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Notes

ON HICKS' COMPOSITE COMMODITY THEOREM: A SUPPLEMENTARY NOTE

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In a previous paper (Osana (1982)), it was shown that the order property, non-satiation, local non-satiation, weak monotonicity, monotonicity, strong monotonicity, weak convexity, convexity, strong convexity, upper semi-continuity, and lower semi-continuity of the preference relation defined on the original commodity space carry over to the preference relation defined on the new commodity space involving a composite commodity. This constitutes an exact formulation of Hicks' Composite Commodity Theorem. The present note supplements the results by showing that some differential properties of the original preference relation are inherited by the induced preference relation.

We shall use the same notation as in the previous paper. Let H be a non-empty finite set, representing the set of commodities. The consumption set X is a non-empty closed subset of R^H that is bounded from below. As we shall deal with the case of differentiable and hence continuous preference relation, we shall assume, in particular, that $X=R^H_+$ (cf. Osana (1982, Theorem 11)). The preference relation Q is a reflexive total transitive binary relation on X. As Q is assumed to be continuous, there is a continuous utility function u on X representing Q. Let I be a non-empty proper subset of H and define $J=H-I$. Given a price vector p in R^I_{++}, the set of possible pairs of consumptions of the commodities in J and expenditures on the commodities in I is defined by

$$X(p)=\{(x,c)\in R^J\times R\colon (x,y)\in X \text{ for some } y\in R^I \text{ such that } p\cdot y=c\}\,.$$

Since $X=R^H_+$, it follows that $X(p)=R^J_+\times R_+$ for every $p\in R^I_{++}$. For each $p\in R^I_{++}$ and each $(x,c)\in X(p)$, let

$$Y(p,x,c)=\{y\in R^I\colon (x,y)\in X \;\&\; p\cdot y=c\}\,.$$

Actually, $Y(p,x,c)$ is independent of x under our assumption, so that we shall write $Y(p,c)=Y(p,x,c)$. For each $(p,x,c)\in R^I_{++}\times R^J_+\times R_+$, let

$$h(p,x,c)=\{y\in Y(p,c)\colon u(x,y)\geqq u(x,y') \text{ for every } y'\in Y(p,c)\}\,,$$

$$v(p,x,c)=u(x,h(p,x,c))\,,$$

$$v^p(x,c)=v(p,x,c)\,.$$

For every $p\in R^I_{++}$, the preference relation $Q(p)$ on $X(p)$ can be expressed as

$$Q(p)=\{((x,c),(x',c'))\in(R^I_+\times R_+)^2\colon v^p(x,c)\geqq v^p(x',c')\}\,.$$

That is, v^p is a utility function on $X(p)$ representing $Q(p)$. Our aim in the present note is to demonstrate that some differential properties of u carry over to v^p. In what follows, we shall assume that

(1) u is twice continuously differentiable on R^H_{++},

(2) $u_z(z)>0$ for every $z\in R^H_{++}$, where $u_z(z)$ is the gradient at z of u,

(3) $a^T u_{zz}(z)a<0$ for every $z\in R^H_{++}$ and every $a\in R^H-\{0\}$ such that $u_z{}^T(z)a=0$, where $u_{zz}(z)$ is the Hessian matrix at z of u,

(4) u is strictly quasi-concave on R^H_+, i.e., $u((1-t)z+tz')>u(z')$ for every $t\in]0,1[$ and every $z,z'\in R^H_+$ such that $u(z)\geqq u(z')$ & $z\neq z'$.

