Title	A BALANCED OUTCOME FUNCTION YIELDING PARETO OPTIMAL ALLOCATIONS AT NASH EQUILIBRIUM POINTS IN THE PRESENCE OF EXTERNALITIES: A CASE OF LINEAR PRODUCTION FUNCTION
Sub Title	
Author	NAKAMURA, Shinsuke
Publisher	Keio Economic Society, Keio University
Publication year	1983
Jtitle	Keio economic studies Vol.20, No.2 (1983.) ,p.27- 43
JaLC DOI	
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Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19830002-0027

A BALANCED OUTCOME FUNCTION YIELDING PARETO OPTIMAL ALLOCATIONS AT NASH EQUILIBRIUM POINTS IN THE PRESENCE OF EXTERNALITIES: A CASE OF LINEAR PRODUCTION FUNCTION*

Shinsuke NAKAMURA

Abstract: The purpose of this paper is to construct a balanced outcome function which attains Pareto optimal allocations in such a way that nobody has any incentives to deceive the government even if there are externalities. With this outcome function, we will show (1) there exists a Nash equilibrium, (2) every Nash equilibrium is Pareto optimal, (3) every Pareto optimal allocation can be attained as a Nash equilibrium if the initial endowments are suitably redistributed, and (4) every Nash equilibrium is individually rational.

1. INTRODUCTION

In the presence of externalities, a Walras equilibrium is not necessarily Pareto optimal and a Pareto optimal allocation is not necessarily sustained by a Walras equilibrium. The purpose of this paper is to construct an outcome function which attains Pareto optimal allocation in such a way that nobody has any incentives to deceive the government even if there are externalities.

In an economy with externalities, Aoki [1] discusses the relation between competitive equilibria and Pareto optimal allocations. But he restricts himself to an economy of very special type in which there is only a single consumer and externalities exist only within each industry. In a more general framework, Osana [7] shows that every Pareto optimal allocation is a competitive equilibrium if some suitable tax-subsidy system is adopted. But these arguments do not consider implementability of the competitive equilibrium.

On the other hand, Hurwicz and Schmeidler [4] construct some mechanisms guaranteeing the existence of Nash equilibrium and the Pareto optimality of the equilibrium for every admissible profile of preferences, when the set of alternatives is finite. In a more practical case when the set of alternatives is a convex and compact subset of some Euclidean space, Rob [8] obtains necessary and sufficient conditions ensuring that every Nash equilibrium is Pareto optimal and derives sufficient conditions for the existence of a Nash equilibrium. However they treat a very abstract model and the relation between markets and mechanisms is not clear.

* The author is grateful to Professors Masao Fukuoka, Michihiro Ohyama, Kunio Kawamata, Hiroaki Osana, and Toru Maruyama for helpful comments.

Furthermore they do not consider individual rationality and the implementability of Pareto optimal allocations as a Nash equilibrium. But typically, our economy is a special case of Rob's. Hence it may appear that our mechanism must satisfy Rob's conditions because these are both necessary and sufficient. But in fact our mechanism does not satisfy Rob's conditions. This is because Rob treats only feasible mechanisms whose ranges are included by the attainable set and our mechanism is not necessarily feasible so that outcomes which are not in equilibria may not be in the attainable set.

Our approach is divided into two parts. First, we will define a price system with some tax-subsidy system which can attain a Pareto optimal allocation but in which economic agents may have incentives to lie. We shall show that the price system has the following four properties: Under standard assumptions,

(1) Existence: There exists an equilibrium.

(2) Non-wastefulness: Every equilibrium is Pareto optimal.

(3) Unbiasedness: Every Pareto optimal allocation can be attained as an equilibrium provided that the initial endowments are suitably redistributed.

(4) Individual Rationality: For all consumers, every equilibrium is at least as good as the initial endowment.

Secondly we define a mechanism or a government which attains the above price equilibria through Nash strategies. A government is typically the ordered set of message spaces and an outcome function into a commodity allocation space. Namely we regard a government as a system which aggregates the messages of economic agents and determines an allocation. Consumers know the outcome function and report their optimal messages given the messages of others. If the allocation attained by the Nash equilibrium coincides with the allocation attained by the price system, then no consumer has any incentives to alter his message and the price system is incentive compatible. Construction of these governments is the main purpose of this paper.

In our context, we assume that there is one firm and externalities occur only among consumer's preferences. But in the field of welfare economics, externalities among firms and between firms and consumers are important. Hence generalization to this direction is desirable. Furthermore this model does not cover the pure exchange economy. For this issue see Nakamura [6].

2. EXTERNALITIES AMONG CONSUMERS' PREFERENCES

In the presence of externalities, the basic theorems of welfare economics do not hold. It is thought that some tax-subsidy system can remedy such failures. But it is not clear what tax-subsidy system is optimal. In this section we consider an optimal tax-subsidy system which solves the problem and discuss the implementability of the system. For simplicity we assume externalities exist only among consumers' preferences. For more general cases, see Osana [7].

