# ON HICKS' COMPOSITE COMMODITY THEOREM

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Hicks (1946) succeeded in simplifying his exposition of consumer theory by employing the so-called composite commodity theorem which states that any group of commodities whose relative prices remain unchanged can be treated as a single commodity. He proved this theorem by showing that the substitution term in the Slutsky equation for a group of commodities with respect to its own price is negative (Hicks (1946, p. 312)). Samuelson (1947, p. 143) proved the same theorem by establishing the negative semi-definiteness of the substitution matrix involving a group of commodities. These properties of consumer demand functions are due to the convexity of preference relations. So it will be useful to have the same theorem in the following form: the preference relation in the new commodity space involving a group of commodities inherits all the relevant properties of the prerference relation in the original commodity space. What Hicks had in mind would be this. In fact, he writes: [So] long as the terms on which money can be converted into other commodities are given, there is no reason why we should not draw up a determinate indifference system between any commodity $X$ and money (that is to say, purchasing power in general) (Hicks (1946, p. 33). In the present paper, we shall present a proof of the composite commodity theorem in this form, which may be regarded as a geometric version of the Hicksian theorem. The Hicksian or Samuelsonian version follows from the geometric version through the usual consumer theory.

## 1. NOTATION AND DEFINITIONS

In consumer theory, we are interested in the properties of a preference relation listed in the following definition.

DEFINITION. Let $T$ be a reflexive total transitive binary relation on a non-empty subset $S$ of a finite-dimensional Euclidean space $E$. Then $T$ is said to be:

*upper semi-continuous* if for every $x \in S$ the set $\{y \in S: (y, x) \in T\}$ is closed in $S$,

*lower semi-continuous* if for every $x \in S$ the set $\{y \in S: (x, y) \in T\}$ is closed in $S$,

*continuous* if $T$ is upper semi-continuous and lower semi-continuous,

*non-satiated* if for every $x \in S$ there is $y \in S$ such that $(x, y) \notin T$,

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*locally non-satiated* if for every $(x, \varepsilon) \in S \times R_{++}$ there is $y \in S$ such that $(x, y) \notin T$ & $\|y - x\| < \varepsilon$,
*weakly monotone* if $(x, y) \in T$ for every $(x, y) \in E \times S$ such that $x \geqq y$,
*monotone* if $(y, x) \in S^2 - T$ for every $(x, y) \in E \times S$ such that $x > y$,
*strongly monotone* if $(y, x) \in S^2 - T$ for every $(x, y) \in E \times S$ such that $x \geqq y$ & $x \neq y$,
*weakly convex* if $((1-t)x + ty, x) \in T$ for every $t \in ]0, 1[$ and every $(y, x) \in T$,
*convex* if $(x, (1-t)x + ty) \in S^2 - T$ for every $t \in ]0, 1[$ and every $(x, y) \in S^2 - T$,
*strongly convex* if $(x, (1-t)x + ty) \in S^2 - T$ for every $t \in ]0, 1[$ and every $(x, y) \in T$ such that $(y, x) \in T$ & $x \neq y$.

Let $H$ be a non-empty finite set, representing the set of commodities. The consumption set $X$ is a non-empty closed subset of $R^H$ that is bounded from below. The preference relation $Q$ is a reflexive total transitive binary relation on $X$. Let $I$ be a non-empty proper subset of $H$ and define $J = H - I$. The prices of the commodities in $I$ will be kept constant so that the commodities in $I$ will form a composite commodity. Given a price vector $p$ in $R^I_{++}$, the set of possible pairs of consumptions of the commodities in $J$ and expenditures on the commodities in $I$ is defined by

$$X(p) = \{(x, c) \in R^J \times R \colon (x, y) \in X \text{ for some } y \in R^I \text{ such that } p \cdot y = c\}\,.$$

For each $p \in R^I_{++}$ and each $(x, c) \in X(p)$, define

$$Y(p, x, c) = \{y \in R^I \colon (x, y) \in X \;\&\; p \cdot y = c\}\,.$$

Note that $X(p) = \{(x, c) \in R^J \times R \colon Y(p, x, c) \neq \varnothing\}$ for every $p \in R^I_{++}$.

