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THE PASINETTI-SAMUELSON-MODIGLIANI MODEL AS A TWO-PERSON NONZERO-SUM DIFFERENTIAL GAME*

David CHAPPELL and Roger W. LATHAM

INTRODUCTION

The purpose of the present paper is to recast the neo-classical formulation of the Pasinetti model [5], proposed by Samuelson and Modigliani [7], as a nonzero-sum differential game treating capitalists and workers as two players. It is assumed that each player wishes to maximise the integral of discounted utility derived from his intertemporal consumption path. Since the sum of the players' criteria is neither zero nor a constant the game is of the nonzero-sum type. Hence the players are not is direct conflict with each other and consequently there may be several solution concepts which are of interest (see Starr and Ho [8]). Here we attempt to characterize the *noncooperative* (Nash) equilibrium solution.

The main virtue of this approach is that the savings propensities, assumed to be constant in Samuelson and Modigliani [7], now become decision variables. In particular it allows us to try to solve for their equilibrium steady state values in terms of the parameters of the model.

On the debit side, there are problems in adopting a differential games approach to such a model. Firstly, because the theory of infinite-horizon, many-player, non-zero sum games is at best incomplete. Secondly, because of the non-linearities involved, we cannot solve for the control variables as functions of the state variables and thus characterize a saddle-point equilibrium (see Starr and Ho [8] or Intriligator (3), p. 387). Instead we derived implicit relationships between the control, state and co-state variables.[1] Nevertheless, we are able to derive some interesting qualitative results on possible long-run steady-state equilibria and stability.

THE SAMUELSON-MODIGLIANI MODEL

The production side of the economy is summarized by

$$Y = F(K, L) = C + \dot{K}, \qquad K = K_c + K_w, \tag{1}$$

* The authors are grateful for helpful comments from Professors J. L. Ford and J. S. Metcalfe but the usual caveat applies.

[1] It is worth pointing out that this is an established practice in the analysis of non-linear deterministic optimal control problems, which may be viewed as a special case of differential games with only one player.

where F is homogeneous of degree one, Y is output K is capital, L is labour, and K_c and K_w are the amounts of capital owned by the capitalists and workers respectively. For obvious capital-theoretic reasons Samuelson and Modigliani assume that the production of the consumption good C and of the investment good $\dot{K}$ involve the same capital-labour factor intensities. But, as pointed out by Harris [2], if this assumption holds the two goods may be regarded as the same except for units of measurement and these can be chosen so that their relative price is unity. Thus one may just as well assume that there is a single commodity which can be both consumed and used as a factor of production.

The labour force grows at the natural rate n. If n includes Harrod-neutral technical progress L is measured in terms of efficiency units. For convenience all of the other variables are expressed in terms of L e.g.

$$y=\frac{Y}{L}, \quad k=\frac{K}{L}, \quad k_c=\frac{K_c}{L} \quad \text{etc.}$$

Thus (1) can be rewritten as

$$y=f(k)\,, \tag{2}$$

which is assumed to satisfy the Inada conditions.

The marginal productivity conditions give

$$\left.\begin{aligned} r&=\frac{P}{K}=\frac{P_c+P_w}{K}=f'(k) \\ w&=\frac{W}{L}=\frac{Y-rK}{L}=f(k)-f'(k)k \end{aligned}\right\} \tag{3}$$

where r is the rate of profit, P is total profits, w is the wage rate, and W is total wages. The savings-investment equations for the two classes are

$$\left.\begin{aligned} &\dot{K}_c=s_cP_c=s_c(rK_c)=s_cK_cf'(k)\,, \\ &\dot{K}_w=s_w(W+P_w)=s_w(W+rK_w)=s_w(Y-rK_c) \\ &\quad=s_w[F(K,L)-f'(k)\cdot K_c] \end{aligned}\right\} \tag{4}$$

where s_i $(i=c, w)$ is the savings propensity of the ith class.

The above assumptions yield the following system of differential equations

$$\dot{k}_c=(s_cf'-n)\cdot k_c\,, \tag{5}$$

$$\dot{k}_w=s_w(f-f'\cdot k_c)-nk_w\,, \tag{6}$$

which have been analysed in some detail by Samuelson and Modigliani.