2.1. *Economy*

We consider an economy with n consumers, one firm, and $l+1$ commodities. A commodity bundle is denoted by (x, y), where $x \in R_+$ (numeraire) and $y \in R^l_+$ (others). Our attention is chiefly directed to consumers and we assume, for simplicity, that the production function is linear, and may be expressed as

$$x+\alpha y=0\,, \qquad \text{where} \quad \alpha \in R^l_+\backslash\{0\}\,. \tag{1}$$

The i-th consumer's preference relation is denoted by $\succsim_i$ which is assumed to be a complete monotone preordering on $R_+ \times R^l_+ \times R^{l(n-1)}_+$ where monotonicity means

$$(x_i, y_i, y_{-i}) \succ_i (x_i', y_i', y_{-1}) \qquad \text{if} \quad (x_i, y_i) \geq (x_i', y_i')\,. \tag{2}$$

His initial endowment is given by $(\omega_i^x, \omega_i^y) \in R_+ \times R^l_+$. Note that we assume implicitly that the consumption set is $R_+ \times R^l_+ \times R^{l(n-1)}_+$.

The attainable set for this economy is

$$A=\left\{(x, y) \in R^n_+ \times R^{ln}_+ \,\middle|\, \sum_i (x_i-\omega_i^x)+\alpha \sum_i (y_i-\omega_i^y)=0\right\}. \tag{3}$$

Pareto optimality and individual rationality are defined as follows.

Definition 1. $(x^*, y^*) \in A$ is Pareto optimal if there is no $(x, y) \in A$ such that
(i) for all i, $(x_i, y) \succsim_i (x_i^*, y^*)$ and
(ii) for some i, $(x_i, y) \succ_i (x_i^*, y^*)$.

Definition 2. $(x, y) \in A$ is individually rational if for all i,

$$(x_i, y) \succsim_i (\omega_i^x; \omega_1^y, \cdots, \omega_n^y)\,.$$

2.2. *Price Equilibrium*

In this section we define a price equilibrium. First we define a transfer system.

Fix two distinct consumers i and j $(i \neq j)$. Let t_{ij} be a transfer rate from i to j. Then if consumer j consumes y_j unit of commodity y, then consumer i pays

$$t_{ij}(y_j-\omega_j^y)$$

for j's consumption.

Similarly, consumer j pays

$$t_{ji}(y_i-\omega_i^y)$$

for i's consumption of commodity y of y_i unit. Thus i's net transfer to j is

$$t_{ij}(y_j-\omega_j^y)-t_{ji}(y_i-\omega_i^y)\,.$$

Hence the sum of transfers paid by i is equal to

$$\sum_{j \neq i} (t_{ij}(y_j-\omega_j^y)-t_{ji}(y_i-\omega_i^y))=\sum_{j \neq i} t_{ij}(y_j-\omega_j^y)+\left(-\sum_{j \neq i} t_{ji}\right)(y_i-\omega_i^y)\,.$$

Thus if we write $-\sum_{j\neq i} t_{ji}$ as t_{ii}, then the total transfer from i can be written as

$$\sum_j t_{ij}(y_j-\omega_j^y),$$

which is very simple.

Formally, the transfer system is defined as follows.

Definition 3. $t\in R^{ln^2}$ is called a transfer system if for every j,

$$\sum_{i\neq j} t_{ij}=-t_{jj}.$$

If $i\neq j$, t_{ij} is a transfer rate from i to j, and t_{ii} can be interpreted as a tax rate for i. Hence the condition means that j's subsidy rate $(=-t_{jj})$ is equal to the sum of transfer rates to j.

The following remark is obvious.

Remark 1. $t\in R^{ln^2}$ is a transfer system if and only if

$$\sum_i\sum_j t_{ij}(y_j-\omega_j^y)=0$$

for every $y\in R_+^{ln}$.

$t_{ij}(y_j-\omega_j^y)$ is an amount of transfer from consumer i to consumer j. Hence this remark means that total transfer is always equal to zero so that the budget constraint of the government is always satisfied.

Given a price $p\in R_+^l\backslash\{0\}$ and a transfer system $t\in R^{ln^2}$, the i-th consumer's budget set is defined by

$$B_i(p,t)=\left\{(x_i,y)\in R_+\times R_+^{ln}\;\middle|\;x_i+py_i\leqq\omega_i^x+p\omega_i^y-\sum_j t_{ij}(y_j-\omega_j^y)\right\}. \tag{4}$$

The budget set has a straightforward interpretation. We can now define a price equilibrium.

Definition 4. $(p^*,t^*,x^*,y^*)\in R_+^l\times R^{ln^2}\times R_+^n\times R^{ln}$ is called a price equilibrium if

(i) $p^*\in R_l^+\backslash\{0\}$, and t^* is a transfer system,

(ii) for every i,

$$(x_i^*,y^*)\in B_i(p^*,t^*) \quad\text{and}$$
$$(x_i^*,y^*)\succsim_i(x_i,y)\quad\text{for every}\quad(x_i,y)\in B_i(p^*,t^*),$$

(iii) for all (x,y) with $x+\alpha y=0$,

$$\sum_i(x_i^*-\omega_i^x)+p^*\sum_i(y_i^*-\omega_i^y)\geqq x+p^*y,$$

(iv) $(x^*, y^*) \in A$.

The corresponding allocation (x^*, y^*) is said to be a price allocation.

Conditions (i), (iii), and (iv) are obvious. (ii) means that consumers maximize their utilities given prices and transfers. For y, condition (ii) means that for each i, y^* is optimal for all consumers under the transfer system t^*.