LEMMA 1. *For every* $p \in R^I_{++}$,
(a) *$Y(p, x, c)$ is non-empty and compact for every $(x, c) \in X(p)$,*
(b) *$X(p)$ is a non-empty closed subset of $R^J \times R$ that is bounded from below,*
(c) *if $X$ is convex then $X(p)$ is convex.*

*Proof.* (a) Clearly, $Y(p, x, c)$ is non-empty and closed in $R^I$. It is bounded from below since $X$ is bounded from below, while it is bounded from above since $p \in R^I_{++}$. So $Y(p, x, c)$ is bounded and hence compact.

(b) Non-emptiness and boundedness from below are obvious. Let $\{(x^\nu, c^\nu)\}$ be a sequence in $X(p)$ converging to $(x^0, c^0) \in R^J \times R$. For every $\nu$ there is $y^\nu \in R^I$ such that $(x^\nu, y^\nu) \in X$ & $p \cdot y^\nu = c^\nu$. Let $d = c^0 + 1$. Without loss of generality, we may assume that $p \cdot y^\nu \leqq d$ for every $\nu$. Since $X$ is bounded from below, there is $a \in R$ such that $a \leqq x_i$ for every $x \in X$ and every $i \in H$. Let $Y = \{y \in R^I \colon p \cdot y \leqq d \;\&\; a \leqq y_i \text{ for every } i \in I\}$. Then $Y$ is compact and $y^\nu \in Y$ for every $\nu$, so that we may assume that $\{y^\nu\}$ converges to some $y^0 \in Y$. Hence $(x^0, y^0) \in X$ by the closedness of $X$, and $p \cdot y^0 = c^0$; therefore, $(x^0, c^0) \in X(p)$. Thus $X(p)$ is closed in $R^J \times R$.

(c) Let $(x^1, c^1), (x^2, c^2) \in X(p)$ and $t \in [0, 1]$. Then there are $y^1, y^2 \in R^I$ such that $(x^1, y^1) \in X$ & $(x^2, y^2) \in X$ & $p \cdot y^1 = c^1$ & $p \cdot y^2 = c^2$. Let $x = (1-t)x^1 + tx^2$ & $y =$

$(1-t)y^1+ty^2$ & $c=(1-t)c^1+tc^2$. Then $(x,y)\in X$ by the convexity of $X$. Clearly, $y\in R^I$ and $p\cdot y=c$ so that $(x,c)\in X(p)$. Hence $X(p)$ is convex.

The set $X(p)$ is interpreted as the consumption set in the new commodity space of lower dimension. The above lemma shows that the fundamental properties of the original consumption set $X$ are inherited by $X(p)$. For each $p\in R^I_{++}$, the preference relation on $X(p)$ is defined by

$$Q(p)=\{((x,c),(x',c'))\in X(p)\times X(p)\text{: There is } (y,y')\in Y(p,x,c) \times Y(p,x',c') \text{ such that (a) } ((x,y),(x',y'))\in Q,$$
$$\text{(b) } ((x,y),(x,z))\in Q \text{ for every } z\in Y(p,x,c), \text{ and}$$
$$\text{(c) } ((x',y'),(x',z))\in Q \text{ for every } z\in Y(p,x',c')\}\ .$$

In the next section, a sufficient condition will be given for $Q(p)$ to be a well-defined preference relation on $X(p)$. The following lemma shows that the same condition simplifies the definition of $Q(p)$.