LEMMA 1. *h is a function of $R^I_{++}\times R^J_+\times R_+$ into R^I_+ which is continuous on $R^I_{++}\times R^J_+\times R_{++}$.*

Proof. Let $(p,x,c)\in R^I_{++}\times R^J_+\times R_+$. Since $Y(p,c)$ is non-empty and compact (cf. Osana (1982, Lemma 1)) and u is continuous on R^H_+, it follows that $h(p,x,c)$ is non-empty. Furthermore, $h(p,x,c)$ is a singleton, since u is strictly quasi-concave on R^H_+. For each $(p,c)\in R^I_{++}\times R_{++}$, let $B(p,c)=\{y\in R^I_+ : p\cdot y\leqq c\}$. Let $(p,c)\in R^I_{++}\times R_{++}$. Then $B(p,c)\subset\{y\in R^I: -a<y<a\}$ for some $a\in R^I_{++}$. Write $K=\{y\in R^I: -a<y<a\}$. Let $b=\min\{a_i p_i: i\in I\}$ and $P=\{(p',c')\in R^I_{++}\times R_{++}$: $c'<\frac{1}{2}(b+c)$ & $p_i'>(b+c)/2a_i$ for every $i\in I\}$. Then P is an open neighborhood in $R^I\times R$ of (p,c). For every $(p',c')\in P$, $B(p',c')\subset K\subset \operatorname{cl} K$. Since B is closed at (p,c), this implies that B is upper hemi-continuous at (p,c) (cf. Hildenbrand (1974, B.III, Proposition 2)). Let G be an open subset of R^I such that $G\cap B(p,c)\neq\varnothing$. There is $y\in G\cap B(p,c)$. If $y=0$ they $y\in G\cap B(p',c')$ for every $(p',c')\in R^I_{++}\times R_{++}$. Suppose $y\neq 0$. Since G is open, there is $t\in]0,1[$ such that $ty\in G$. Hence $p\cdot ty<p\cdot y\leqq c$ so that there is an open neighborhood P' in $R^I\times R$ of (p,c) such that $ty\in B(p',c')$ for every $(p',c')\in P'$. That is, $G\cap B(p',c')\neq\varnothing$ for every $(p',c')\in P'$. Thus B is lower hemicontinuous at (p,c) and hence continuous at (p,c). Let $x\in R^J_+$. Since $h(p,x,c)=\{y\in B(p,c)$: $u(x,y)\geqq u(x,y')$ for every $y'\in B(p,c)\}$ and u is continuous at (x,y), it follows that h is upper hemi-continuous at (p,x,c) (cf. Hildenbrand (1974, B.III, Theorem 3)). Since h is single-valued, this implies that h is continuous at (p,x,c). That is, h is continuous on $R^I_{++}\times R^J_+\times R_{++}$.

Let $D=\{(p,x,c)\in R^I_{++}\times R^J_{++}\times R_{++}: h(p,x,c)\in R^I_{++}\}$. Then D is open in $R^I\times R^J\times R$, since h is continuous on $R^I_{++}\times R^J_{++}\times R_{++}$.

LEMMA 2. *There is a unique function μ of D into R_{++} such thai $u_y(x,h(p,x,c))=\mu(p,x,c)p$ for every $(p,x,c)\in D$.*

Proof. Let $(p,x,c)\in D$. For each $y\in R^I_{++}$, let $f(y)=u(x,y)$ and $g(y)=c-p\cdot y$. Then f and g are continuously differentiable on R^I_{++}. Note that $g(h(p,x,c))=0$ and $g_y(h(p,x,c))=-p$ so that rank $g_y(h(p,x,c))=1$. Since $f(h(p,x,c))\geqq f(y)$ for every $y\in R^I_{++}$ such that $g(y)=0$, there is a unique real number $\mu(p,x,c)$ such that $f_y(h(p,x,c))=\mu(p,x,c)p$. By assumption, $f_y(h(p,x,c))=u_y(x,h(p,x,c))>0$ and $p>0$ so that $u(p,x,c)>0$.