It should be noted that price equilibrium is different from the equilibrium introduced by Osana [7] in which y_i^* is optimal given not only prices and transfers but also others' consumption $(y_1^*, \cdots, y_{i-1}^*, y_{i+1}^*, \cdots, y_n^*)$. In fact our definition of price equilibrium is stronger than that of the equilibrium in Osana [7], so that we can assure non-wastefulness (Theorem 2) and individual rationality (Theorem 4).

2.3. *Mechanism and Nash Equilibrium*

We define a mechanism as follows.

Definition 5 (message space). For every i, let

$$M^i = R^{ln} \qquad \text{and} \qquad M = \prod_i M^i .$$

Definition 6 (outcome function). For all $m = (m_{i1}, \cdots, m_{in})_i \in M$, let

$$y_j(m) = \sum_i m_{ij}$$

$$x_i(m) = \omega_i^x + \alpha\omega_i^y - \alpha y_i(m) - \sum_j t_{ij}(m)(y_j(m) - \omega_j^y) ,$$

where $t_{ij}(m) = m_{i+1,j} - m_{i+2,j}$.

Thus m_{ij} can be interpreted as an additional demand of j reported i. The following remark is obvious.

Remark 2. This game is balanced, or

$$\sum_i (x_i(m) - \omega_i^x) + \alpha \sum_i (y_i(m) - \omega_i^y) = 0$$

for every $m \in M$.

A Nash equilibrium is defined as follows.

Definition 7. $m^* \in M$ is called a Nash equilibrium if for each i,

$$(x_i(m^*), y(m^*)) \succsim_i (x_i(m_1^*, \cdots, m_{i-1}^*, m_i, m_{i+1}^*, \cdots, m_n^*),$$
$$y(m_1^*, \cdots, m_{i-1}^*, m_i, m_{i+1}^*, \cdots, m_n^*))$$

for every $m_i \in M^i$.

The corresponding allocation is said to be a Nash allocation.

2.4. *Theorems and an Example*

THEOREM 1 (existence). *Assume* $\succsim_i$ *is convex and continuous, and* $(\omega_i^x, \omega_i^y) \in R_{++} \times R^l_{++}$ *for every i. If* $\alpha > 0$, *then there exists a price equilibrium.*

THEOREM 2 (non-wastefulness). *Every price allocation is Pareto optimal.*

THEOREM 3 (unbiasedness). *Suppose* $\succsim_i$ *is convex and continuous for every i. Then every Pareto optimal allocation* $(x^*, y^*) \in R^n_{++} \times R^{ln}_{++}$ *can be attained as a price allocation, provided that the initial endowments are suitably redistributed.*

It should not be confused with unbiasedness and full implementability in the sense of Dasgupta, Hammond and Maskin [2] which, in the context of this model, implies that the set of price allocations coincides with the set of Pareto optimal allocations, which is not true in this case.

THEOREM 4 (individual rationality). *Every price allocation is individually rational.*

The above theorems show the price equilibrium has the desired properties. But the implementability of the price equilibrium is not considered. Hence there remains a kind of "Free Rider Problem" or a problem of incentive compatibility. The following theorems show the implementability of price equilibria through Nash strategies.

From now on we assume that there are three or more consumers so that $n \geqq 3$. The case $n = 2$ is left as an open problem.

THEOREM 5. *Every Nash allocation is a price allocation.*

THEOREM 6. *Every price allocation is a Nash allocation.*

COROLLARY. *In Theorems* 1, 2, 3, *and* 4, *one can replace price equilibrium (allocation) by Nash equilibrium (allocation).*

Before presenting a formal proof, we first consider a simple economy with two commodities and three persons. We assume $\succsim_i$ is represented by a twice continuously differentiable and quasi-concave utility function u^i which is increasing with regard to his consumption (x_i, y_i). Furthermore we assume interior maximum.

Pareto Optimality

$$\text{Maximize}\, u^1(x_1, y_1, y_2, y_3)$$

subject to

$$u^2(x_2, y_1, y_2, y_3) = \bar{u}^2 ,$$

$$u^3(x_3, y_1, y_2, y_3) = \bar{u}^3 ,$$

and

$$x_1+x_2+x_3-\omega^x+\alpha(y_1+y_2+y_3-\omega^y)=0\,,$$

where

$$\omega^x=\omega_1^x+\omega_2^x+\omega_3^x$$

and

$$\omega^y=\omega_1^y+\omega_2^y+\omega_3^y\,.$$

Let

$$\begin{aligned}L\equiv u^1&(x_1,y_1,y_2,y_3)\\&+\lambda_2(u^2(x_2,y_1,y_2,y_3)-\bar{u}^2)+\lambda_3(u^3(x_3,y_1,y_2,y_3)-\bar{u}^3)\\&-\mu(x_1+x_2+x_3-\omega^x+\alpha(y_1+y_2+y_3-\omega^y))\,.\end{aligned}$$

So

$$\begin{aligned}&u_x^1=\mu\,,\qquad \lambda_2u_x^2=\mu\,,\qquad \lambda_3u_x^3=\mu\,,\\&u_1^1+\lambda_2u_1^2+\lambda_3u_1^3=\mu\alpha\,,\\&u_2^1+\lambda_2u_2^2+\lambda_3u_2^3=\mu\alpha\,,\\&u_3^1+\lambda_2u_3^2+\lambda_3u_3^3=\mu\alpha\,,\end{aligned}$$

where $u_x^i=\partial u^i/\partial x_i$, $u_1^i=\partial u^i/\partial y_1$, $u_2^i=\partial u^i/\partial y_2$, and $u_3^i=\partial u^i\partial y_3$.