LEMMA 2. *If $Q$ is upper semi-continuous then, for every $p\in R^I_{++}$, $Q(p)=\{((x,c),(x',c'))\in X(p)\times X(p)$: There is $y\in Y(p,x,c)$ such that $((x,y),(x',z))\in Q$ for every $z\in Y(p,x',c')\}$.*

*Proof.* Let $T$ denote the set on the right-hand side. Then clearly $Q(p)\subset T$. Let $((x,c),(x',c'))\in T$. Then there is $y''\in Y(p,x,c)$ such that $((x,y''),(x',z))\in Q$ for every $z\in Y(p,x',c')$. Since, by Lemma 1, $Y(p,x,c)$ and $Y(p,x',c')$ are non-empty and compact, it follows that there is $(y,y')\in Y(p,x,c)\times Y(p,x',c')$ such that (1) $((x,y),(x,z))\in Q$ for every $z\in Y(p,x,c)$ and (2) $((x',y'),(x',z))\in Q$ for every $z\in Y(p,x',c')$. Hence $((x,y),(x,y''))\in Q$ & $((x,y''),(x',y'))\in Q$ so that, by transitivity, (3) $((x,y),(x',y'))\in Q$. It follows from (1), (2), and (3) that $((x,c),(x',c'))\in Q(p)$. Thus $T\subset Q(p)$ so that $Q(p)=T$.

## 2. INHERITANCE OF VARIOUS PROPERTIES

THEOREM 1 (Complete Preordering). *If $Q$ is upper semi-continuous then, for every $p\in R^I_{++}$, $Q(p)$ is a reflexive total transitive binary relation on $X(p)$.*

*Proof.* Let $(x,c),(x',c')\in X(p)$. By Lemma 1, $Y(p,x,c)$ and $Y(p,x',c')$ are non-empty and compact so that there is $(y,y')\in Y(p,x,c)\times Y(p,x',c')$ such that $((x,y),(x,z))\in Q$ for every $z\in Y(p,x,c)$ and $((x',y'),(x',z))\in Q$ for every $z\in Y(p,x',c')$. If $((x,y),(x',y'))\in Q$ then $((x,c),(x',c'))\in Q(p)$. Suppose $((x,y),(x',y'))\notin Q$. Since $Q$ is reflexive and total, it follows that $((x',y'),(x,y))\in Q$ so that $((x',c'),(x,c))\in Q(p)$. Hence $Q(p)$ *is reflexive and total.*

Suppose $((x,c),(x',c'))\in Q(p)$ & $((x',c'),(x'',c''))\in Q(p)$. Then there is $(y,y')\in Y(p,x,c)\times Y(p,x',c')$ such that $((x,y),(x',z))\in Q$ for every $z\in Y(p,x',c')$ and $((x',y'),(x'',z))\in Q$ for every $z\in Y(p,x'',c'')$. Hence $((x,y),(x',y'))\in Q$ so that

$((x,y),(x'',z))\in Q$ for every $z\in Y(p,x'',c'')$. Thus $((x,c),(x'',c''))\in Q(p)$, establishing the transitivity of $Q(p)$.

THEOREM 2 (Non-Satiation). *If $Q$ is upper semi-continuous and non-satiated then, for every $p\in R^I_{++}$, $Q(p)$ is non-satiated.*

*Proof.* Let $(x,c)\in X(p)$. Since $Y(p,x,c)$ is non-empty and compact, there is $y\in Y(p,x,c)$ such that $((x,y),(x,z))\in Q$ for every $z\in Y(p,x,c)$. Since $Q$ is non-satiated, there is $(x',y'')\in X$ such that $((x,y),(x',y''))\notin Q$. Let $c'=p\cdot y''$. Then $y''\in Y(p,x',c')$ and there is $y'\in Y(p,x',c')$ such that $((x',y'),(x',z))\in Q$ for every $z\in Y(p,x',c')$. Hence $((x',y'),(x',y''))\in Q$ so that $((x,y),(x',y'))\notin Q$. Suppose $((x,c),(x',c'))\in Q(p)$. Then there is $y^*\in Y(p,x,c)$ such that $((x,y^*),(x',z))\in Q$ for every $z\in Y(p,x',c')$. Hence $((x,y^*),(x',y'))\in Q$ so that $((x,y),(x,y^*))\notin Q$. But, since $y^*\in Y(p,x,c)$, it follows that $((x,y),(x,y^*))\in Q$, which is a contradiction. Thus $((x,c),(x',c'))\notin Q(p)$.

