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OUTCOME FUNCTIONS YIELDING WALRASIAN ALLOCATIONS AT NASH EQUILIBRIUM POINTS IN A PRIVATE OWNERSHIP ECONOMY*

Shinsuke NAKAMURA

Abstract: It is known that the outcome function realizing Walrasian equilibrium correspondence can easily be constructed provided an auctioneer is permitted. Schmeidler and Hurwicz have shown that, even if an auctioneer is not permitted, there are some mechanisms implementing Walras correspondence. Their formulations were restricted, however, to the cases of pure exchange economy. In this note, we construct mechanisms which implement Walras correspondence in a private ownership economy which is not particularly restricted to the pure exchange economy. In these mechanisms prices are set by firms or consumers and no auctioneer is required.

1. INTRODUCTION

In this note we discuss implementability of Walrasian correspondence through Nash strategies in a private ownership economy. It is well-known that the outcome function realizing Walrasian equilibrium correspondence can easily be constructed provided an auctioneer is permitted.

Schmeidler and Hurwicz have shown that, even if an auctioneer is not permitted, there are some mechanisms implementing Walras correspondence. In particular, Schmeidler (1980) constructs a mechanism which is balanced and implements Walras allocations at strong equilibrium points. But his mechanism lacks continuity and he considers only a neo-classical economy. Hurwicz (1979) constructs a mechanism which is continuous and balanced, and implements Walras correspondence in a general pure exchange economy.

Their formulations were restricted, however, to the cases of pure exchange economy. And in an economy with productions this implementation problem of Walrasian equilibria is left as an open problem. We construct continuous mechanisms which are implementing Walras correspondence in a private ownership economy that is not particularly restricted to a pure exchange case. In these mechanisms prices are set by firms or consumers and no auctioneer is required.

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2. FIRMS AS PLAYERS

Consider an economy with $l+1$ commodities which is denoted by (x, y) where $x \in R$ (numeraire) and $y \in R^l$ (others).

Let I be a finite set of consumers and J be a finite set of firms. For all $i \in I$ let $X_i \subset R \times R^l$ denote the consumption set of i in terms of his net demand and $\succsim_i$ be the complete and monotone preordering on X_i. For all $j \in J$ let $Y_j \subset R \times R^l$ denote the production possibilities of j. Let $\theta_{ij} \in R_+$ be such that

$$\textstyle\sum_i \theta_{ij} = 1 \qquad \text{for all} \quad j .$$

This represents a profit share.

In this economy we construct mechanisms which implement a Walras correspondence. First, we consider a mechanism where firms are players and price setters. In an economy where firms are players it is natural that prices are set by firms.

DEFINITION 2.1. Let

$$M^i = R^l \qquad \text{for every} \quad i ,$$

$$M^j = R^l_+ \times R \times R^l \qquad \text{for every} \quad j \in J ,$$

and

$$M = \textstyle\prod_i M^i \times \prod_j M^j .$$

The dimension of this message space is equal to $\#I \times l + \#J \times (2l+1)$ which is larger than the dimension required in the usual Walras mechanism. But it is important that the dimension of our message space is finite.

DEFINITION 2.2. For every $m = ((y_i)_i, (p_j, z_j^x, z_j^y)_j) \in M$ let

$$y_i(m) = y_i - y_{i+1} + \textstyle\sum_j z_j^y / \#I \tag{1}$$

$$x_i(m) = -(\textstyle\sum_k p_k / \#J) y_i(m) + \sum_j \theta_{ij}((\sum_k p_k / \#J) z_j^y(m) + z_j^x(m)) \tag{2}$$

$$z_j^y(m) = z_j^y \tag{3}$$

$$z_j^x(m) = z_j^x - (p_j - p_{j+1})^2 + (p_{j+1} - p_{j+2})^2 \tag{4}$$

$$p_j(m) = \textstyle\sum_{k \neq j} p_k / (\#J - 1) . \tag{5}$$

y_i denotes the reported demand by consumer i and z_j and p_j denote the reported supply and reported price by firm j, respectedly. But to implement Walrasian correspondence, we need some rule to ration. Definition 2.2 means this rule. Equation (1) means that the aggregate excess demand of commodity y of the whole economy is always equal to zero. Equation (2) assures the budget constraint of the i-th consumer. With regard to firms, there is no rationing in principle. But firms set prices. Hence the second term of (4) is the tax for the j-th firm to determine the different prices from the other's. Equation (5) decides the prices faced on the j-th firm as the mean of the other's reported prices.

DEFINITION 2.3. For every $m \in M$ let

$$M^i(m_{-i}) = \{s \in M^i \mid (x_i(s, m_{-i}), y_i(s, m_{-i})) \in X_i\},$$

$$M^j(m_{-j}) = \{s \in M^j \mid (z_j^x(s, m_{-j}), z_j^y(s, m_{-j})) \in Y_j\},$$

and

$$M(m) = \prod_i M^i(m_{-i}) \times \prod_j M^j(m_{-j}).$$

A Nash equilibrium can be defined as follows.