Hence

$$\begin{aligned}&u_1^1/u_x^1+u_1^2/u_x^2+u_1^3/u_x^3=\alpha\,,\\&u_2^1/u_x^1+u_2^2/u_x^2+u_2^3/u_x^3=\alpha\,,\\&u_3^1/u_x^1+u_3^2/u_x^2+u_3^3/u_x^3=\alpha\,.\end{aligned}\tag{5}$$

Price Equilibrium

$$\text{Maximize}\,u^i(x_i,y_1,y_2,y_3)$$

subject to

$$x_i+py_i=\omega_i^x+p\omega_i^y-t_{i1}(y_1-\omega_1^y)-t_{i2}(y_2-\omega_2^y)-t_{i3}(y_3-\omega_3^y)\,.$$

Let

$$\begin{aligned}L\equiv u^i&(x_i,y_1,y_2,y_3)\\&-\lambda(x_i-\omega_i^x+p(y_i-\omega_i^y)+t_{i1}(y_1-\omega_1^y)+t_{i2}(y_2-\omega_2^y)+t_{i3}(y_3-\omega_3^y))\,.\end{aligned}$$

Therefore at an interior maximum, we must have

$$u_x^i=\lambda\,,\qquad u_i^i=\lambda(p+t_{ii})\,,\qquad u_j^i=\lambda t_{ij}\qquad\text{if}\quad i\neq j\,,$$

so that

$$u_i^i/u_x^i = p + t_{ii}\,, \qquad u_j^i/u_x^i = t_{ij} \quad \text{if} \quad i \neq j\,, \qquad \text{and} \tag{6}$$
$$p = \alpha \text{ (by profit maximization)}\,.$$

Thus

$$u_i^1/u_x^1 + u_i^2/u_x^2 + u_i^3/u_x^3 = p + t_{1i} + t_{2i} + t_{3i} = p = \alpha\,, \qquad i = 1,\,2,\,3\,.$$

Hence price equilibrium is Pareto optimal.

Nash Equilibrium

$$\begin{aligned}
&\mathrm{Max}_{m_{i1}, m_{i2}, m_{i3}}\, u^i(x_i(m), y_1(m), y_2(m), y_3(m)) \\
&\quad \equiv u^i(\omega_i^x + \alpha\omega_i^y - \alpha(m_{1i} + m_{2i} + m_{3i}) \\
&\qquad - (m_{i+1,1} - m_{i+2,1})(m_{11} + m_{21} + m_{31}) \\
&\qquad - (m_{i+1,2} - m_{i+2,2})(m_{12} + m_{22} + m_{32}) \\
&\qquad - (m_{i+1,3} - m_{i+2,3})(m_{13} + m_{23} + m_{33}), \\
&\qquad\ m_{11} + m_{21} + m_{31}, m_{12} + m_{22} + m_{32}, m_{13} + m_{23} + m_{33})\,.
\end{aligned}$$

Hence the first order conditions for a maximum are

$$\partial u^i/\partial m_{ii} = -(\alpha + m_{i+1,i} - m_{i+2,i})u_x^i + u_i^i = 0\,, \qquad \text{so that}$$
$$u_i^i/u_x^i = \alpha + m_{i+1,i} - m_{i+2,i}\,. \tag{7}$$

And

$$\partial u^i/\partial m_{ij} = -(m_{i+1,j} - m_{i+2,j})u_x^i + u_j^i = 0 \qquad \text{if} \quad i \neq j\,, \quad \text{so that}$$
$$u_j^i/u_x^i = m_{i+1,j} - m_{i+2,j} \qquad \text{if} \quad i \neq j\,. \tag{8}$$

Thus if we write

$$t_{ij} \equiv m_{i+1,j} - m_{i+2,j}\,, \qquad i,j = 1,2,3,$$

then

$$u_i^i/u_x^i = \alpha + t_{ii}$$
$$u_j^i/u_x^i = t_{ij} \qquad \text{if} \quad i \neq j\,.$$

By (7) and (8),

$$u_i^1/u_x^1 + u_i^2/u_x^2 + u_i^3/u_x^3 = \alpha + (m_{2i} - m_{3i}) + (m_{3i} - m_{1i}) + (m_{1i} - m_{2i}) = \alpha\,,$$

for all $i = 1, 2, 3$.

Hence Nash allocation coincides with price allocation.

3. PROOFS

Proof or Reamrk 1. Obvious.

Proof of Remark 2. Obvious.

Proof of Theorem 1. Let

$$X_i = R_+ \times \{(0, \cdots, 0)\} \times R_+^{ln} \times \{(0, \cdots, 0)\}$$
$$\subset R_+ \times R_+^{ln(i-1)} \times R_+^{ln} \times R_+^{ln(n-i)},$$

$$Y = \left\{(x, y_1, \cdots, y_n) \in R \times R^{ln^2} \,\middle|\, y_1 = \cdots = y_n \text{ and } x + \alpha \sum_j y_{ij} = 0 \text{ for all } i\right\},$$

and

$$\tilde{\omega}_i = (\omega_i^x, 0, \cdots, 0, \omega^y, 0, \cdots, 0) \in X_i .$$

We extend $\succsim_i$ on X_i in the natural way, that is

$$(x_i, 0, \cdots, 0, y, 0, \cdots, 0) \succsim_i (x_i', 0, \cdots, 0, y', 0, \cdots, 0)$$

if and only if

$$(x_i, y) \succsim_i (x_i', y') .$$

Denote

$$\tilde{A} = \left\{(u, v) \in \prod_i X_i \times Y \,\middle|\, \sum_i u_i \leqq \sum_i \tilde{\omega}_i + v\right\}.$$

Then $\tilde{A}$ is compact since $\alpha > 0$ and X_i's are lower bounded.