THEOREM 3 (Local Non-Satiation). *If $Q$ is upper semi-continuous and locally non-satiated then, for every $p\in R^I_{++}$, $Q(p)$ is locally non-satiated.*

*Proof.* Let $(x,c)\in X(p)$ and $\varepsilon\in R_{++}$. Since $Y(p,x,c)$ is non-empty and compact, there is $y\in Y(p,x,c)$ such that $((x,y),(x,z))\in Q$ for every $z\in Y(p,x,c)$. Since $Q$ is locally non-satiated, there is $(x',y'')\in X$ such that $((x,y),(x',y''))\notin Q$ & $\|(x',y'')-(x,y)\| < \varepsilon/\max\{1,\sum_{i\in I}p_i\}$, where the maximum norm is used. Let $c'=p\cdot y''$. Then, as in the proof of Theorem 2, we can see that $((x,c),(x',c'))\notin Q(p)$. Since

$$|c'-c|=|p\cdot(y''-y)|=|\textstyle\sum_{i\in I}p_i(y_i''-y_i)|\leqq \sum_{i\in I}p_i|y_i''-y_i|$$
$$< \textstyle\sum_{i\in I}p_i\varepsilon/\max\{1,\sum_{i\in I}p_i\}\leqq\varepsilon\,,$$

it follows that

$$\|(x',c')-(x,c)\|=\max\{\max_{i\in J}|x_i'-x_i|,\ |c'-c|\}$$
$$< \max\{\varepsilon/\max\{1,\textstyle\sum_{i\in I}p_i\},\varepsilon\}=\varepsilon\,.$$

THEOREM 4 (Weak Monotonicity). *If $Q$ is upper semi-continuous and weakly monotone then, for every $p\in R^I_{++}$, $Q(p)$ is weakly monotone.*

*Proof.* Let $((x,c),(x',c'))\in(R^J\times R)\times X(p)$ & $(x,c)\geqq(x',c')$. Then there is $y'\in Y(p,x',c')$ such that $((x',y'),(x',z))\in Q$ for every $z\in Y(p,x',c')$. For each $i\in I$, let $y_i=y_i'+(c-c')/p_i\#I$. Then $p\cdot y=p\cdot y'+c-c'=c$ & $(x,y)\geqq(x',y')$ so that $y\in Y(p,x,c)$ & $((x,y),(x',y'))\in Q$ and hence $((x,y),(x',z))\in Q$ for every $z\in Y(p,x',c')$. Thus $((x,c),(x',c'))\in Q(p)$.

THEOREM 5 (Monotonicity). *If $Q$ is upper semi-continuous and monotone then, for every $p\in R^I_{++}$, $Q(p)$ is monotone.*

*Proof.* Let $((x,c),(x',c'))\in(R^J\times R)\times X(p)$ & $(x,c)>(x',c')$. Then there is

$y' \in Y(p, x', c')$ such that $((x', y'), (x', z)) \in Q$ for every $z \in Y(p, x', c')$. For each $i \in I$, let $y_i = y_i' + (c - c')/p_i \# I$. Then $(x, y) > (x', y')$ & $y \in Y(p, x, c)$ & $((x', y'), (x, y)) \notin Q$. Suppose $((x', c'), (x, c)) \in Q(p)$. Then there is $y'' \in Y(p, x', c')$ such that $((x', y''), (x, z)) \in Q$ for every $z \in Y(p, x, c)$. Hence $((x', y'), (x', y'')) \in Q$ & $((x', y''), (x, y)) \in Q$ so that $((x', y'), (x, y)) \in Q$, which is a contradiction. Thus $((x', c'), (x, c)) \notin Q(p)$.

THEOREM 6 (Strong Monotonicity). *If $Q$ is upper semi-continuous and strongly monotone then, for every $p \in R^I_{++}$, $Q(p)$ is strongly monotone.*

*Proof.* Almost the same as the proof of Theorem 5.

