $\beta_{kj}^*<\beta_j$ for all k, then $\beta_j^*<\beta_j$ since β_j^* is a weighted average of β_{kj}^*.

The risk premium on the market portfolio of risky divisible assets in the extended model is smaller than that suggested by the classic CAPM, as shown in (40). However, the relationship between the risk premium $(E(\tilde{\mu}_j^d)-r)$ on the *individual asset j* as suggested by the extended model and the risk premium on the same asset as suggested by the classic CAPM, depends on the relationship between β_j^* and β_j. From (38), if

$$\frac{V_M^d}{V_M^d+V_M^i}\beta_j^*<\beta_j ,$$

then the risk premium required on asset j under the extended model is smaller than that required under the CAPM.

4. OPTIMAL DECISION RULES FOR EVALUATING SINGLE RISKY PROJECTS

Thus far we derived market equilibrium conditions for the price structure of *risky divisible assets*. However, optimal decision rules for the evaluation of single *risky indivisible assets* can also be derived, as shown next.

Several researchers (e.g., Hamada [8]; Levy and Sarnat [13]; Lintner [16]; Mossin [25]; Rubinstein [26]; Stapleton [28]; Tuttle and Litzenberger [31]) applied the stock valuation condition of the CAPM to corporate capital budgeting decisions under conditions of risk. If l denotes firm l, the expression

$$E(\tilde{\mu}_l)=r+\pi\operatorname{cov}(\tilde{\mu}_l, \tilde{\mu}_M) , \qquad \text{where} \quad \pi\equiv(E(\tilde{\mu}_M)-r)/\sigma^2(\tilde{\mu}_M) ,$$

provides the stock value of firm l. Therefore, a shareholders' wealth-maximizing firm will undertake a risky project z if, and only if, it contributes to the firm's stock value more than it costs the shareholders, that is, if and only if

$$E(\tilde{\mu}_z)\geq r+\pi\operatorname{cov}(\tilde{\mu}_z, \tilde{\mu}_M) \tag{41}$$

where $\tilde{\mu}_z$ is the random return on project z. In fact, the term $r+\pi\operatorname{cov}(\tilde{\mu}_z, \tilde{\mu}_M)$ is interpreted in this case as the appropriate risk-adjusted discount rate for project z, or the cost of capital for this investment project.

The same approach can be employed for evaluating risky indivisible assets. Let l denote a firm whose stock is traded in the market:

$$E(\tilde{\mu}_l^d)=r+\pi^*\operatorname{cov}(\tilde{\mu}_l^d, \tilde{\mu}_M^*)$$

is the expression for firm l's stock value (see (28)), where π^* and $\tilde{\mu}_M^*$ are defined by (29) and (30), respectively. The firm will undertake investment in risky indivisible asset z if and only if

$$E(\tilde{\mu}_z^i)\geq r+\pi^*\operatorname{cov}(\tilde{\mu}_z^i, \tilde{\mu}_M^*) . \tag{42}$$

Therefore, under the extended model the cost of capital for evaluating risky project z must be modified in order to take into account the covariations of the returns on the project with the returns on the "market portfolio" of risky indivisible assets.[4]

The market price per unit of risk π^* (see (29)), can be interpreted as the risk-standardized cost of capital. It is a market parameter, which is appropriate for all firms and for evaluating all risky capital projects. If $\operatorname{cov}(\tilde{\mu}_z^i, \tilde{\mu}_M^*) > 0$, then risky project z should be undertaken if and only if

$$\frac{E(\tilde{\mu}_z^i) - r}{\operatorname{cov}(\tilde{\mu}_z^i, \tilde{\mu}_M^*)} \geq \pi^* . \tag{43}$$

The random return $\tilde{\mu}_M^*$ on the "overall market portfolio" can be written as (see (30))

$$\tilde{\mu}_M^* = [\gamma_l(\tilde{\mu}_l^d + \tilde{\mu}_l^i)]/\sum \gamma_k + \left[\sum_{k \neq l} \gamma_k(\tilde{\mu}_k^d + \tilde{\mu}_k^i)\right] \Big/ \sum \gamma_k . \tag{44}$$

Firm's l existing investments represent only an insignificant portion of the market's total existing investments. Therefore, the term $[\gamma_l(\tilde{\mu}_l^d + \tilde{\mu}_l^i)]/\sum \gamma_k$ is likely to be insignificant in comparison to $\tilde{\mu}_M^*$ and, consequently, $\operatorname{cov}(\tilde{\mu}_z^i, [\gamma_l(\tilde{\mu}_l^d + \tilde{\mu}_l^i)]/\sum \gamma_k)$ is likely to be insignificant in comparison to $\operatorname{cov}(\tilde{\mu}_z^i, \tilde{\mu}_M^*)$. Therefore, decision rule (43) can be written as

$$\frac{E(\tilde{\mu}_z^i) - r}{\operatorname{cov}\left(\tilde{\mu}_z^i, \left[\sum_{k \neq l} \gamma_k(\tilde{\mu}_k^d + \tilde{\mu}_k^i)\right] \Big/ \sum \gamma_k\right)} \geq \pi^* .$$

In other words, the contribution of project z to the firm's variance has very little effect on decision rule (43), and it can be ignored. Therefore, risky project z should be evaluated solely in terms of its own expected return $E(\tilde{\mu}_z^i)$ and systematic risk $\operatorname{cov}(\tilde{\mu}_z^i, \tilde{\mu}_M^*)$, without reference to the firm's existing investments. In fact, the diversifiable unsystematic risk in the firm's investment portfolio is eliminated indirectly by the investors via their own diversification, so that the firm need not diversify. (Proofs of the proposition that portfolio diversification and corporate risk diversification are perfect substitutes were offered by Mossin [25] and by Levy and Sarnat [13].)