CAPITALISTS AND WORKERS

In this paper it is assumed that capitalists

$$\max_{\{s_c\}} \int_0^{\infty} U_c(c_c)^{-\delta_c t}dt \qquad (7)$$[2]

where $s_c \in [0, 1]$ and $c_c = (1 - s_c)f' \cdot k_c$, and workers

$$\max_{\{s_w\}} \int_0^{\infty} U_w(c_w)e^{-\delta_w t}dt \qquad (8)$$

where $s_w \in [0, 1]$ and $c_w = (1 - s_w)(f - f' \cdot k_c)$. U_i and δ_i are the instantaneous utility function and positive discount rate of the ith class ($i = c, w$), respectively. Both classes optimize subject to the constraints imposed by the differential equation system (5) and (6) and it is assumed that the initial values of k_c and k_w, i.e. $k_c(0)$ and $k_w(0)$, are given.

Introducing costate variables p_c, q_c, p_w, q_w, which are assumed to be nonnegative $\forall\, t \in [0, \infty)$ and to satisfy the Pontryagin constraint qualification (see [6] p. 81), the Hamiltonians for the two classes are:

$$H^c = e^{-\delta_c t}\{U_c(c_c) + p_c[(s_c f' - n)k_c] + q_c[s_w(f - f'k_c) - nk_w]\} \qquad (9)$$

$$H^w = e^{\delta_w t}\{U_w(c_w) + p_w[(s_c f' - n)k_c] + q_w[s_w(f - f'k_c) - nk_w]\} \qquad (10)$$

Then the necessary conditions for a Nash equilibrium (see Starr and Ho [8] and Intriligator [3] p. 387) are that for each $t \in [0, \infty)$ the control variables yield a Nash equilibrium of the *static* nonzero-sum game where the pay-offs are H^c and H^w. Thus for capitalists

$$\frac{\delta H^c}{\delta s_c} = e^{-\delta_c t} f' \cdot k_c\{-U_c'(c_c) + p_c\} \gtreqless 0$$

and for workers

$$\frac{\delta H^w}{\delta s_w} = e^{-\delta_w t}(f - f' \cdot k_c)\{-U_w(c_w) + q_w\} \gtreqless 0\,.$$

Notice that if $\delta H^c/\delta s_c > 0$, $s_c = 1$ and if $\delta H^w/\delta s_w < 0$, $s_w = 0$, which is the classical savings function in its extreme form. However, in the sequel it is assumed that there are interior solutions such that

$$U_c'(c_c) = p_c\,, \qquad s_c \in (0, 1)\,, \qquad (11)$$

and

$$U_w'(c_w) = q_w\,, \qquad s_w \in (0, 1)\,, \qquad (12)$$

Assuming that U_c and U_w are strictly concave these are maxima rather than

[2] The reader may be puzzled by the formulation i.e. that capitalists derive satisfaction from their consumption *per efficiency unit of labour*. But since L follows an exogenously determined monotone increasing time path this is mathematically equivalent to the capitalists' deriving satisfaction from their consumption of the good in terms of its own units.

minima and can be solved as follows. From (11)

$$s_c = s_c(k_c, k_w, p_c)\,, \tag{13}$$

where

$$\frac{\delta s_c}{\delta k_c} = \frac{(1-s_c)(f''k_c + f')}{f'k_c}, \qquad \frac{\delta s_c}{\delta k_w} = \frac{(1-s_c)(f''k_c)}{f'k_c},$$

$$\frac{\delta s_c}{\delta p_c} = \frac{1}{U_c'' \cdot f'k_c},$$

and from (12)

$$s_w = s_w(k_c, k_w, q_w) \tag{14}$$

where

$$\frac{\delta s_w}{\delta k_c} = \frac{(1-s_w)(-f''k_c)}{(f - f'k_c)}, \qquad \frac{\delta s_w}{\delta k_w} = \frac{(1-s_w)(f' - f''k_c)}{(f - f'k_c)},$$

$$\frac{\delta s_w}{\delta q_w} = \frac{-1}{U_w''[f - f' \cdot k_c]}.$$

The canonical equations, in addition to (5) and (6), are

$$\frac{d}{dt}(e^{-\delta_c t} p_c) = -\frac{\delta H^c}{\delta k_c} - \frac{\delta H^c}{\delta s_w} \cdot \frac{\delta s_w}{\delta k_c} \tag{15}$$

$$\frac{d}{dt}(e^{-\delta_c t} q_c) = -\frac{\delta H^c}{\delta k_w} - \frac{\delta H^c}{\delta s_w} \cdot \frac{\delta s_w}{\delta k_w} \tag{16}$$

$$\frac{d}{dt}(e^{-\delta_w t} p_w) = -\frac{\delta H^w}{\delta k_c} - \frac{\delta H^w}{\delta s_c} \cdot \frac{\delta s_c}{\delta k_c} \tag{17}$$

$$\frac{d}{dt}(e^{-\delta_w t} q_w) = -\frac{\delta H^w}{\delta k_w} - \frac{\delta H^w}{\delta s_c} \cdot \frac{\delta s_c}{\delta k_w} \tag{18}$$