THEOREM 7 (Weak Convexity). *If $Q$ is upper semi-continuous and weakly convex then, for every $p \in R^I_{++}$, $Q(p)$ is weakly convex.*

*Proof.* Let $t \in ]0, 1[$ & $((x', c'), (x, c)) \in Q(p)$ & $(x^*, c^*) = (1-t)(x, c) + t(x', c')$. Then there is $y' \in Y(p, x', c')$ such that $((x', y'), (x, z)) \in Q$ for every $z \in Y(p, x, c)$. Since $Y(p, x, c)$ and $Y(p, x^*, c^*)$ are non-empty and compact, there is $(y, y^*) \in Y(p, x, c) \times Y(p, x^*, c^*)$ such that $((x, y), (x, z)) \in Q$ for every $z \in Y(p, x, c)$ and $((x^*, y^*), (x^*, z)) \in Q$ for every $z \in Y(p, x^*, c^*)$. Let $z^* = (1-t)y + ty'$. Since $((x', y'), (x, y)) \in Q$, the weak convexity of $Q$ implies that $((x^*, z^*), (x, y)) \in Q$ and hence $z^* \in Y(p, x^*, c^*)$. So $((x^*, y^*), (x^*, z^*)) \in Q$, which implies that $((x^*, y^*), (x, y)) \in Q$. Thus $((x^*, y^*), (x, z)) \in Q$ for every $z \in Y(p, x, c)$. Therefore, $((x^*, c^*), (x, c)) \in Q(p)$.

THEOREM 8 (Convexity). *If $Q$ is upper semi-continuous and convex then, for every $p \in R^I_{++}$, $Q(p)$ is convex.*

*Proof.* Let $t \in ]0, 1[$ & $((x, c), (x', c')) \in (X(p))^2 - Q(p)$ & $(x^*, c^*) = (1-t)(x, c) + t(x', c')$. Then $((x', c'), (x, c)) \in Q(p)$ so that there is $y'' \in Y(p, x', c')$ such that $((x', y''), (x, z)) \in Q$ for every $z \in Y(p, x, c)$. Since $Y(p, x', c')$ is non-empty and compact, there is $y' \in Y(p, x', c')$ such that $((x', y'), (x', z)) \in Q$ for every $z \in Y(p, x', c')$. Hence $((x', y'), (x, z)) \in Q$ for every $z \in Y(p, x, c)$. Also $Y(p, x, c)$ and $Y(p, x^*, c^*)$ are non-empty and compact so that there is $(y, y^*) \in Y(p, x, c) \times Y(p, x^*, c^*)$ such that $((x, y), (x, z)) \in Q$ for every $z \in Y(p, x, c)$ and $((x^*, y^*), (x^*, z)) \in Q$ for every $z \in Y(p, x^*, c^*)$. Let $z^* = (1-t)y + ty'$. If $((x, y), (x', y')) \in Q$ then $((x, c), (x', c')) \in Q(p)$, which is a contradiction. Hence $((x, y), (x', y')) \notin Q$ so that, by the convexity of $Q$, $((x, y), (x^*, z^*)) \in X^2 - Q$ and hence $z^* \in Y(p, x^*, c^*)$. Therefore $((x^*, y^*), (x^*, z^*)) \in Q$ so that $((x, y), (x^*, y^*)) \notin Q$. Suppose $((x, c), (x^*, c^*)) \in Q(p)$. Then there is $z' \in Y(p, x, c)$ such that $((x, z'), (x^*, z)) \in Q$ for every $z \in Y(p, x^*, c^*)$. Hence $((x, y), (x, z')) \in Q$ & $((x, z'), (x^*, y^*)) \in Q$ so that $((x, y), (x^*, y^*)) \in Q$, a contradiction. Thus $((x, c), (x^*, c^*)) \notin Q(p)$.

