Title	A VARIATIONAL PROBLEM IN THE THEORY OF OPTIMAL ECONOMIC GROWTH : An Existence Theorem
Sub Title	
Author	MARUYAMA, Toru
Publisher	Keio Economic Society, Keio University
Publication year	1983
Jtitle	Keio economic studies Vol.20, No.1 (1983.) ,p.23- 31
JaLC DOI	
Abstract	
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19830001-0023

A VARIATIONAL PROBLEM IN THE THEORY OF OPTIMAL ECONOMIC GROWTH
—An Existence Theorem—

Toru MARUYAMA

1. INTRODUCTION

The theory of optimal economic growth is one of the most attractive themes in the recent developments in mathematical economics. The basic problem is to find out an optimal path of economic growth (or capital accumulation) in the sense that it maximizes certain economic welfare over time under some technological constraint. Being stimulated by the ingenious idea of Ramsey [12], a lot of economists, including P. A. Samuelson and T. C. Koopmans, have been working on this field and various mathematical theories of optimal control such as the Pontrjagin's maximum principle have been successfully introduced to economic analysis.

Recently, Chichilnisky [3] tried to prove rigorously the existence of an optimal path of economic growth relying upon an effective use of the weighted Sobolev space. And Takekuma [13] also gave another interesting version of the existence proof.

In Maruyama [10], the author also tried to add a further new insight to this existence problem, and established a sufficient condition for the existence of an optimal economic growth path in the case of *finite* planning time horizon. However it is quite apparent that we must encounter with various difficulties if we try to extend our analysis to the case of *infinite* time horizon. The purpose of this paper is to revise the earlier treatment of this problem in [11] and to show a way to overcome such difficulties. The key point is to transform the problem with an infinite measure space to the equivalent one with a finite measure space by means of the weighted Sobolev space. The author is indebted to Berkovitz [1] and Chichilnisky [3] for the important ideas embodied in the proof.

2. THE PROBLEM

Let us begin by specifying some notations and their economic interpretations. First the following items are assumed to be given.

$\mathbf{R}_+ = [0, \infty)$ planning time horizon.
u: $\mathbf{R}_+ \times \mathbf{R}^l_+ \to \mathbf{R}_+$ welfare function at each time.
f: $\mathbf{R}_+ \times \mathbf{R}^l_+ \to \mathbf{R}^l_+$ production function at each time.

$\delta > 0$ the discount rate of the welfare in the future.
$\lambda \in (0, 1)^l$ the vector of the depreciation rates of l capital goods.

Furthermore we have a couple of variable mappings to be optimized;

k: $\mathbf{R}_+ \to \mathbf{R}^l_+$ path of capital accumulation.
s: $\mathbf{R}_+ \to [0, 1]^l$ path of the vector whose components are saving rates of each goods.

For any vector $x \in \mathbf{R}^l$, we designate by M_x the diagonal matrix of the form

$$M_x = \begin{bmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_l \end{bmatrix}$$

where x_i $(1 \leqq i \leqq l)$ is the i-th coordinate of x.

Then the problem of optimal economic growth can be formulated as follows:

Maximize

$$J(k, s) = \int_0^{\infty} u[t, (I - M_{s(t)})f(t, k(t))]e^{-\delta t}dt \tag{1}$$

(P) *subject to*

$$\dot{k}(t) = M_{s(t)}f(t, k(t)) - M_\lambda k(t) \tag{2}$$

$$k(0) = \bar{l} \quad \text{(given vector)} \tag{3}$$

(I is the identity matrix.)

Define w: $\mathbf{R}_+ \times \mathbf{R}^l_+ \times [0, 1]^l \to \mathbf{R}_+$ and g: $\mathbf{R}_+ \times \mathbf{R}^l_+ \times [0, 1]^l \to \mathbf{R}^l$ by

$$w(t, k, s) = u[t, (I - M_s)f(t, k)]$$

and

$$g(t, k, s) = M_s f(t, k) - M_\lambda k$$

respectively. Furthermore let ν be a finite measure on $\mathbf{R}_+$ defined by

$$\nu(E) = \int_E e^{-\delta t}dt$$

for every Lebesgue-measurable set E in $\mathbf{R}_+$. Then the problem (P) can be rewritten in the form:

(P′)

Maximize

$$J(k,s)=\int_0^\infty w(t,k(t),s(t))d\nu \tag{1'}$$

subject to

$$\dot{k}(t)=g(t,k(t),s(t)) \tag{2'}$$

$$k(0)=\bar{k} \tag{3'}$$

Throughout this paper, we shall assume the following conditions to be satisfied.