DEFINITION 2.4. $m^* \in M(m^*)$ is called a Nash equilibrium if
(i) for every i,

$$(x_i(m^*), y_i(m^*)) \succsim_i (x_i(m_i, m^*_{-i}), y_i(m_i, m^*_{-i}))$$

for every $m_i \in M^i(m^*_{-i})$ and
(ii) for every j,

$$p_j(m^*) z_j^y(m^*) + z_j^x(m^*) \geqq p_j(m_j, m^*_{-j}) z_j^y(m_j, m^*_{-j}) + z_j^x(m_j, m^*_{-j})$$

for every $m_j \in M^j(m^*_{-j})$.
The corresponding allocation is called a Nash allocation.

Then one can assert the following theorems.

THEOREM 2.1. *This game is balanced, i.e., for every* $m \in M$

(i) $$\textstyle\sum_i x_i(m) = \sum_j z_j^x(m) \quad \textit{and}$$

(ii) $$\textstyle\sum_i y_i(m) = \sum_j z_j^y(m).$$

THEOREM 2.2. *If* $\#J \geqq 3$, *then every Nash allocation is a Walras allocation.*

THEOREM 2.3. *If* $\#J \geqq 3$, *then every Walras allocation is a Nash allocation.*

3. FIRMS AS NON-PLAYERS

Next, we consider an economy where only consumers are players and hence consumers set prices. Firms are non-player participants. For in a private ownership economy, there exists a thought such that consumers ultimately decide the production plan through shareholders' meetings. (See Grossman and Stiglitz (1977) for example.)

DEFINITION 3.1. Let

$$M^i = R^l_+ \times R^l \times (R \times R^l)^J \qquad \text{and} \qquad M = \prod_i M^i.$$

DEFINITION 3.2. For every $m=((p_i, y_i, (z^x_{ij}, z^y_{ij})_j)_i)\in M$, let

$$y_i(m)=y_i-y_{i+1}+\sum_j z^y_{i+1,j} \tag{6}$$

$$x_i(m)=-p_{i+1}y_i(m)-(p_i-p_{i+1})^2+\sum_j \theta_{ij}(p_{i+1}z^y_j(m)+z^x_j(m)) \tag{7}$$

where

$$z^x_j(m)=\sum_i z^x_{ij} \quad \text{and} \quad z^y_j(m)=\sum_i z^y_{ij}\,.$$

Equation (6) corresponds to Eq. (1) and assures the balance of demand and supply of commodity y. Equation (7) corresponds to Eq. (2).

DEFINITION 3.3. For every $m\in M$, let

$$M^i(m_{-i})=\{s\in M^i \mid (x_i(s, m_{-i}), y_i(s, m_{-i}))\in X_i \text{ and } (z_j(s, m_{-i}), z_j(s, m_{-i}))\in Y_j \text{ for every } j\}\,,$$

and

$$M(m)=\prod_i M^i(m_{-i})\,.$$

A Nash equilibrium in this case can be defined as follows.

DEFINITION 3.4. $m^*\in M(m^*)$ is called a Nash equilibrium if for every i,

$$(x_i(m^*), y_i(m^*))\succsim_i (x_i(m_i, m^*_{-i}), y_i(m_i, m^*_{-i}))$$

for every $m_i\in M^i(m^*_{-i})$. The corresponding allocation is called a Nash allocation.

Then the following two theorems hold.

THEOREM 3.1. *Every Nash allocation is a Walras allocation.*

THEOREM 3.2. *Every Walras allocation is a Nash allocation.*

This mechanism lacks balancedness. Hence it is desirable to solve this defect.

4. PROOFS

Proof of Theorem 2.1. Obvious.

Proof of Theorem 2.2. Let m^* be a Nash equilibrium. Then

$$p_j(m^*)=p_{j'}(m^*)\equiv p^{**} \quad \text{for every } j \text{ and } j'\in J\,.$$

Define

$$y^{**}_i=y_i(m^*)\,,\qquad x^{**}_i=x_i(m^*)\,,\qquad z^{**}_j=z_j(m^*)\,.$$

Then the balance of demand and supply is obvious. First, we will show utility maximization. Let

$$B_i(p^{**})=\{(x, y)\in X_i \mid x+p^*y\leqq\sum_j \theta_{ij}(p^{**}z^{y**}_j+z^{x**}_j)\}\,.$$

Then $(x_i^{**}, y^{**}) \in B_i(p^{**})$.
Fix $(x, y) \in B_i(p^{**})$. Without loss of generality we may assume

$$x + p^{**}y = \sum_j \theta_{ij}(p^{**}z_j^{y**} + z_j^{x**}) .$$

Let

$$y_i = y + y_{i+1}^* - \sum_j z_j^{y*} .$$

Then

$$y_i((p_i^*, y_i), m_{-i}^*) = y , \qquad \text{and hence} \quad x_i((p_i^*, y_i), m_{-i}^*) = x .$$

Hence

$$(x_i^{**}, y_i^{**}) \succsim_i (x, y) .$$

Next we will show profit maximization. Let $z_j \in Y_j$. Then

$$p_j((p_j^*, z_j), m_{-j}^*) = p^{**} \qquad \text{and} \qquad z_j((p_j^*, z_j), m_{-j}^*) = z_j .$$