Using (11), (12), (13) and (14), these differential equations may readily be expressed as

$$\dot{p}_c = \{q_c \cdot f'' \cdot k_c - p_c[f'' \cdot k_c + f' - (n + \delta_c)]\} \tag{19}$$

$$\dot{q}_c = \{q_c[f'' \cdot k_c - f' + (n + \delta_c)] - p_c f'' \cdot k_c\} \tag{20}$$

$$\dot{p}_w = \{q_w f'' \cdot k_c - p_w[f'' \cdot k_c + f' - (n + \delta_w)]\} \tag{21}$$

$$\dot{q}_w = \{q_w[f'' \cdot k_c - f' + (n + \delta_w)] - p_w f'' \cdot k_c\} \tag{22}$$

THE STEADY STATE SOLUTION

Necessary conditions for a steady state solution of the dynamic system defined by (5), (6), (19), (20), (21), and (22) are:

$$s_c^* f'(k^*) = n \tag{23a}$$

or

$$k_c^* = 0 \tag{23b}$$

$$s_w^*[f(k^*) - f'(k^*) \cdot k_c^*] = nk_w^* , \tag{24}$$

$$q_c^* f''(k^*) \cdot k_c^* = p_c^*[f''(k^*) \cdot k_c^* + f'(k^*) - (n + \delta_c)] , \tag{25}$$

$$q_c^*[f''(k^*) \cdot k_c^* - f'(k^*) + (n + \delta_c)] = p_c^* f''(k^*) k_c^* , \tag{26}$$

$$q_w^* f''(k^*) \cdot k_c^* = p_w^*[f''(k^*) \cdot k_c^* + f'(k^*) - (n + \delta_w)] , \tag{27}$$

$$q_w^*[f''(k^*) \cdot k_c^* - f'(k^*) + (n + \delta_w)] = p_w^* f''(k^*) \cdot k_c^* , \tag{28}$$

where the asterisks denote steady state values. From (25) and (26) we can obtain

$$q_c^* = \frac{p_c^* \cdot [f''(k^*) k_c^* + f'(k^*) - (n + \delta_c)]}{f''(k^*) \cdot k_c^*} = \frac{p_c^* \cdot f''(k^*) \cdot k_c^*}{[f''(k^*) k_c^* - f'(k^*) + (n + \delta_c)]}$$

Now p_c^* (which is positive because $U_c'(c_c)$ is assumed to be positive in (11)) can be cancelled to give

$$[f'(k^*) - (n + \delta_c)]^2 - (f''(k^*) \cdot k_c^*)^2 = -(f''(k^*) \cdot k_c^*)^2 ,$$

implying

$$f'(k^*) = n + \delta_c . \tag{29}$$

Following the same procedure with regard to (27) and (28) yields

$$f'(k^*) = n + \delta_w . \tag{30}$$

From (29) and (30) it is clear that δ_w must equal δ_c. This is obviously restrictive *but it is a necessary condition for a Nash equilibrium steady state to exist.* It follows that this is a modified golden rule path à la Cass [1]. Henceforth this common discount rate will be denoted by δ. Substituting (29) and (30) back into (25) through to (28) gives

$$p_i^* = q_i^* , \qquad i = c, w , \tag{31}$$

i.e. both capitalists' and workers' shadow values of capitalist and worker investment must be equal in a Nash equilibrium steady state.

Now suppose that $k_c^* \neq 0$ and consider (23a). Although this looks like a Pasinetti equilibrium it is different in character because s_c^* and k^* are to be determined simultaneously. Furthermore combining (23a) with (30) permits us to solve for s_c^* in terms of the parameters of the problem i.e.

$$s_c^* = \frac{n}{n + \delta} , \tag{32}$$

which is clearly less than unity.

Next consider the determination of s_w^* and k_w^* (or k_c^* since k^* is determined from (30)). Combining (23a) and (24) and rearranging gives

$$(s_c^* - s_w^*) = \frac{s_w^*[f(k^*) - f'(k^*) \cdot k^*]}{k_w^* f'(k^*)} > 0 .$$

In other words s_c^* *necessarily bounds* s_w^* *from above* if $k_c^* \neq 0$. In addition rearranging (24) gives

$$s_w^* = k_w^* n[f(k^*) - (k^* - k_w^*) f'(k^*)]^{-1} \tag{33}$$

Since k^* is determined in (30), (33) gives a relationship between s_w^* and k_w^* i.e.

$$s_w^* = s_w^*(k_w^*) , \tag{34}$$

which has the following properties

$$s_w^*(0) = 0 , \qquad s_w^*(\infty) = \frac{n}{n+\delta} , \qquad s_w^{*\prime}(0) = n[f(k^*) - f'(k^*) \cdot k^*]^{-1} > 0 ,$$

$$s_w^{*\prime}(\infty) = 0 , \qquad s_w^{*\prime}(k_w) > 0 , \qquad s_w^{*\prime\prime}(k_w) < 0 ,$$

and is depicted in Fig. 1. Thus in a Nash equilibrium steady state the larger is the amount of capital owned by the workers the higher must be their propensity to save in order to sustain it.