Assumption 1. u is measurable on $\mathbf{R}_+\times\mathbf{R}^l_+$. Furthermore u is upper semi-continuous and concave in the last l-dimensional vector.

Assumption 2. f is continuous on $\mathbf{R}_+\times\mathbf{R}^l_+$.

Assumption 3. There exists $C>0$ such that

$$k_i\geqq C \quad \text{implies} \quad \sup_{t\in\mathbf{R}_+} f_i(t,k)\leqq\lambda_i k_i$$

for any $i\ (=1,2,\cdots,l)$, where k_i (resp. f_i) is the i-th coordinate of k (resp. f).

Assumption 4. There exists a couple of positive constants, α and β, such that

$$0<\beta<\delta/2 \qquad \text{and}$$

$$\|f(t,k)\|<\alpha\|k\|e^{\beta t} \quad \text{for all } t\in\mathbf{R}_+ \text{ and } k\in\mathbf{R}^l_+ .$$

Assumption 5. There exist a non-negative ν-integrable function θ: $\mathbf{R}_+\to\mathbf{R}$ and a vector $b\in\mathbf{R}^l$ such that

$$w(t,k,s)-\langle b,g(t,k,s)\rangle\leqq\theta(t) \qquad \text{for every} \quad (t,k,s) .$$

3. BOUNDEDNESS OF ADMISSIBLE PATHS

We denote by S the set of all the measurable mappings s: $\mathbf{R}_+\to[0,1]^l$, and we also denote by $W_\delta^{1,2}$ the weighted Sobolev space on $\mathbf{R}_+$ with the weight function $e^{-\delta t}$. (Cf. Kufner [7] or Kufner et al. [8] pp. 417–423.)

Definition. A pair $(k,s)\in W_\delta^{1,2}\times S$ is said to be an *admissible pair* if it satisfies (2) and (3). And $k\in W_\delta^{1,2}$ is called an *admissible path* if there exists an $s\in S$ such that (k,s) is an admissible pair. The set of all the admissible pairs is denoted by A, and the set of all the admissible paths by A_k.

Thanks to Assumption 3, the following lemma can be proved in the same manner as in Proposition 1 of Maruyama [10].

LEMMA 1. $$\sup_{k\in A_k}\|k\|_{\infty,\nu}<lC .$$

Proof. Since

$$\dot{k}(t)+M_\lambda k(t)=M_{s(t)}f(t,k(t))\geqq 0\,,$$

we must have

$$\dot{k}_i(t)\geqq -\lambda_i k_i(t) \qquad \text{for all} \quad i\,.$$

Hence

$$k_i(t)\geqq \bar{k}_i e^{-\lambda_i t}\geqq 0 \qquad \text{for all} \quad i\,. \tag{4}$$

On the other hand, since

$$f_i(t,k(t))\leqq \lambda_i k_i(t) \qquad \text{if} \quad k_i(t)\geqq C$$

by Assumption 3, we must have

$$\dot{k}_i(t)=s_i(t)f_i(t,k(t))-\lambda_i k_i(t)\leqq 0 \qquad \text{if} \quad k_i(t)\geqq C\,.$$

Consequently

$$k_i(t)\leqq C \qquad \text{for all} \quad i\,. \tag{5}$$

By (4) and (5),

$$0\leqq k_i(t)\leqq C \qquad \text{for all} \quad t \quad \text{and} \quad i\,.$$

Hence

$$\sup_{k\in A_k}\|k\|_{\infty,\nu}<lC\,. \qquad \text{Q.E.D.}$$

Furthermore, taking account of Assumption 4, we have

$$\begin{aligned}\|\dot{k}(t)\| &\leqq \|M_{s(t)}f(t,k(t))\|+\|M_\lambda k(t)\|\\ &\leqq \alpha\|k(t)\|e^{\beta t}+lC\\ &\leqq lC(\alpha e^{\beta t}+1)\,.\end{aligned} \tag{6}$$

The right-hand side is ν-integrable. Hence

$$\sup_{k\in A_k}\|\dot{k}\|_{1,\nu}\leqq N_1 \qquad \text{for some} \quad 0<N_1<\infty\,. \tag{7}$$

Similarly

$$\sup_{k\in A_k}\|\dot{k}\|_{2,\nu}\leqq N_2 \qquad \text{for some} \quad 0<N_2<\infty\,. \tag{8}$$

The following proposition can easily be derived from Lemma 1 and (8).

PROPOSITION 1. *A_k is bounded in $W_\delta^{1,2}$.*

COROLLARY 1. *A_k is weakly sequentially compact in $W_\delta^{1,2}$.*

Proof. Since $W_\delta^{1,2}$ is a Hilbert space, the boundedness of A_k implies that it is weakly sequentially compact. Q.E.D.

