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NECESSARY AND SUFFICIENT CONDITIONS FOR QUASI-TRANSITIVITY AND TRANSITIVITY OF SPECIAL MAJORITY RULES

Satish K. JAIN

Abstract: It is shown that for every special majority rule (i) value-restriction, limited agreement and weakly antagonistic preferences constitute a set of necessary and sufficient conditions for quasi-transitivity of the social preference relation (ii) strong value restriction, a condition stronger than both value-restriction and extremal restriction, is necessary and sufficient for transitivity of the social preference relation.

INTRODUCTION

Inada, Sen and Pattanaik have obtained necessary and sufficient conditions for quasi-transitivity and transitivity of the social preference relation generated by the simple majority rule. In this paper we obtain the corresponding results for the class of special majority rules. The motivation for this study is twofold. Some of the special majority rules, especially the two-thirds majority rule, are widely used in national and international decision-making bodies, particularly in the context of constitutional amendments. Therefore, it is important to characterize for these rules the configurations of individual preferences which yield rational social preferences. Secondly, the study of special majority rules is important from a theoretical standpoint as they are closely related to the simple majority rule. Most of the properties which the simple majority rule satisfies are shared by the special majority rules.

We show that for every special majority rule value-restriction, limited agreement and weakly antagonistic preferences constitute a set of necessary and sufficient conditions for quasi-transitivity of the social preference relation. Thus, conditions for quasi-transitivity of special majority rules are the same as that of the simple majority rule. For transitivity of social preference relation generated by any special majority rule, a condition introduced in this paper called strong value restriction is shown to be both necessary and sufficient. Strong value restriction is a more demanding requirement than either value-restriction or extremal restriction. Therefore, the extremal restriction which is necessary and sufficient for transitivity of the simple majority rule is necessary but not sufficient for transitivity of the special majority rules.

RESTRICTIONS ON PREFERENCES

The set of social alternatives would be denoted by S. The cardinality n of S would be assumed to be finite and greater than 2. The set of individuals and the number of individuals are designated by L and N respectively. $N(\ )$ would stand for the number of individuals holding the preferences specified in the parentheses, and N_k for the number of individuals holding the k-th preference ordering. Each individual $i \in L$ is assumed to have an ordering R_i defined over S. The symmetric and asymmetric parts of R_i are denoted by I_i and P_i respectively. The social preference relation is denoted by R and its symmetric and asymmetric components by I and P respectively.

Special Majority Rules:

$$\forall x, y \in S: \quad xRy \quad \text{iff} \quad N(yP_ix) \leqslant p[N(xP_iy) + N(yP_ix)],$$

where p is a fraction such that $1/2 < p < 1$. For $p = 2/3$ we obtain the familiar two-thirds majority rule.

An individual is defined to be concerned with respect to a triple iff he is not indifferent over every pair of alternatives belonging to the triple; otherwise he is unconcerned. For individual i, in the triple $\{x, y, z\}$, x is best iff $(xR_iy \wedge xR_iz)$; medium iff $(yR_ixR_iz \vee zR_ixR_iy)$; worst iff $(yR_ix \wedge zR_ix)$; uniquely best iff $(xP_iy \wedge xP_iz)$; uniquely medium iff $(yP_ixP_iz \vee zP_ixP_iy)$; and uniquely worst iff $(yP_ix \wedge zP_ix)$.

Now we define several restrictions which specify the permissible sets of individual orderings. All these restrictions are defined over triples of alternatives.

Value-Restriction (VR): It holds over a triple iff there is an alternative in the triple such that all concerned individuals agree that it is not best or it is not medium or it is not worst.

Limited Agreement (LA): It holds over $\{x, y, z\}$ iff there exist distinct $a, b \in \{x, y, z\}$ such that $\forall i \in L$: aR_ib.

Dichotomous Preferences (DP): It holds over a triple iff no individual has a strong ordering over the triple.