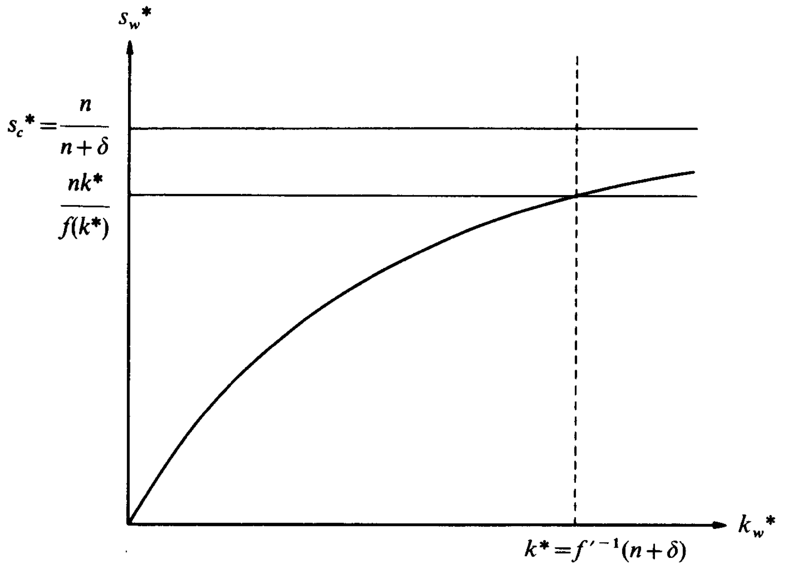

Fig. 1.

However it is important to note that we cannot solve for both s_w^* and k_w^* simultaneously in terms of parameters. Hence the steady state distribution of the ownership between the two calsses is left undetermined. This is because our analysis has so far been confined to the study of steady states. In a fully dynamic analysis more information is available i.e. the initial level and distribution of the capital stock.

Finally consider the steady state equilibrium when $k_c^*=0$ i.e. (23b). Then $c_c^*=0$, $s_w^*=nk^*/f(k^*)$, and $c_w^*=f(k^*)-nk^*$. Prima facie this looks like a Meade equilibrium (see [4]) but here s_w^* is not a parameter.

STABILITY

In this section the local stability of the two possible types of steady state equilibrium corresponding to $k_c^* \neq 0$ and $k_c^*=0$ is examined. First consider the differential equation system defined by (19), (20), (21), (22), (5), (6) and eliminate s_w and s_c by using (13) and (14). the resulting system involves six variables i.e. p_c, q_c, p_w, q_w, k_c, and k_w. Taking a linear approximation around any particular stationary solution, using a Taylor's series expansion and ignoring all terms of higher order than the first, yields