Thus

$$p^{**}z_j^{y**} + z_j^{x**} \geqq (1, p^{**})z_j . \qquad \text{Q.E.D.}$$

Proof of Theorem 2.3. Let

$$(p^*, (x_i^*, y_i^*)_i, (z_j^{x*}, z_j^{y*})_j)$$

be a Walras equilibrium.
Define m^{**} as

$$p_j^{**} = p^* , \qquad z_j^{x**} = z_j^{x*} , \qquad z_j^{y**} = z_j^{y*} , \qquad \text{and}$$

$$y_i^{**} \quad \text{be such that} \quad y_i^{**} - y_{i+1}^{**} = y_i^* - \sum_j z_j^{y**} .$$

Then the assertion follows.

Proof of Theorem 3.1. Let m^* be a Nash equilibrium. Then

$$p_i^* = p_{i'}^* \equiv p^* \qquad \text{for every} \quad i \quad \text{and} \quad i' \in I .$$

Let

$$x_i^* = x_i(m^*) , \qquad y_i^* = y_i(m^*) , \qquad \text{and} \qquad z_j^* = z_j(m^*) .$$

Then balance of demand and supply is obvious.

First, we will show utility maximization. Let

$$B_i(p^*) = \{(x, y) \in X_i \mid x + p^*y \leqq \sum_j \theta_{ij}(1, p^*)z_j^*\} .$$

Note that $(x_i^*, y_i^*) \in B_i(p^*)$. Let $(x, y) \in B_i(p^*)$.
We may assume that

$$x + p^*y = \sum_j \theta_{ij}(1, p^*)z_j^* .$$

Define

$$y_i = y + y^*_{i+1} - \sum_j z^{y*}_{i+1,j}.$$

Then

$$y = y_i((p^*_i, y_i, z^*_i), m^*_{-i}) \qquad \text{and hence} \quad x = x_i((p^*_i, y_i, z^*), m^*_{-i}).$$

Thus

$$(x^*_i, y^*_i) \succsim_i (x, y).$$

Next, we will show profit maximization. Suppose that there exists $(z^x_j, z^y_j) \in Y_j$ such that

$$z^x_j + p^* z^y_j > z^{x*}_j + p^* z^{y*}_j.$$

Choose $i \in I$ such that $\theta_{ij} > 0$. Let

$$z^x_{ij} = z^x_j - \sum_{k \neq j} z^{x*}_k \qquad \text{and} \qquad z^y_{ij} = z^y_j - \sum_{k \neq j} z^{y*}_k.$$

Define $z_i = (z_{ij}, z^*_{i,-j})$.
Then

$$y_i((p^*_i, y^*_i, z_i), m^*_{-i}) = y^*_i \qquad \text{and}$$

$$x_i((p^*_i, y^*_i, z_i), m^*_{-i}) > x^*_i,$$

which contradicts the fact that $\succsim_i$ is monotone, since m^* is a Nash equilibrium. Q.E.D.

Proof of Theorem 3.2. Let (p^*, x^*, y^*, z^*) be a Walras equilibrium.

Define m^{**} as

$$p^{**}_i = p^*, \qquad z^{**}_{ij} = z^*_j/\#I,$$

and y^{**}_i be such that

$$y^{**}_i - y^{**}_{i+1} = y^*_i - \sum_j z^{y*}_{i+1,j}.$$

Then

$$y_i(m^{**}) = y^*_i, \qquad z_j(m^{**}) = z^*_j, \qquad \text{and hence} \quad x_i(m^{**}) = x^*_i$$

by monotonicity.
Let $(p_i, y_i, z_i) \in M^i(m^{**}_{-i})$. Note that

$$(x_i((p^*, y_i, z_i), m^{**}_{-i}), y_i((p^*, y_i, z_i), m^{**}_{-i}))$$
$$\succsim_i (x_i((p_i, y_i, z_i), m^{**}_{-i}), y_i((p_i, y_i, z_i), m^{**}_{-i})).$$

Hence we may assume $p_i = p^*$. By profit maximization,

$$\begin{aligned}&x_i((p^*, y_i, z_i), m^{**}_{-i}) + p^* y_i((p^*, y_i, z_i), m^{**}_{-i})\\ &= \textstyle\sum_j \theta_{ij}(p^* z_j^y((p^*, y_i, z_i), m^{**}_{-i}) + z_j^x((p^*, y_i, z_i), m^{**}_{-i}))\\ &\leqq \textstyle\sum_j \theta_{ij}(p^* z_j^{y*} + z_j^{x*})\,.\end{aligned}$$

Hence

$$(x_i((p^*, y_i, z_i), m^{**}_{-i}), y_i((p^*, y_i, z_i), m^{**}_{-i}))$$

satisfies the budget constraint. Thus

$$(x_i(m^{**}), y_i(m^{**})) \succsim_i (x_i((p^*, y_i, z_i), m^{**}_{-i}), y_i((p^*, y_i, z_i), m^{**}_{-i}))\,.$$

Q.E.D.

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