4. EXISTENCE THEOREM

PROPOSITION 2. $\gamma \equiv \sup_{(k,s)\in A} J(k,s)$ *is finite.*

Proof. Let $\{(k_n, s_n)\}$ be a sequence in A such that

$$\lim_n J(k_n, s_n) = \gamma .$$

By Assumption 5,

$$\begin{aligned} w(t, k_n(t), s_n(t)) &\leqq \theta(t) + \langle b, g(t, k_n(t), s_n(t))\rangle \\ &= \theta(t) + \langle b, \dot{k}_n(t)\rangle . \end{aligned}$$

Taking account of (7), we obtain

$$\begin{aligned} \int_0^\infty w(t, k_n(t), s_n(t))dv &\leqq \int_0^\infty \theta(t)dv + \int_0^\infty \langle b, \dot{k}_n(t)\rangle dv \\ &\leqq \int_0^\infty \theta(t)dv + N_1\|b\| \\ &< \infty \qquad \text{for all} \quad n . \end{aligned} \tag{9}$$

Thus we can conclude that γ must be finite. Q.E.D.

Let us define the correspondence Ω: $\mathbf{R}_+ \times \mathbf{R}^l_+ \twoheadrightarrow \mathbf{R}^l \times \mathbf{R}_+$ by

$$\begin{aligned} \Omega(t,k) = \{(\xi, \eta) \in \mathbf{R}^l \times \mathbf{R}_+ \mid \xi = g(t,k,s) \text{ and} \\ 0 \leqq \eta \leqq w(t,k,s) \text{ for some } s \in [0,1]^l\} . \end{aligned} \tag{10}$$

Thanks to our Assumption 1 and Assumption 2, it is quite easy to prove that Ω is a compact-convex-valued upper hemi-continuous (u.h.c.) correspondence. Therefore the correspondence

$$k \twoheadrightarrow \Omega(\tilde{t}, k) = \overline{\text{co}}\, \Omega(\tilde{t}, k) \tag{11}$$

is also a compact-convex-valued u.h.c. correspondence for each fixed $\tilde{t} \in \mathbf{R}_+$. If we denote

$$K(\tilde{t}; \tilde{k}, \varepsilon) = \{(\tilde{t}, k) \in \mathbf{R}_+ \times \mathbf{R}^l_+ \mid \|k - \tilde{k}\| < \varepsilon\}$$

$((\tilde{t}, \tilde{k}) \in \mathbf{R}_+ \times \mathbf{R}^l_+)$, then we obtain the following result as a consequence of the u.h.c. of the correspondence (11).

PROPOSITION 3. *For each* $(\tilde{t}, \tilde{k}) \in \mathbf{R}_+ \times \mathbf{R}^l_+$,

$$\Omega(\tilde{t}, \tilde{k}) = \bigcap_{\varepsilon > 0} \overline{\text{co}}\, \Omega(K(\tilde{t}; \tilde{k}, \varepsilon)) .$$

Thus we have just finished up the preparation for the next crucial proposition. Let $\{(k_n, s_n)\}$ be a sequence in A such that

$$\lim_n J(k_n, s_n) = \gamma \,. \tag{12}$$

Then, by Corollary 1, there exists a weakly convergent subsequence (no change in notations) of $\{k_n\}$; i.e.

$$k_n \longrightarrow k^* \qquad \text{weakly in} \quad W_\delta^{1,2} \,. \tag{13}$$

PROPOSITION 4. *There exists an ν-integrable function ζ: $\mathbf{R}_+ \to \mathbf{R}$ such that*

$$\int_0^\infty \zeta(t) d\nu \geqq \gamma \tag{14}$$

and

$$(\dot{k}^*(t), \zeta(t)) \in \Omega(t, k^*(t)) \quad a.e. \tag{15}$$

Proof. (13) implies that $k_n \longrightarrow k^*$ strongly in $L^2(\nu)$. Hence we can assume, without loss of generality, that

$$k_n(t) \longrightarrow k^*(t) \qquad \text{a.e.} \quad (\nu) \,. \tag{16}$$

On the other hand, (13) implies that

$$\dot{k}_n \longrightarrow \dot{k}^* \qquad \text{weakly in} \quad L^2(\nu) \,. \tag{17}$$