Weakly Antagonistic Preferences (WAP)[1]: It holds over $\{x, y, z\}$ iff

$$\forall a, b, c \in \{x, y, z\}: [(\exists i: aP_ibP_ic) \rightarrow \forall i: (aP_ibP_ic \vee cP_ibP_ia \vee aI_ic)].$$

Strong Value Restriction (SVR): It is satisfied over a triple iff there exists (i) an

[1] WAP is logically equivalent to the union of Inada's Antagonistic Preferences (AP) and Dichotomous Preferences (DP). VR, LA and DP have the property that if a set of R_i satisfies any of them then the condition holds over every subset of R_i as well. AP does not possess this property. WAP, however, satisfies this property. In the context of derivation of maximal configurations which would yield rational social preferences it is convenient to deal with conditions which possess this property.

alternative such that it is best in every R_i or (ii) an alternative such that it is worst in every R_i or (iii) an alternative such that it is uniquely medium in every concerned R_i or (iv) a pair of distinct alternatives such that every individual is indifferent between the alternatives of the pair. More formally, SVR holds over $\{x, y, z\}$ iff there exist distinct $a, b, c \in \{x, y, z\}$ such that $[\forall i\colon (aR_ib \wedge aR_ic) \vee \forall i\colon (bR_ia \wedge cR_ia) \vee \forall$ concerned $i\colon (bP_iaP_ic \vee cP_iaP_ib) \vee \forall i\colon aI_ib]$.

CONDITIONS FOR QUASI-TRANSITIVITY

LEMMA 1. *For every special majority rule, a sufficient condition for quasi-transitivity of the social preference relation is that DP holds over every triple of alternatives.*

Proof. Satisfaction of DP over a triple $\{x, y, z\}$ implies that the set of permissible orderings must be a subset of the following 7 orderings,

1. xP_iyI_iz 2. yI_izP_ix
3. yP_ixI_iz 4. xI_izP_iy
5. zP_ixI_iy 6. xI_iyP_iz
7. xI_iyI_iz

Because of symmetry it is sufficient to show that xPy and yPz imply xPz.

$$xPy \rightarrow N_1 + N_4 > p(N_1 + N_2 + N_3 + N_4)$$

$$yPz \rightarrow N_3 + N_6 > p(N_3 + N_4 + N_5 + N_6)$$

Combining the two inequalities we obtain,

$$\begin{aligned} &N_1 + N_3 + N_4 + N_6 > p(N_1 + N_2 + N_5 + N_6) + 2p(N_3 + N_4) \\ &\quad \rightarrow N_1 + N_6 > p(N_1 + N_2 + N_5 + N_6) + (2p - 1)(\mathrm{N}_3 + N_4) \\ &\quad \rightarrow N_1 + N_6 > p(N_1 + N_2 + N_5 + N_6), \quad \text{as} \quad p > 1/2 \\ &\quad \rightarrow xPz\,. \end{aligned}$$

THEOREM 1. *For every special majority rule, a sufficient condition for quasi-transitivity of the social preference relation is that WAP is satisfied over every triple of alternatives.*

Proof. If no individual has a strong ordering over $\{x, y, z\}$ then quasi-transitivity follows from Lemma 1. For non-trivial fulfilment of WAP assume, without any loss of generality, that someone has the ordering xP_iyP_iz. Then it follows that the set of permissible orderings must be a subset of the following 5 orderings,

$$\begin{array}{ll} 1. & xP_i yP_i z \\ 2. & zP_i yP_i x \\ 3. & yP_i xI_i z \\ 4. & xI_i zP_i y \\ 5. & xI_i yI_i z \end{array}$$

Quasi-transitivity is violated iff exactly one of the following two cycles holds with at least 2 of the R being P,

$$xRy \wedge yRz \wedge zRx \quad \text{(Forward cycle)}$$

$$yRx \wedge xRz \wedge zRy \quad \text{(Backward cycle)}.$$

Suppose the forward cycle holds with at least 2 of the R being P. First suppose that zPx obtains

$$\begin{aligned} zPx &\rightarrow N_2 > p(N_1+N_2) \\ &\rightarrow N_2 > \frac{p}{1-p} N_1 \\ &\rightarrow N_2 > N_1, \quad \text{as} \quad p > \frac{1}{2}. \end{aligned}$$

Now,

$$\begin{aligned} (xRy \wedge yRz) &\rightarrow N_2+N_3 \leqslant p(N_1+N_2+N_3+N_4) \\ &\quad \text{and} \quad N_2+N_4 \leqslant p(N_1+N_2+N_3+N_4) \\ &\rightarrow N_1+N_3 \leqslant p(N_1+N_2+N_3+N_4) \\ &\quad \text{and} \quad N_1+N_4 \leqslant p(N_1+N_2+N_3+N_4), \\ &\quad \text{as} \quad N_2 > N_1 \\ &\rightarrow zRy \wedge yRx \\ &\rightarrow xIy \wedge yIz. \end{aligned}$$