$$\begin{bmatrix} \dot{p}_c \\ \dot{q}_c \\ \dot{p}_w \\ \dot{q}_w \\ \dot{k}_c \\ \dot{k}_w \end{bmatrix} \simeq \begin{bmatrix} \frac{\partial \dot{p}_c}{\partial p_c} & \frac{\partial \dot{p}_c}{\partial q_c} & \frac{\partial \dot{p}_c}{\partial p_w} & \frac{\partial \dot{p}_c}{\partial q_w} & \frac{\partial \dot{p}_c}{\partial k_c} & \frac{\partial \dot{p}_c}{\partial k_w} \\ \frac{\partial \dot{q}_c}{\partial p_c} & \frac{\partial \dot{q}_c}{\partial q_c} & \frac{\partial \dot{q}_c}{\partial p_w} & \frac{\partial \dot{q}_c}{\partial q_w} & \frac{\partial \dot{q}_c}{\partial k_c} & \frac{\partial \dot{q}_c}{\partial k_w} \\ \frac{\partial \dot{p}_w}{\partial p_c} & \frac{\partial \dot{p}_w}{\partial q_c} & \frac{\partial \dot{p}_w}{\partial p_w} & \frac{\partial \dot{p}_w}{\partial q_w} & \frac{\partial \dot{p}_w}{\partial k_c} & \frac{\partial \dot{p}_w}{\partial k_w} \\ \frac{\partial \dot{q}_w}{\partial p_c} & \frac{\partial \dot{q}_w}{\partial q_c} & \frac{\partial \dot{q}_w}{\partial p_w} & \frac{\partial \dot{q}_w}{\partial q_w} & \frac{\partial \dot{q}_w}{\partial k_c} & \frac{\partial \dot{q}_w}{\partial k_w} \\ \frac{\partial \dot{k}_c}{\partial p_c} & \frac{\partial \dot{k}_c}{\partial q_c} & \frac{\partial \dot{k}_c}{\partial p_w} & \frac{\partial \dot{k}_c}{\partial q_w} & \frac{\partial \dot{k}_c}{\partial k_c} & \frac{\partial \dot{k}_c}{\partial k_w} \\ \frac{\partial \dot{k}_w}{\partial p_c} & \frac{\partial \dot{k}_w}{\partial q_c} & \frac{\partial \dot{k}_w}{\partial p_w} & \frac{\partial \dot{k}_w}{\partial q_w} & \frac{\partial \dot{k}_w}{\partial k_c} & \frac{\partial \dot{k}_w}{\partial k_w} \end{bmatrix} \begin{bmatrix} p_c - p_c^* \\ q_c - q_c^* \\ p_w - p_w^* \\ q_w - q_w^* \\ k_c - k_c^* \\ k_w - k_w^* \end{bmatrix} \tag{35}$$

where all the partial derivatives are evaluated at the stationary solution.

If $k_c^* \neq 0$ the above matrix becomes

$$\begin{bmatrix} -k_c^*f'' & k_c^*f'' & 0 & 0 & -p_c^*f'' & -p_c^*f'' \\ -k_c^*f'' & k_c^*f'' & 0 & 0 & -q_c^*f'' & -q_c^*f'' \\ 0 & 0 & -k_c^*f'' & k_c^*f'' & -p_w^*f'' & -p_w^*f'' \\ 0 & 0 & -k_c^*f'' & k_c^*f'' & -q_w^*f'' & -q_w^*f'' \\ -U_c''^{-1} & 0 & 0 & 0 & \delta + k_c^*f'' & k_c^*f'' \\ 0 & 0 & 0 & -U_w''^{-1} & -k_r^*f'' & \delta - k_c^*f'' \end{bmatrix} \tag{36}$$

of which the characteristic equation is

$$\lambda^3(\lambda-\delta)\left\{\lambda^2-\delta\lambda-\left[\frac{q_w^*}{U_w''}+\frac{q_c^*}{U_c''}\right]f''\right\}=0 \tag{37}$$

Hence the six characteristic roots are zero (of multiplicity three), δ, and the pair

$$\frac{1}{2}\left\{\delta\pm\sqrt{\delta^2+4\left[\frac{q_w^*}{U_w''}+\frac{p_c^*}{U_c''}\right]f''}\right\} \tag{38}$$

which are real and of opposite sign. Thus when $k_c^*\neq 0$ all the roots are real but only one is negative.

On the other hand if $k_c^*=0$ capitalists cease to exist in the steady state and it can readily be shown that the characteristic equation is

$$\mu^3(\mu-\delta)\left\{\mu^2-\delta\mu-\frac{q_w^*}{U_w''}f''\right\}=0 \tag{39}$$

Again all the roots are real but only one is negative.

Therefore in both cases we know, from Coddington and Levinson [9] (Ch. 13 especially Theorem 4.1) that there exists a one-dimensional space curve in the six-dimensional space containing the stationary point in question. Any trajectory starting on this space curve tends asymptotically towards the equilibrium point while any trajectory starting from a point off this curve moves exponentially away from the equilibrium point as time proceeds. It is also worth noting that cyclical behaviour cannot occur in either case because all the characteristic roots are real.

CONCLUSION

The aim of this paper has been to characterize noncooperative equilibrium steady state growth paths in the context of the Pasinetti-Samuelson-Modigliani model and to examine their stability. In a steady state the discount rates of the two classes must be equal and we have a modified golden rule. If $k_c^*\neq 0$ the capitalists' savings propensity is a function of the rate of growth of the labour force and the common discount rate and necessarily bounds the workers' savings propensity from above. The workers' savings propensity, on the other hand, is a function of the capital stock which they own. If $k_c^*=0$ the workers' savings propensity is a function of the equilibrium level of the capital stock and the rate of growth of the labour force. In neither case is stability assured. However in contrast to the Samuelson-Modigliani analysis cycles are not possible.

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