5. SUMMARY AND CONCLUDING REMARKS

The main results of this study are summarized here, and some concluding remarks are made.

(a) If some risky indivisible assets are included in the investor's optimal

[4] The second order effects of the acceptance of project z on $\tilde{\mu}_M^*$ and on π^* can be ignored, since the influence of a single investment project on the market-wide parameters $\tilde{\mu}_M^*$ and π^* is likely to be insignificant.

portfolio, then his demand for risky divisible assets is modified according to the following function:

$$x^k = \frac{1}{2}\left(\frac{\partial \sigma^2}{\partial \mu}\right)_k V\left[\mu^d - 2\left(\frac{\partial \mu}{\partial \sigma^2}\right)_k \operatorname{cov}(\tilde{\mu}^d, \tilde{\mu}_k^i) - re\right]. \tag{45}$$

The adjustments that he makes in his evaluation of the risk premium required on risky assets are determined by (1) the covariations of the returns on risky divisible assets with the returns on risky indivisible assets, (2) the investor's wealth, and (3) the investor risk-aversion.

The major implication of equation (45) is that investors do not necessarily hold identical portfolios of risky assets. This result is consistent with the real-world behavior of investors, as documented by many empirical studies that show that investors differ in their investment strategy. (See, e.g., Blume, Crockett, and Friend [2]; Blume and Friend [3].) Therefore, the relaxation of the assumption that all assets are perfectly divisible eliminates one of the most unattractive implications of the classic CAPM (i.e., that all investors hold identical portfolios). In this sense, the present extended model bridges the gap between the theoretical CAPM and empirical findings.

(b) The equilibrium structure of risky divisible asset prices is given by the expression

$$E(\tilde{\mu}_j^d) = r + \pi^* \operatorname{cov}(\tilde{\mu}_j^d, \tilde{\mu}_M^*), \tag{46}$$

where

$$\pi^* \equiv \frac{E(\tilde{\mu}_M^d) - r}{\sum_k (W_k/V_M^d) \operatorname{cov}(\tilde{\mu}_k^d, \tilde{\mu}_k^d + \tilde{\mu}_k^i)}, \qquad \tilde{\mu}_M^* \equiv \frac{\sum_k \gamma_k(\tilde{\mu}_k^d + \tilde{\mu}_k^i)}{\sum_k \gamma_k},$$

or, equivalently, by

$$E(\tilde{\mu}_j^d) = r + \frac{V_M^d}{V_M^d + V_M^i}[E(\tilde{\mu}_M^d) - r]\beta_j^*, \qquad \beta_j^* \equiv \left(\sum_k \gamma_k \beta_{kj}^*\right) \Big/ \sum_k \gamma_k. \tag{47}$$

The major implications of these expressions are:

(1) There is a linear relationship between each asset's expected return and systematic risk.

(2) The asset's systematic risk and the market price per unit of risk are modified in order to reflect the covariations between the returns on the investors' portfolios of risky divisible assets and the returns on their portfolios of risky indivisible assets. These modifications also reflect the fact that investors may hold different portfolios of risky assets.

(3) The extended model suggests that the market risk premium implied by the classic CAPM, is overstated, since $[V_M^d/(V_M^d + V_M^i)][E(\tilde{\mu}_M^d) - r] < [E(\tilde{\mu}_M) - r]$. This provides theoretical explanations for one of the major discrepancies between the theoretical CAPM and empirical findings.

(c) The criterion for evaluating a single risky project z (i.e., a risky indivisible asset), is

$$E(\tilde{\mu}_z^i) \geq r + \pi^* \operatorname{cov}(\tilde{\mu}_z^i, \tilde{\mu}_M^*) .$$

Thus, the "cost of capital" for evaluating risky projects should be adjusted to reflect covariations of the returns on the projects with the returns on the "market portfolio" of all existing risky projects. The appropriate cost of capital is equal to the risk-free rate r plus a risk premium which depends only on the project's adjusted systematic risk, and each project should be evaluated solely in terms of its own expected return and adjusted systematic risk.

Finally, note that expressions (45)–(47) in the extended model have the same basic structure of the well-known expressions of the classic CAPM, despite the facts that investors in the extended model face different investment frontiers and may hold widely different portfolios of risky assets. In this sense, the present model is a generalization of the classic CAPM, and it also provides theoretical explanations for some of the discrepancies between the theoretical CAPM and empirical findings. In fact, the CAPM can easily be derived from the extended model. If investment opportunities in indivisible assets are not considered, then

$$\tilde{\mu}_j^d \equiv \tilde{\mu}_j , \qquad V_M^d = \sum_k W_k = W_0 , \qquad \tilde{\mu}_k^d \equiv \tilde{\mu}_M ,$$

$$\tilde{\mu}_M^d = \sum_k (W_k / V_M^d) \tilde{\mu}_k^d = \tilde{\mu}_M , \qquad \gamma_k = W_k [E(\tilde{\mu}_M) - r] ,$$

and expressions (45) and (46), respectively, become:

$$x^k = \frac{1}{2} \left(\frac{\partial \sigma^2}{\partial \mu} \right)_k V(\mu - re) ,$$

$$E(\tilde{\mu}_j) = r + \frac{E(\tilde{\mu}_M) - r}{\sigma^2(\tilde{\mu}_M)} \cdot \operatorname{cov}(\tilde{\mu}_j, \tilde{\mu}_M) .$$

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