Therefore, if zPx holds then it is impossible for the forward cycle to hold with at least 2 of R being P. The only remaining possibility is $xPy \wedge yPz \wedge xIz$. However,

$$\begin{aligned} xPy \wedge yPz &\rightarrow N_1+N_4 > p(N_1+N_2+N_3+N_4) \\ &\quad \text{and} \quad N_1+N_3 > p(N_1+N_2+N_3+N_4) \\ &\rightarrow 2N_1 > 2p(N_1+N_2)+(2p-1)(N_3+N_4) \\ &\rightarrow N_1 > p(N_1+N_2), \quad \text{as} \quad p > 1/2 \\ &\rightarrow xPz, \end{aligned}$$

which contradicts xIz. Therefore it is impossible for the forward cycle to hold with at least 2 of R being P. Analogously it can be shown that the backward cycle cannot hold with at least 2 of R being P. So R must be quasi-transitive.

LEMMA 2. *A set of orderings violates all three restrictions VR, LA and WAP iff it includes one of the following six 3-ordering sets, except for a formal interchange of alternatives,*[2]

(A)	xP_iyP_iz yP_izP_ix zP_ixP_iy	(B)	xP_iyP_iz yP_izP_ix zP_ixI_iy
(C)	xP_iyP_iz yP_izP_ix zI_ixP_iy	(D)	xP_iyP_iz yP_izI_ix zP_ixI_iy
(E)	xP_iyP_iz yI_izP_ix zP_ixI_iy	(F)	xP_iyP_iz yI_izP_ix zI_ixP_iy

Proof. It is well known that a set of orderings violates VR iff it contains a set of 3 concerned orderings forming a Latin Square,[3]

Latin Square I	Latin Square II
xR_iyR_iz	xR_izR_iy
yR_izR_ix	zR_iyR_ix
zR_ixR_iy	yR_ixR_iz

There are in all 54 such 3-ordering sets. However, it is sufficient to consider the following 11 sets as the remaining ones can be obtained from these by a formal interchange of alternatives,

(1)	xP_iyP_iz yP_izP_ix zP_ixP_iy	(2)	xP_iyP_iz yP_izP_ix zP_ixI_iy
(3)	xP_iyP_iz yP_izP_ix zI_ixP_iy	(4)	xP_iyP_iz yP_izI_ix zP_ixI_iy

[2] As union of VR, LA and WAP is logically equivalent to the union of VR, LA and extremal restriction, as has been noted by Inada, this lemma is logically equivalent to Sen's lemma in [7]. Sen obtains 8 3-ordering sets instead of our 6 sets. It can, however, be checked that 2 of them are redundant as they can be obtained by a formal interchange of alternatives. The proof given here is more economical as the number of configurations which have to be checked is much smaller than in Sen's proof.

[3] See Ward [9] and Majumdar [4].