THEOREM 9 (Strong Convexity). *If $Q$ is upper semi-continuous and strongly convex then, for every $p \in R^I_{++}$, $Q(p)$ is strongly convex.*

*Proof.* Let $t \in ]0,1[$ & $((x,c),(x',c')) \in Q(p)$ & $((x',c'),(x,c)) \in Q(p)$ & $(x,c) \neq (x',c')$ & $(x^*,c^*) = (1-t)(x,c) + t(x',c')$. Then there is $(y,y') \in Y(p,x,c) \times Y(p,x',c')$ such that $((x,y),(x',z)) \in Q$ for every $z \in Y(p,x',c')$ and $((x',y'),(x,z)) \in Q$ for every $z \in Y(p,x,c)$. Since $Y(p,x^*,c^*)$ is non-empty and compact, there is $y^* \in Y(p,x^*,c^*)$ such that $((x^*,y^*),(x^*,z)) \in Q$ for every $z \in Y(p,x^*,c^*)$. Let $z^* = (1-t)y + ty'$. Since $(x,c) \neq (x',c')$, it follows that either $x \neq x'$ or $c \neq c'$. If $c \neq c'$ then $y \neq y'$. Hence $(x,y) \neq (x',y')$. Since $((x,y),(x',y')) \in Q$ & $((x',y'),(x,y)) \in Q$, the strong convexity of $Q$ implies that $((x,y),(x^*,z^*)) \in X^2 - Q$ and hence $z^* \in Y(p,x^*,c^*)$. Therefore $((x^*,y^*),(x^*,z^*)) \in Q$ so that $((x,y),(x^*,y^*)) \notin Q$. Then, as in the proof of Theorem 8, we can see that $((x,c),(x^*,c^*)) \notin Q(p)$.

THEOREM 10 (Upper Semi-Continuity). *If $Q$ is upper semi-continuous then, for every $p \in R^I_{++}$, $Q(p)$ is upper semi-continuous.*

*Proof.* Let $(x,c) \in X(p)$. Let $\{(x^\nu,c^\nu)\}$ be a sequence in the set $\{(x',c') \in X(p)$: $((x',c'),(x,c)) \in Q(p)\}$ converging to some $(x^0,c^0) \in X(p)$. By the definition of $Q(p)$, for every $\nu$ there is $y^\nu \in Y(p,x^\nu,c^\nu)$ such that $((x^\nu,y^\nu),(x,z)) \in Q$ for every $z \in Y(p,x,c)$. Since $\{c^\nu\}$ converges to $c^0$, we may assume without loss of generality that, for every $\nu$, $y^\nu$ belongs to the set $\{y' \in R^I_+ : p \cdot y \leqq c^0 + 1\}$, which is compact. Hence, without loss of generality, we may assume that $\{y^\nu\}$ converges to some $y^0$ in the set. For every $\nu$, $p \cdot y^\nu = c^\nu$ & $(x^\nu,y^\nu) \in X$; therefore, $p \cdot y^0 = c^0$ and, by the closedness of $X$, $(x^0,y^0) \in X$ so that $y^0 \in Y(p,x^0,c^0)$. Let $z \in Y(p,x,c)$. Then $((x^\nu,y^\nu),(x,z)) \in Q$ for every $\nu$ so that, by the upper semi-continuity of $Q$, $((x^0,y^0),(x,z)) \in Q$. That is, $((x^0,y^0),(x,z)) \in Q$ for every $z \in Y(p,x,c)$. Since $Y(p,x^0,c^0)$ is non-empty and compact, there is $y^* \in Y(p,x^0,c^0)$ such that $((x^0,y^*),(x^0,z)) \in Q$ for every $z \in Y(p,x^0,c^0)$. In particular, $((x^0,y^*),(x^0,y^0)) \in Q$. Hence $((x^0,y^*),(x,z)) \in Q$ for every $z \in Y(p,x,c)$. Thus $((x^0,c^0),(x,c)) \in Q(p)$ so that $Q(p)$ is upper semi-continuous.

THEOREM 11 (Lower Semi-Continuity). *Suppose $X = R^H_+$. If $Q$ is continuous then, for every $p \in R^I_{++}$, $Q(p)$ is lower semi-continuous.*