Hence there is a convex and compact set $K \subset R \times R^{ln^2}$ such that

$$\text{proj}_{X_i} \tilde{A} \subset \text{int}\, K \qquad \text{and} \qquad \text{proj}_Y \tilde{A} \subset \text{int}\, K .$$

Let $\tilde{X}_i = X_i \cap K$ and $\tilde{Y} = Y \cap K$.
Then $\tilde{X}_i$ and $\tilde{Y}$ are convex and compact.

Fix $\nu \in N$. Let

$$P_\nu = \{q \in R^{ln^2} \mid \|q\| \leqq \nu\} .$$

For all $q \in P_\nu$ let

$$\tilde{B}_i(q) = \{(x_i, 0, \cdots, 0, y, 0, \cdots, 0) \in \tilde{X}_i \mid x_i + q_i y \leqq \omega_i^x + q_i \omega^y\}$$

for all i.

Since $\omega_i^x \in R_{++}$. $\tilde{B}_i$ is a continuous correspondence of P_ν into $\tilde{X}_i$. If we define

$$\xi_i(q) = \{u_i^* \in \tilde{X}_i \mid u_i^* \text{ is } \succsim_i\text{-greatest subject to } u \in \tilde{B}_i(q)\} ,$$

then by Berge's maximal principle, ξ_i is upper hemi-continuous. Furthermore we can show that ξ_i is convex- and compact-valued.

For every $q \in P_\nu$ let

$$\zeta(q) = \{v \in \tilde{Y} \mid (1, q)v = \max(1, q)\tilde{Y}\} .$$

Then ζ is upper hemi-continuous and convex- and compact-valued. For all $q \in P_\nu$ define

$$z(q) = \sum_i \xi_i(q) - \zeta(q) - \sum_i \tilde{\omega}_i .$$

By Gale-Nikaido's lemma (see Debreu [3], pp. 82–83), there exist

$$q^\nu \in P_\nu \qquad \text{and} \qquad z^\nu \in z(q^\nu)$$

such that

$$(1, q)z^\nu \leqq 0$$

for all $q \in P_\nu$.

Since $z^\nu \in z(q^\nu)$, there exist

$$u_i^\nu \in \xi_i(q^\nu) \qquad \text{and} \qquad v^\nu \in \zeta(q^\nu)$$

such that

$$z^\nu = \sum_i u_i^\nu - v^\nu - \sum_i \tilde{\omega}_i .$$

Fix i. Suppose $\|q_i^\nu\| \to \infty$. Since $u_i^\nu \in \tilde{X}_i$, we may assume $\{u_i^\nu\}$ converges to some $u_i^* \in \tilde{X}_i$. Since $u_i^\nu \in \xi_i(q^\nu)$, it follows that for every $\nu \in N$,

$$(1, q^\nu)u_i \leqq (1, q^\nu)\tilde{\omega}_i \qquad \text{and}$$

$$(1, q^\nu)u \leqq (1, q^\nu)\tilde{\omega}_i \quad \text{implies} \quad u_i^\nu \succsim_i u .$$

Namely for every $\nu \in N$,

(i) $(1/\|q_i^\nu\|, q^\nu/\|q_i^\nu\|)u_i^\nu \leqq (1/\|q_i^\nu\|, q^\nu/\|q_i^\nu\|)\tilde{\omega}_i$ and

(ii) $(1/\|q_i^\nu\|, q^\nu/\|q_i^\nu\|)u \leqq (1/\|q_i^\nu\|, q^\nu/\|q_i^\nu\|)\tilde{\omega}_i$ implies $u_i^\nu \succsim_i u$.

Since $\|q_i^\nu/\|q_i^\nu\|\| = 1$, we may assume $\{q_i^\nu/\|q_i^\nu\|\}$ converges to some q_i^*, and $1/\|q_i^\nu\| \to 0$.

Let

$$q^* = (0, \cdots, 0, q_i^*, 0, \cdots, 0) \in R^{ln(i-1)} \times R^{ln} \times R^{ln(n-i)} .$$

By (i)

(iii) $(0, q^*)u_i^* \leqq (0, q^*)\tilde{\omega}_i$.

Let $u \in \tilde{X}_i$ be such that $u \succ_i u_i^*$. By (ii) for sufficiently large ν,

$$(1/\|q_i^\nu\|, q^\nu/\|q_i^\nu\|)u > (1/\|q_i^\nu\|, q^\nu/\|q_i^\nu\|)\tilde{\omega}_i .$$

Hence

$$(0, q^*)u \geqq (0, q^*)\tilde{\omega}_i .$$

This means that

$$(0, q^*)u < (0, q^*)\tilde{\omega}_i \qquad \text{implies} \quad u_i^* \succsim_i u\,.$$

Since $\|q_i^*\| = 1$, $(\omega_i^x, \omega^y) \in R_{++} \times R_{++}^{ln}$, and $\succsim_i$ is continuous, we can show that

$$(0, q^*)u \leqq (0, q^*)\tilde{\omega}_i \qquad \text{implies} \quad u_i^* \succsim_i u\,.$$

But this contradicts the fact that $\succsim_i$ is monotone.