LEMMA 3. *h and μ are continuously differentiable on D.*

Proof. Let $Z = R^I_{++} \times R_{++} \times D$. Then Z is an open subset of $R^I \times R \times R^I \times R^J \times R$. Let $(p^*, x^*, c^*) \in D$. Let $(y^*, m^*) = (h(p^*, x^*, c^*), \mu(p^*, x^*, c^*))$. Then $(y^*, m^*, p^*, x^*, c^*) \in Z$. For each $(y, m, p, x, c) \in Z$, let

$$G(y, m, p, x, c) = \begin{bmatrix} u_y(x, y) - mp \\ c - p \cdot y \end{bmatrix}.$$

Then

$$\det [G_y(y^*, m^*, p^*, x^*, c^*),\ G_m(y^*, m^*, p^*, x^*, c^*)]$$
$$= (m^*)^{-2} \begin{vmatrix} u_{yy}(x^*, y^*) & u_y(x^*, y^*) \\ u_y{}^T(x^*, y^*) & 0 \end{vmatrix} \neq 0$$

so that, by the implicit function theorem, there is an open nieghborhood U in $R^I \times R^J \times R$ of (p^*, x^*, c^*), a continuously differentiable function g of U into R^I, and a continuously differentiable function θ of U into R such that $g(p^*, x^*, c^*) = y^*$ & $\theta(p^*, x^*, c^*) = m^*$ & $(g(p, x, c), \theta(p, x, c), p, x, c) \in Z$ & $G(g(p, x, c), \theta(p, x, c), p, x, c) = 0$ for every $(p, x, c) \in U$. Hence, for every $(p, x, c) \in U$, $g(p, x, c) \in R^I_{++}$ & $\theta(p, x, c) \in R_{++}$ & $u_y(x, g(p, x, c)) = \theta(p, x, c)p$ & $c = p \cdot g(p, x, c)$. For each $x \in R^J_+$ and each $y \in R^I_+$, let $f^x(y) = u(x, y)$. Then, for every $x \in R^J_+$, f^x is quasi-concave on R^I_+. For every $(p, x, c) \in U$, $f_y{}^x(g(p, x, c)) = u_y(x, g(p, x, c)) = \theta(p, x, c)p$ so that $g(p, x, c) \in h(p, x, c)$ (cf. Arrow and Enthoven (1961, Theorem 3)) and hence, by Lemma 1, $h(p, x, c) = g(p, x, c) \in R^I_{++}$. Without loss of generality, we may assume that $U \subset R^I_{++} \times R^J_{++} \times R_{++}$. Then $U \subset D$ so that, by Lemma 2, $\mu(p, x, c) = \theta(p, x, c)$ for every $(p, x, c) \in U$. Thus h and μ are continuously differentiable at (p^*, x^*, c^*).

LEMMA 4. *v is twice continuously differentiable on D.*

Proof. Note that $\sum_{i \in I} p_i h^i(p, x, c) = c$ for every $(p, x, c) \in D$. Let $(p, x, c) \in D$. Then $p^T h_p(p, x, c) = -h^T(p, x, c)$ & $p^T h_x(p, x, c) = 0$ & $p^T h_c(p, x, c) = 1$ so that $v_p{}^T(p, x, c) = u_y{}^T(x, h(p, x, c)) h_p(p, x, c) = \mu(p, x, c) p^T h_p(p, x, c) = -\mu(p, x, c) h^T(p, x, c)$ & $v_x{}^T(p, x, c) = u_x{}^T(x, h(p, x, c)) + u_y{}^T(x, h(p, x, c)) h_x(p, x, c) = u_x{}^T(x, h(p, x, c)) + \mu(p, x, c) p^T h_x(p, x, c) = u_x{}^T(x, h(p, x, c))$ & $v_c(p, x, c) = u_y{}^T(x, h(p, x, c)) h_c(p, x, c) = \mu(p, x, c) p^T h_c(p, x, c) = \mu(p, x, c)$. Since h and μ are continuously differentiable at (p, x, c) and u_x is continuously differentiable at $(x, h(p, x, c))$, it follows that v_p, v_x, and v_c are continuously differentiable at (p, x, c). Thus v is twice continuously differentiable at (p, x, c).