Therefore, by the well-known Mazur's Theorem (cf. Dunford–Schwartz [4] p. 422 or Maruyama [9] Corollary 5.3), we can find out, for each $j \in \mathbf{N}$, some finite elements

$$k_{n_j+1}, k_{n_j+2}, \cdots, k_{n_j+m(j)}$$

in $\{k_n\}$ and

$$\alpha_{ij} \geqq 0 \,, \qquad 1 \leqq i \leqq m(j)$$

$$\sum_{i=1}^{m(j)} \alpha_{ij} = 1$$

such that

$$\left\| \dot{k}^* - \sum_{i=1}^{m(j)} \alpha_{ij} \dot{k}_{n_j+i} \right\|_{2,\nu} \leqq \frac{1}{j} \,,$$

$$n_{j+1} > n_j + m(j) \,. \tag{18}$$

We denote

$$\begin{aligned} \psi_j(t) &= \sum_{i=1}^{m(j)} \alpha_{ij} \dot{k}_{n_j+i}(t) \\ &= \sum_{i=1}^{m(j)} \alpha_{ij} g(t, k_{n_j+i}(t), s_{n_j+i}(t)) \,. \end{aligned} \tag{19}$$

By (18), we can assume, without loss of generality, that

$$\psi_j(t) \longrightarrow \dot{k}^*(t) \qquad \text{a.e.} \quad (\nu)\,. \tag{20}$$

Define a sequence of functions $\{\zeta_j: \mathbf{R}_+ \to \mathbf{R}\}$ by

$$\zeta_j(t) = \sum_{i=1}^{m(j)} \alpha_{ij} w(t, k_{n_j+i}(t), s_{n_j+i}(t))\,. \tag{21}$$

And if we define

$$\zeta(t) = \limsup \zeta_j(t)\,,$$

then $\zeta(t)$ is bounded as proved in Proposition 2.

Since

$$w(t, k_{n_j+i}(t), s_{n_j+i}(t)) - \langle b, g(t, k_{n_j+i}(t), s_{n_j+i}(t))\rangle \leqq \theta(t) \qquad \text{for all} \quad t$$

by Assumption 5, we get

$$\begin{aligned}\zeta_j(t) - \langle b, \psi_j(t)\rangle &= \sum_{i=1}^{m(j)} \alpha_{ij}\{w(t, k_{n_j+i}(t), s_{n_j+i}(t)) - \langle b, g(t, k_{n_j+i}(t), s_{n_j+i}(t))\rangle\} \\ &\leqq \sum_{i=1}^{m(j)} \alpha_{ij}\theta(t) = \theta(t) \qquad \text{for all} \quad t\,. \end{aligned}\tag{22}$$

Hence

$$\begin{aligned}\zeta_j(t) &\leqq \theta(t) + \langle b, \psi_j(t)\rangle \\ &\leqq \theta(t) + \|b\| lC(\alpha e^{\beta t} + 1)\end{aligned}\tag{23}$$

by (6). The right-hand side of (23) is ν-integrable. Thus applying the Fatou's lemma,

$$\int_0^\infty \zeta(t) d\nu \geqq \limsup \int_0^\infty \zeta_j(t) d\nu\,. \tag{24}$$

By a simple calculation,

$$\begin{aligned}\limsup \int_0^\infty \zeta_j(t) d\nu &= \limsup \sum \alpha_{ij} \int_0^\infty w(t, k_{n_j+i}(t), s_{n_j+i}(t)) d\nu \\ &= \limsup \sum \alpha_{ij} J(k_{n_j+i}, s_{n_j+i}) \\ &= \gamma\,.\end{aligned}\tag{25}$$

Combining (25) with (24), we get (14).

It remains to show (15). For each fixed $t \in \mathbf{R}_+$, we can assume that

$$\zeta_j(t) \longrightarrow \zeta(t)\,. \tag{26}$$

Taking account of (16), we can find out some $n_0 \in \mathbf{N}$, for each $\varepsilon > 0$, such that

$$\|k_n(t)-k^*(t)\|<\varepsilon \qquad \text{for all} \quad n \geqq n_0 . \tag{27}$$

Therefore

$$(t, k_n(t)) \in K(t; k^*(t), \varepsilon) \qquad \text{for all} \quad n \geqq n_0 . \tag{28}$$

Consequently we have, for sufficiently large j,

$$(g(t, k_{n_j+i}(t), s_{n_j+i}(t)), w(t, k_{n_j+i}(t), s_{n_j+i}(t))) \in \Omega(K(t; k^*(t), \varepsilon)) \tag{29}$$

which implies

$$(\psi_j(t), \zeta_j(t)) \in \text{co}\, \Omega(K(t; k^*(t), \varepsilon)) . \tag{30}$$

Furthermore by (20) and (26),

$$(\dot{k}^*(t), \zeta(t)) \in \overline{\text{co}}\, \Omega(K(t; k^*(t), \varepsilon)) . \tag{31}$$

Since (31) holds for arbitrary $\varepsilon>0$,

$$(\dot{k}^*(t), \zeta(t)) \in \bigcap_{\varepsilon>0} \overline{\text{co}}\, \Omega(K(t; k^*(t), \varepsilon)) = \Omega(t, k^*(t)) . \tag{32}$$

The last equality comes from Proposition 3. This completes the proof. Q.E.D.