Hence for every i, $\|q_i^\nu\| \not\to \infty$, so that we may assume $\{q_i^\nu\}$ converges to some $\bar{q}_i$ for every i. On the other hand, since z^ν belongs to a compact set, we may assume $\{z^\nu\}$ converges to some $\bar{z}$. Since z is an upper hemi-continuous correspondence,

$$\bar{z} \in z(\bar{q})\,.$$

Since $P_\nu \subset P_{\nu+1}$ for every $\nu \in N$,

$$\bigcup_\nu P_\nu \supset R^{ln^2}\,, \qquad \text{and} \qquad \lim z^\nu = \bar{z}\,,$$

it follows that

(iv) $(1, q)\bar{z} \leqq 0$ for every $q \in R^{ln^2}$.

Hence if we write

$$\bar{z} = (\bar{x}, \bar{y}_1, \cdots, \bar{y}_n) \in R \times R^{ln^2}\,,$$

we can show that

(v) $\bar{x} \leqq 0$ and $\bar{y}_i = 0$ for every i.

By the definition of $\bar{z}$, there exist

$$\bar{u}_i \in \xi_i(\bar{q}) \qquad \text{and} \qquad \bar{v} \in \zeta(\bar{q})$$

such that

$$\bar{z} = \sum_i \bar{u}_i - \bar{v} - \sum_i \tilde{\omega}_i\,.$$

Denote

$$\bar{u}_i = (\bar{x}_i, 0, \cdots, 0, \bar{y}_i, 0, \cdots, 0) \qquad \text{and} \qquad \bar{v} = (\bar{x}, \bar{y}, \cdots, \bar{y})\,.$$

Then by (v)

(vi) $\sum_i \bar{x}_i \leqq \bar{x} + \sum_i \omega_i^x$

$$\bar{y}_i = \bar{y} + \omega^y \equiv \hat{y} \qquad \text{for every} \quad i\,.$$

By the standard arguments, we can show that

$$\bar{u}_i \quad \text{is } \succsim_i\text{-greatest subject to} \quad u \in X_i \quad \text{and}$$

$$(1, \bar{q})u \leqq (1, \bar{q})\tilde{\omega}_i\,, \quad \text{and}$$

$$(1, \bar{q})\bar{v} = \max\,(1, \bar{q})Y\,.$$

By the definition of Y, we can assert

$$\sum_i \bar{q}_{ij} = \alpha \qquad \text{for every } j \qquad \text{and} \qquad (1, \bar{q})\bar{v} = 0 .$$

So

$$(1, \bar{q})\bar{z} = -(1, \bar{q})\bar{v} + \sum_i (1, \bar{q})(\bar{u}_i - \bar{\omega}_i) = 0$$

by monotonicity. Hence by (vi)

$$\sum_i \bar{x}_i = \bar{x} + \sum_i \omega_i^x ,$$

so that $(\bar{x}_1, \cdots, \bar{x}_n, \hat{y}) \in A$.

Define $p^* = \alpha$,

$$t_{ij}^* = \bar{q}_{ij} \qquad \text{if } i \neq j , \qquad \text{and} \qquad t_{jj}^* = \bar{q}_{jj} - \alpha .$$

Then $(p^*, (t_{ij}^*), (\bar{x}_1, \cdots, \bar{x}_n), \hat{y})$ is a price equilibrium, since

$$\bar{q}_{jj} = p^* + t_{jj}^* . \qquad \text{Q.E.D.}$$

Proof of Theorem 2. Suppose the price allocation (x^*, y^*) is not Pareto optimal. Then there is $(x, y) \in A$ such that

(i) for each i, $(x_i, y) \succsim_i (x_i^*, y^*)$ and

(ii) for some i, $(x_i, y) \succ_i (x_i^*, y^*)$.

By (ii) and utility maximization,

(iii) for some i, $x_i + p^* y_i > \omega_i^x + p^* \omega_i^y - \sum_j t_{ij}^* (y_j - \omega_j^y)$.

Suppose there is i such that

$$x_i + p^* y_i < \omega_i^x + p^* \omega_i^y - \sum_j t_{ij}^* (y_j - \omega_j^y) .$$

Then there is $x_i' \in R_+$ such that $x_i' > x_i$ and

$$x_i' + p^* y_i \leqq \omega_i^x + p^* \omega_i^y - \sum_j t_{ij}^* (y_j - \omega_j^y) .$$

So

$$(x_i', y) \succ_i (x_i, y) \succsim_i (x_i^*, y^*) \qquad \text{and} \qquad (x_i', y) \in B_i(p^*, t^*) ,$$

a contradiction. Hence for every i

$$x_i + p^* y_i \geqq \omega_i^x + p^* \omega_i^y - \sum_j t_{ij}^* (y_j - \omega_j^y) .$$

By (iii),

$$\sum_i (x_i - \omega_i^x) + p^* \sum_i (y_i - \omega_i^y) > -\sum_i \sum_j t_{ij}^* (y_j - \omega_j^y) = 0 .$$

But profit maximization implies $p^* = \alpha$, so that

$$\sum_i (x_i - \omega_i^x) + \alpha \sum_i (y_i - \omega_i^x) > 0\,,$$

which contradicts the fact that $(x, y) \in A$. Q.E.D.