Let $P^* = \{p \in R^I\colon (p, x, c) \in D$ for some $(x, c) \in R^J \times R\}$. For each $p \in P^*$, let $D(p) = \{(x, c) \in R^J \times R\colon (p, x, c) \in D\}$.

THEOREM. *For every $p \in P^*$, (1) v^p is twice continuously differentiable on $D(p)$, (2) $v_z{}^p(z) > 0$ for every $z \in D(p)$, and (3) $a^T v^p_{zz}(z) a < 0$ for every $z \in D(p)$ and every $a \in R^J \times R - \{0\}$ such that $v_z{}^p(z) \cdot a = 0$.*

Proof. (1) follows from Lemma 4 and (2) follows from the proof of Lemma 4. (3) Take any $(x, c) \in R^J \times R - \{0\}$ such that $(v_x, v_c) \cdot (x, c) = 0$. Then

$$\begin{aligned}
&[x^T, c]\begin{bmatrix} v_{xx} & v_{xc} \\ v_{cx} & v_{cc} \end{bmatrix}\begin{bmatrix} x \\ c \end{bmatrix} \\
&= x^T u_{xx} x + x^T u_{xy} h_x x + c\mu_x{}^T x + x^T u_{xy} h_c c + c\mu_c c \\
&= x^T u_{xx} x + x^T u_{xy}(h_x x + h_c c) + p^T(h_x x + h_c c)\mu_x{}^T x + c\mu_c c \\
&= x^T u_{xx} x + x^T u_{xy}(h_x x + h_c c) + (h_x x + h_c c)^T p \mu_x{}^T x + c\mu_c c \\
&= x^T u_{xx} x + x^T u_{xy}(h_x x + h_c c) + (h_x x + h_c c)^T (u_{yx} + u_{yy} h_x) x + c\mu_c c \\
&= x^T u_{xx} x + x^T u_{xy}(h_x x + h_c c) + (h_x x + h_c c)^T u_{yx} x + (h_x x + h_c c)^T u_{yy} h_x x \\
&\quad + (h_x x + h_c c)^T p \mu_c p^T (h_x x + h_c c) \\
&= x^T u_{xx} x + x^T u_{xy}(h_x x + h_c c) + (h_x x + h_c c)^T u_{yx} x + (h_x x + h_c c)^T u_{yy} h_x x \\
&\quad + (h_x x + h_c c)^T u_{yy} h_c p^T (h_x x + h_c c) \\
&= x^T u_{xx} x + x^T u_{xy}(h_x x + h_c c) + (h_x x + h_c c)^T u_{yx} x \\
&\quad + (h_x x + h_c c)^T u_{yy}(h_x x + h_c p^T h_x x + h_c p^T h_c c) \\
&= x^T u_{xx} x + x^T u_{xy}(h_x x + h_c c) + (h_x x + h_c c)^T u_{yx} x + (h_x x + h_c c)^T u_{yy}(h_x x + h_c c) \\
&= [x^T\ (h_x x + h_c c)^T]\begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix}\begin{bmatrix} x \\ h_x x + h_c c \end{bmatrix} < 0\,,
\end{aligned}$$

the last inequality being due to Assumption (3) and the fact that

$$\begin{aligned}
(u_x, u_y) \cdot (x, h_x x + h_c c) &= u_x{}^T x + u_y{}^T (h_x x + h_c c) = v_x{}^T x + \mu p^T (h_x x + h_c c) \\
&= v_x{}^T x + \mu c = v_x{}^T x + v_c c = (v_x, v_c) \cdot (x, c) = 0\,.
\end{aligned}$$

The theorem asserts that, if corner solutions are ruled out in the choice of commodity bundles in the commodity subspace forming a composite commodity, then twice continuous differentiability, positive marginal utilities, and strong quasi-concavity are preserved for the new utility function defined in the new commodity space involving a composite commodity.

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