(5) xP_iyP_iz (6) xP_iyP_iz
yP_izI_ix yI_izP_ix
zI_ixP_iy zP_ixI_iy

(7) xP_iyP_iz (8) xP_iyI_iz
yI_izP_ix yP_izI_ix
zI_ixP_iy zP_ixI_iy

(9) xP_iyI_iz (10) xP_iyI_iz
yP_izI_ix yI_izP_ix
zI_ixP_iy zI_ixP_iy

(11) xI_iyP_iz
yI_izP_ix
zI_ixP_iy

(1), (2), (3), (4), (6) and (7) are the same as A, B, C, D, E and F respectively. Consider (5). Both LA and WAP are satisfied. To violate LA one has to include $(yP_izP_ix \vee yI_izP_ix \vee zP_iyP_ix \vee zP_iyI_ix \vee zP_ixP_iy)$. Inclusion of any of these orderings excepting that of zP_iyP_ix would imply a violation of WAP also and in each case one of the six sets would be contained in the set of R_i. If we include zP_iyP_ix then WAP is violated iff a concerned ordering not already contained in the set is included. If a strong ordering is included then the set contains B or C. If a weak ordering is included then D or F is contained. Now consider (8) which satisfies WAP but violates LA. To violate WAP a strong ordering must be included. Because of symmetry it suffices to consider the case when xP_iyP_iz is included. With the inclusion of xP_iyP_iz the set contains D. The case of (11) is similar. Next we consider (9). Both WAP and LA are satisfied. To violate LA we have to include $(yP_izP_ix \vee yI_izP_ix \vee zP_iyP_ix \vee zP_iyI_ix \vee zP_ixP_iy)$. If yP_izP_ix or zP_iyP_ix or zP_ixP_iy is included then WAP is also violated and the set includes D or E. If yI_izP_ix or zP_iyI_ix is included then WAP continuous to be satisfied. WAP would be violated iff a strong ordering is included. Inclusion of a strong ordering makes the set contain D or E or F. Demonstration for the case (10) is analogous. Proof is completed by noting that all the six sets violate all three restrictions.

THEOREM 2. *For every special majority rule, a necessary and sufficient condition for quasi-transitivity of the social preference relation is that* $(VR \vee LA \vee WAP)$ *is satisfied over every triple of alternatives.*

Proof. Sen [7] has shown that for the class of binary social decision rules satisfying neutrality, monotonicity and the strict Pareto-criterion, both VR and LA are sufficient conditions for quasi-transitivity of the social R. As all special majority rules are binary social decision rules satisfying monotonicity, neutrality and the strict Pareto-criterion, the sufficiency of VR and LA follows as a corollary of Sen's theorems. Sufficiency of WAP has been shown in Theorem 1. In what follows we show that if a set of orderings violates all three restrictions then there exists an assignment of individuals such that R violates quasi-transitivity, establishing the necessity part. If a set of orderings violates all the three restrictions then by Lemma 2 it must include one of the six sets (A)–(F) mentioned in the statement of the lemma. Therefore, it suffices to show that for each of the six sets there exists an assignment such that R violates quasi-transitivity.

For (A) take $N_1=pN$, $N_2=N_3=(1-p)N/2$, for (B) $N\geqslant 1/p(1-p)$, $N_1=p^2N+1$, $N_2=p(1-p)N$, $N_3=(1-p)N-1$, for (C) $N\geqslant 1/p(1-p)$, $N_1=p(1-p)N$, $N_2=p^2N+1$, $N_3=(1-p)N-1$, for (D) $M\geqslant p/(1-p)$, $N>(M+p)/p(1-p)$, $N_1=pN-M$, $N_2=M+1$, $N_3=(1-p)N-1$, for (E) $M\geqslant p/(1-p)$, $N>(Mp+1)/(1-p)^2$, $N_1=(1-p)N-1$, $N_2=M+1$, $N_3=pN-M$ and for (F) $M\geqslant p/(1-p)$, $N>(M+p)/p(1-p)$, $N_1=pN-M$, $N_2=(1-p)N-1$, $N_3=M+1$. This results, for (A), (B), (D) and (F) in $xPy\wedge yPz\wedge\sim(xPz)$, for (C) in $yPz\wedge zPx\wedge\sim(yPx)$ and for (E) in $zPx\wedge xPy\wedge\sim(zPy)$.

CONDITIONS FOR TRANSITIVITY

THEOREM 3. *For every special majority rule, a necessary and sufficient condition for transitivity of the social preference relation is that the strong value restriction holds over every triple of alternatives.*

Proof.

Sufficiency:

Suppose transitivity is violated. Then there are x, y, z such that $xRy\wedge yRz\wedge zPx$. Let N_c denote the number of individuals who are concerned with respect to the triple $\{x,y,z\}$.

$$xRy\rightarrow N(yP_ix)\leqslant p[N(xP_iy)+N(yP_ix)]$$

$$\rightarrow N(xP_iy)\geqslant(1-p)[N(xP_iy)+N(yP_ix)]$$

$$\rightarrow N(xP_iy)+N\text{ (concerned } i\text{: } xI_iy)$$

$$\geqslant(1-p)N_c+pN\text{ (concerned } i\text{: } xI_iy)$$

$$\rightarrow N\text{ (concerned } i\text{: } xR_iy)\geqslant(1-p)N_c \tag{1}$$

Similarly,

$$yRz\rightarrow N\text{ (concerned } i\text{: } yR_iz)\geqslant(1-p)N_c \tag{2}$$

$$zPx\rightarrow N(zP_ix)>p[N(xP_iz)+N(zP_ix)]$$

$$\rightarrow N\text{ (\textit{concerned} } i\text{: } zR_ix)>pN_c \tag{3}$$

$$\text{(1) and (3)}\rightarrow\exists\text{concerned } i\text{: } zR_ixR_iy \tag{4}$$

$$\text{(2) and (3)}\rightarrow\exists\text{concerned } i\text{: } yR_izR_ix \tag{5}$$