By Proposition 4, it has been verified that the value

$$\int_0^\infty \zeta(t) dv$$

can be attained under the path $k^*(t)$ if $s^*(t) \in [0, 1]^l$ is suitably chosen at each $t \in \mathbf{R}_+$. Finally we shall prove that $s^*(t)$ can be chosen so as to be measurable. Although this point is almost obvious in our simple case, it may be suggestive, for the sake of the further sophistication of the problem, to provide a proof based on the Filippov's implicit function theorem (cf. Castaing–Valadier [2] Theorem III. 38 or Maruyama [9] Theorem 6.23).

Define the mapping $\rho: \mathbf{R}_+ \times [0, 1]^l \to \mathbf{R}^l \times \mathbf{R}_+$ and the correspondence $\Phi: \mathbf{R}_+ \twoheadrightarrow \mathbf{R}^l \times \mathbf{R}_+$ by

$$\rho(t, s) = (g(t, k^*(t), s), w(t, k^*(t), s))$$

$$\Phi(t) = \rho(t, [0, 1]^l) .$$

Furthermore if we define the correspondence Γ: $\mathbf{R}_+ \twoheadrightarrow \mathbf{R}^l \times \mathbf{R}_+$ by

$$\Gamma(t) = \{(\dot{k}^*(t), \eta) \in \mathbf{R}^l \times \mathbf{R}_+ \mid \zeta(t) \leqq \eta\} ,$$

then by Proposition 4, we must have

$$\Gamma(t) \cap \Phi(t) \neq \varnothing .$$

Since every condition required for the Filippov's theorem is trivially satisfied, there exists a measurable mapping s^*: $\mathbf{R}_+ \to [0, 1]^l$ such that

$$\rho(t, s^*(t)) = (g(t, k^*(t), s^*(t)), w(t, k^*(t), s^*(t))) \in \Gamma(t)$$

i.e.

$$\dot{k}^*(t) = g(t, k^*(t), s^*(t))$$

$$\zeta(t) \leqq w(t, k^*(t), s^*(t)) .$$

Therefore

$$\int_0^\infty \zeta(t)dt = \gamma ,$$

and we can conclude that the pair $(k^*(t), s^*(t))$ is optimal.

THEOREM 1. *Under Assumptions* 1–5, *the problem* (P) *has an optimal solution.*

Keio University
and
University of California,
Berkeley

REFERENCES

[1] Berkovitz, L. D., Existence and lower closure theorems for abstract control problems, *SIAM Journal of Control and Optimization*, **12**, 1974, 27–42.

[2] Castaing, C. and M. Valadier, *Convex analysis and measurable multifunctions* (Springer, Berlin), 1977.

[3] Chichilnisky, G., Nonlinear functional analysis and optimal economic growth, *Journal of Mathematical Analysis and Applications*, **61**, 1977, 504–520.

[4] Dunford, N. and J. T. Schwartz, *Linear operators*, Part 1 (Interscience, New York), 1958.

[5] Ioffe, A. D., On lower semicontinuity of integral functionals I, *SIAM Journal of Control and Optimization*, **15**, 1977, 521–538.

[6] Ioffe, A. D., On lower semicontinuity of integral functionals II, *SIAM Journal of Control and Optimization*, **15**, 1977, 991–1000.

[7] Kufner, A., *Weighted Sobolev spaces* (Teubner, Leipzig), 1980.

[8] Kufner, A., O. John and S. Fůcík, *Function spaces* (Nordhoff, Leyden), 1977.

[9] Maruyama, T., *Functional analysis* (Keio Tsushin, Tokyo), (in Japanese), 1980.

[10] Maruyama, T., A variational problem relating to the theory of optimal economic growth, *Proceedings of the Japan Academy*, **57A**, 1981, 381–386.

[11] Maruyama, T., Optimal economic growth with infinite planning time horizon, *Proceeding of the Japan Academy*, **57A**, 1981, 469–472.

[12] Ramsey, F. P., A Mathematical theory of saving, *Economic Journal*, **38**, 1928, 543–559.

[13] Takekuma, S., A sensitivity analysis on optimal economic growth, *Journal of Mathematical Economics*, **7**, 1980, 193–208.