Proof of Theorem 3. Let

$D = \{(x, y_1, \cdots, y_n) \in R \times R^{ln^2} \mid$

There is $(x_1, \cdots, y_n) \in R^n$ such that $x = \sum_i x_i$ and

for every i, $(x_i + x_i^*, y_i + y^*) \succ_i (x_i^*, y^*)\}$

and

$F = \{(x, y_1, \cdots, y_n) \in R \times R^{ln^2} \mid y_1 = \cdots = y_n$ and

$x + \alpha \sum_j y_{ij} = 0$ for every $i\}$.

Then D and F are convex and $D \cap F = \varnothing$ since (x^*, y^*) is Pareto optimal. Hence there are

$$(q^x; q_1^y, \cdots, q_n^y) \in R \times R^{ln^2} \backslash \{0\} \text{ and } r \in R$$

such that

$$q^x x + \sum_i q_i^y y_i \leqq r \qquad \text{for every} \quad (x, y_1, \cdots, y_n) \in F\,,$$

so that

(i) $q^x x + \sum_i q_i^y y \leqq r$

where $y_i = y$ for all i, and

(ii) $q^x x + \sum_i q_i^y y_i \geqq r$

for every $(x, y_1, \cdots, y_n) \in \operatorname{cl} D$.

By monotonicity $q^x \geqq 0$.

Since $(x^*, \underbrace{y^*, \cdots, y^*}_{n\text{-times}}) \in \operatorname{cl} D \cap F$, it follows that

$$q^x x^* + \sum_i q_i^y y^* = r\,.$$

Hence by (i), for all $(x, y) \in R \times R^{ln}$ with $x + \alpha \sum_j y_j = 0$,

$$q^x x^* + \sum_i q_i^y y^* \geqq q^x x + \sum_i q_i^y y\,.$$

so for all $(x, y) \in R \times R^{ln}$ with $x + (\underbrace{\alpha, \cdots, \alpha}_{n\text{-times}}) y = 0$,

$$q^x x^* + \sum_i q_i^y y^* \geqq q^x x + \sum_i q_i^y y\,.$$

Hence with $q^x \geqq 0$, we may assume

$$q^x = 1 \qquad \text{and} \qquad \sum_i q_i^y = (\underbrace{\alpha, \cdots, \alpha}_{n\text{-times}}),$$

so that $r = 0$.

Define $p^* = \alpha$,

$$t_{ij}^* = q_{ij}^y \quad \text{if} \quad i \neq j,$$

$$t_{ii}^* = q_{ii}^y - \alpha, \qquad \text{and}$$

$$(\omega_i^x, \omega_i^y) = (x_i^*, y_i^*) \quad \text{for every } i \text{ and } j.$$

Then

$$\sum_i t_{ij}^* = \sum_{i \neq j} q_{ij}^y + q_{ii}^y - \alpha = 0 \qquad \text{for every} \quad j.$$

Since profit maximization and attainability are obviously satisfied, we will show utility maximization. Fix i. Note that

$$(x_i^*, y^*) \in B_i(p^*, t^*).$$

Let $(x_i, y) \succ_i (x_i^*, y^*)$. Then

$$\Bigg(x_i - \omega_i^x + \sum_{j \neq i} (x_j^* - \omega_j^x), \underbrace{y^* - \omega^y, \cdots, y^* - \omega^y}_{(i-1)\text{-times}}, y - \omega^y,$$

$$\underbrace{y^* - \omega^y, \cdots, y^* - \omega^y}_{(n-i)\text{-times}}\Bigg) \in \mathrm{cl}\, D.$$

Hence

$$(x_i - \omega_i^x) + \sum_{j \neq i} (x_j^* - \omega_j^x) + q_i^y(y - \omega^y) + \sum_{j \neq i} q_j^y(y^* - \omega^y) \geqq 0.$$

Since $(x^*, y^*) = (\omega^x, \omega^y)$,

$$(x_i - \omega_i^x) + q_i^y(y - \omega^y) \geqq 0,$$

which is equivalent to

$$x_i + p^* y_i \geqq \omega_i^x + p^* \omega_i^y - \sum_j t_{ij}^*(y_j - \omega_j^y).$$

Hence

$$x_i + p^* y_i < \omega_i^x + p^* \omega_i^y - \sum_j t_{ij}^*(y_j - \omega_j^y)$$

implies

$$(x_i^*, y^*) \succsim_i (x_i, y) .$$

By continuity and $(\omega_i^x, \omega^y) \in R_{++} \times R_{++}^{ln}$, it follows that

$$(x_i, y) \in B_i(p^*, t^*)$$

implies

$$(x_i^*, y^*) \succsim_i (x_i, y) . \qquad \text{Q.E.D.}$$

Proof of Theorem 4. Since

$$(\omega_i^x; \omega_1^y, \cdots, \omega_n^y) \in B_i(p^*, t^*)$$

for every i, the assertion is obvious. Q.E.D.