$(4)\rightarrow\exists\, i\colon zP_iy$ (6)

$yRz \wedge (6)\rightarrow\exists\, i\colon yP_iz$ (7)

$(5)\rightarrow\exists\, i\colon yP_ix$ (8)

$xRy \wedge (8)\rightarrow\exists\, i\colon xP_iy$ (9)

$zPx\rightarrow\exists\, i\colon zP_ix$ (10)

(4) through (10) imply that SVR is violated. Thus violation of transitivity implies violation of SVR, i.e., SVR is a sufficient condition for transitivity.

Necessity:

It can be easily checked that SVR is violated over a triple $\{x, y, z\}$ iff the set of R_i contains one of the following 10 sets of orderings, except for a formal interchange of alternatives,

(A)	xP_iyP_iz yP_izP_ix	(B)	xP_iyP_iz zP_ixI_iy
(C)	xP_iyP_iz yI_izP_ix	(D)	xP_iyP_iz zP_iyP_ix yP_ixI_iz
(E)	xP_iyP_iz zP_iyP_ix xI_izP_iy	(F)	xP_iyP_iz yP_ixI_iz xI_izP_iy
(G)	xP_iyI_iz yP_ixI_iz xI_izP_iy	(H)	xI_iyP_iz yP_ixI_iz xI_izP_iy
(I)	xP_iyI_iz yP_izI_ix zP_ixI_iy	(J)	xI_iyP_iz yI_izP_ix zI_ixP_iy

Therefore, for proving the necessity of SVR it suffices to show that for each of these sets there exists an assignment of individuals which results in intransitive social preference relation.

Take for (A), (B) and (C), $N_1=N_2=N/2$, for (D) and (E), $M\geqslant p/(1-p)$, $N\geqslant M/(2p-1)$, $N_1=pN-M$, $N_2=(1-p)N-1$, $N_3=M+1$, for (F), (G) and (H), $N_1=(2p-1)N$, $N_2=N_3=(1-p)N$, and for (I) and (J), $N\geqslant(1+p)/p(2p-1)$, $N_1=(p/(1+p))N+1$, $N_2=((1-p)/(1+p))N$, $N_3=(p/(1+p))N-1$. This results, for (A), (C) and (D) in $xIy \wedge yPz \wedge xIz$, for (B), (E) and (I) in $xPy \wedge yIz \wedge xIz$, and for (F), (G), (H) and (J) in $xIy \wedge yIz \wedge xPz$.

Jawaharlal Nehru University

REFERENCES

[1] Arrow, K. J., Social Choice and Individual Values (2nd ed.), Wiley, New York, 1963.

[2] Inada, K., "A note on the simple majority decision rule," *Econometrica*, **32**, 1964.

[3] Inada, K., "The simple majority decision rule," *Econometrica*, **37**, 1969.

[4] Majumdar, T., "Sen's general theorem on transitivity of majority decisions—an alternative approach," in *Growth and Choice* (T. Majumdar, ed.), Oxford University Press, London, 1969.

[5] Pattanaik, P. K., Voting and Collective Choice, Cambridge University Press, Cambridge, 1971.

[6] Sen, A. K., "Quasi-transitivity, rational choice and collective decisions," *Review of Economic Studies*, **36**, 1969.

[7] Sen, A. K., Collective Choice and Social Welfare, Holden-Day, San Francisco, 1970.

[8] Sen, A. K., and P. K. Patanaik, "Necessary and sufficient conditions for rational choice under majority decision," *Journal of Economic Theory*, **1**, 1969.

[9] Ward, B., "Majority voting and alternative forms of public enterprise," in *The Public Economy of Urban Communities* (J. Margolis, ed.), Johns Hopkins Press, Baltimore, 1965.