*Proof.* Note that $X(p) = R^J_+ \times R_+$. For each $c \in R_+$, let $K(c) = \{y \in R^I_+ : p \cdot y = c\}$. Then $K(c) = Y(p,x,c)$ for every $(x,c) \in X(p)$. Let $(x,c) \in X(p)$. Let $\{(x^\nu,c^\nu)\}$ be a sequence in the set $\{(x',c') \in X(p)$: $((x,c),(x',c')) \in Q(p)\}$ converging to some $(x^0,c^0) \in X(p)$. By the definition of $Q(p)$, for every $\nu$ there is $y^\nu \in K(c)$ such that $((x,y^\nu),(x^\nu,z)) \in Q$ for every $z \in K(c^\nu)$. Since $K(c)$ is non-empty and compact, there is $y \in K(c)$ such that $((x,y),(x,z)) \in Q$ for every $z \in K(c)$. Hence it suffices to show that $((x,y),(x^0,z)) \in Q$ for every $z \in K(c^0)$. Note that $((x,y),(x^\nu,z)) \in Q$ for every $z \in K(c^\nu)$. Since $K(c^0)$ is non-empty and compact, there is $y^0 \in K(c^0)$ such that $((x^0,y^0),(x^0,z)) \in Q$ for every $z \in K(c^0)$.

Case 1. $y^0 = 0$. Then $c^0 = 0$. Take any $i \in I$. For each $\nu$, let $z^\nu_i = c^\nu / p_i$ and $z^\nu_j = 0$ for every $j \in I - \{i\}$. Then $\{z^\nu\}$ converges to $0 = y^0$. Since $z^\nu \in K(c^\nu)$ for every $\nu$, it follows that $((x, y), (x^{\nu}, z^{\nu})) \in Q$ for every $\nu$. By the lower semi-continuity of $Q$, $((x, y), (x^0, y^0)) \in Q$ so that $((x, y), (x^0, z)) \in Q$ for every $z \in K(c^0)$.

Case 2. $y^0 \in R^I_+ - \{0\}$. Then $y^0_i > 0$ for some $i \in I$. For each $\nu$, let $z^{\nu}_i = y^0_i + (c^{\nu} - c^0)/p_i$ and $z^{\nu}_j = y^0_j$ for every $j \in I - \{i\}$. Then $\{z^{\nu}\}$ converges to $y^0$ and $p \cdot z^{\nu} = c^{\nu}$ for every $\nu$. There is a positive integer $\mu$ such that $z^{\nu} \geqq 0$ for every $\nu > \mu$. Hence $z^{\nu} \in K(c^{\nu})$ for every $\nu > \mu$, so that $((x, y), (x^{\nu}, z^{\nu})) \in Q$ for every $\nu > \mu$. By the lower semi-continuity of $Q$, $((x, y), (x^0, y^0)) \in Q$ so that $((x, y), (x^0, z)) \in Q$ for every $z \in K(c^0)$. This completes the proof of Theorem 11.

COROLLARY (Continuity). *Suppose $X = R^H_+$. If $Q$ is continuous then, for every $p \in R^I_{++}$, $Q(p)$ is continuous.*

*Proof.* Immediate from Theorems 10 and 11.

Under the assumptions of Theorem 11 or its Corollary, there is a continuous utility function for $Q(p)$. In that case, we may be interested in its differential properties. We shall discuss this problem in a sequel of the present paper.

## 3. A GRAPHICAL EXPOSITION

We have seen how various properties of the original preference relation $Q$ carry over to the induced preference relation $Q(p)$ on the new commodity space $X(p)$. We shall now visualize the situation graphically.

Assume that $\#H = 3$ and $\#I = 2$. Specifically, we shall write $H = \{1, 2, 3\}$ and $I = \{2, 3\}$. In Figure 1, the horizontal plane with hight $x$ and an indifference surface are depicted. Look at the intersection of the plane and the surface. To other indifference surfaces correspond other intersections. The projections on the $y_2$–$y_3$ plane of some of these intersections are depicted in Figure 2. In this figure, we draw a straight line orthogonal to $p$ whose distance from the origin is equal to $c$. These is a curve tangent