Proof of Theorem 5. Let m^* be a Nash equilibrium. Define

$$p^* = \alpha , \quad t_{ij}^* = t_{ij}(m^*) , \quad x_i^* = x_i(m^*) , \quad \text{and} \quad y_i^* = y_i(m^*) .$$

Then attainability and profit maximization are obvious. We will show utility maximization. Fix i. Note that

$$(x_i^*, y^*) \in B_i(p^*, t^*) .$$

Let $(x_i, y) \in B_i(p^*, t^*)$.

We may assume

$$x_i + p^* y_i = \omega_i^x + p^* \omega_i^y - \sum_j t_{ij}^*(y_j - \omega_j^y) .$$

Define

$$m_{ij} = y_j - \sum_{k \neq i} m_{kj}^*$$

for every j.

Then $m_i \in M^i$ and

$$y_j = y_j(m_1^*, \cdots, m_{i-1}^*, m_i, m^*_{i+1}, \cdots, m_n^*) \text{ for all } j ,$$

it follows that

$$x_i = x_i(m_1^*, \cdots, m_{i-1}^*, m_i, m_{i+1}^*, \cdots, m_n^*) .$$

Since m^* is a Nash equilibrium,

$$\begin{aligned}(x_i^*, y^*) = (x_i(m^*), y(m^*)) &\succsim_i (x_i(m_1^*, \cdots, m_{i-1}^*, m_i, m_{i+1}^*, \cdots, m_n^*), \\ &\quad y(m_1^*, \cdots, m_{i-1}^*, m_i, m_{i+1}^*, \cdots, m_n^*)) \\ &= (x_i, y) .\end{aligned} \qquad \text{Q.E.D.}$$

Proof and Theorem 6. Let (p^*, t^*, x^*, y^*) be a price allocation. Then

$$p^* = \alpha .$$

Fix $j \in \{1, \cdots, n\}$ and $k \in \{1, \cdots, l\}$.
Consider a following equation system.

$$\begin{bmatrix} 1, & 1, & \cdots, & 1, & 1 \\ 1, & -1, & 0, & \cdots, & 0 \\ & & \cdots & & \\ & & \cdots & & \\ 0, & \cdots, & 0, & 1, & -1 \end{bmatrix} \begin{bmatrix} m^*_{1jk} \\ m^*_{2jk} \\ \cdot \\ \cdot \\ m^*_{njk} \end{bmatrix} = \begin{bmatrix} y^*_{jk} \\ t^*_{njk} \\ t^*_{1jk} \\ \cdot \\ t^*_{n-2,jk} \end{bmatrix}$$

The above equations have a unique solution $(m^*_{ijk})_i$. Let

$$m^*_i = (m^*_{ijk})_{jk} \qquad \text{and} \qquad m^* = (m^*_i)_i .$$

Then

$$t_{ij}(m^*) = t^*_{ij} \qquad \text{and} \qquad y_j(m^*) = y^*_j .$$

Hence

$$x_i(m^*) = x^*_i$$

by monotonicity.

Let $m_i \in M^i$. Then

$$(x_i(m^*_1, \cdots, m^*_{i-1}, m_i, m^*_{i+1}, \cdots, m^*_n), \mathrm{y}(\mathrm{m}^*_1, \cdots, m^*_{i-1}, m_i, m^*_{i+1}, \cdots, m^*_n))$$

satisfies the budget constraint. Hence

$$(x_i(m^*), y(m^*)) = (x^*_i, y^*) \succsim_i$$
$$(x_i(m^*_1, \cdots, m^*_{i-1}, m_i, m^*_{i+1}, \cdots, m^*_n), \mathrm{y}(\mathrm{m}^*_1, \cdots, m^*_{i-1}, m_i, m^*_{i+1}, \cdots, m^*_n))$$

so that m^* is a Nash equilibrium. Q.E.D.

Proof of Corollary. In view of Theorems 2.5 and 2.6, an allocation is a price allocation if and only if it is a Nash allocation. Hence the assertion is obvious. Q.E.D.

Keio University

REFERENCES

[1] M. Aoki, Marshallian external economies and optimal tax-subsidy structure, *Econometrica*, **39** (1971), 35–53.

[2] P. Dasgupta, P. Hammond, and E. Maskin, The implementation of social choice rules: Some general results on incentive compatibility, *Rev. Econ. Studies*, **46** (1979), 185–216.

[3] G. Debreu, "Theory of Value," Wiley, New York, 1959.

[4] L. Hurwicz and D. Schmeidler, Construction of outcome functions guaranteeing existence and Pareto optimality of Nash equilibria, *Econometrica*, **46** (1978), 1447–1474.

[5] L. Hurwicz, Outcome functions yielding Walrasian and Lindahl allocations at Nash equilib-

rium points, *Rev. Econ. Studies*, **46** (1979), 217–225.

[6] S. Nakamura, A balanced outcome function yielding Pareto optimal allocations at Nash equilibrium points in the presence of externalities: A case of pure exchange economy, mimeographed, Apr. 1983.

[7] H. Osana, Optimal tax-subsidy system for an economy with Marshallian externalities, *Econometrica*, **45** (1977), 329–340.

[8] R. Rob, A condition guaranteeing the optimality of public choice, *Econometrica*, **49** (1981), 1605–1613.

[9] M. Walker, A simple incentive compatible scheme for attaining Lindahl allocations, *Econometrica*, **49** (1981), 65–71.