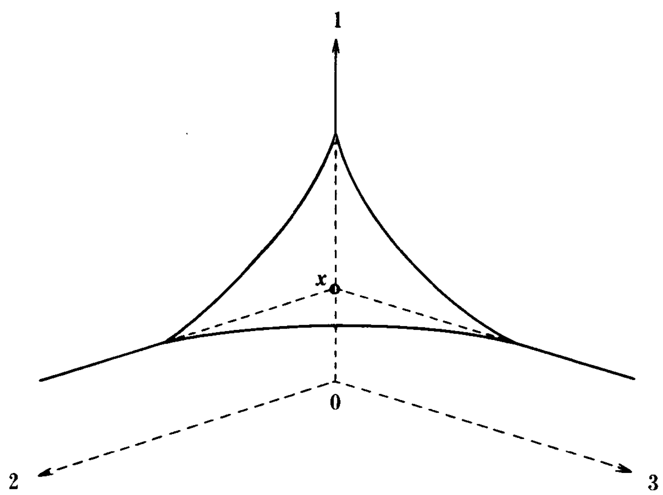


Fig. 1.

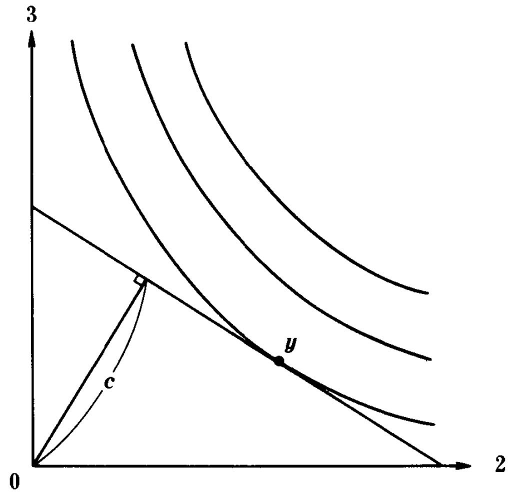


Fig. 2.

to this straight line; the tangent point is the point $y$ defined by condition (b) in the definition of $Q(p)$. We regard the indifference surface corresponding to the tangent curve as representing the satisfaction level of $(x, c)$. Next, consider another consumption level $x'$ of the first commodity. To determine the indifference curve passing through $(x, c)$, we have to find the quantity $c'$ of the composite commodity such that $(x', c')$ is indifferent to $(x, c)$. For this purpose, we look at the intersection of the indifference surface identified above and the horizontal plane with height $x'$. There is a straight line which is orthogonal to $p$ and is tangent to this curve. The distance of the straight line from the origin is the value of $c'$ we want.

There is a simple way to visualize the shapes of indifference curves in the $x$–$c$ plane. In Figure 1, consider the vertical plane whose intersection with the $y_2$–$y_3$ plane owns the origin and the point $p$. This is the $x$–$c$ plane on which indifference curves should be drawn. Take any indifference surface. The projection of the corresponding upper contour set on the plane forms the upper contour set in the $x$–$c$ plane. The boundary of the latter set is the indifference curve we want.[1] Clearly, this is convex, provided the original indifference surface is convex.

## 4. AN EXAMPLE

In this final section, we construct an example showing how important the upper semi-continuity of the original preference relation is for the composite commodity theorem. Let $X = R^3_+$. For each $(x, y, z) \in X$, define

$$u(x, y, z) = \begin{cases} x + 1/(1-z) & \text{if } z < 1, \\ x & \text{if } z \geqq 1. \end{cases}$$

Let $Q = \{((x, y, z), (x', y', z')) \in X^2 : u(x, y, z) \geqq u(x', y', z)\}$ & $p = (1, 1)$ & $(x, c) = (1, 1)$.

[1] See Katzner (1970, p. 145).

Then $Y(p,x,c)=\{(y,z)\in R_+^2: y+z=1\}$.

Suppose there is $(y,z)\in Y(p,x,c)$ such that $((x,y,z),(x,y',z'))\in Q$ for every $(y',z')\in Y(p,x,c)$. If $y>0$ then $z=1-y<1$ so that $u(x,y,z)=x+1/y<x+2/y=u(x,\tfrac{1}{2}y,1-\tfrac{1}{2}y)$; therefore, $(\tfrac{1}{2}y,1-\tfrac{1}{2}y)\in Y(p,x,c)$ & $((x,y,z),(x,\tfrac{1}{2}y,1-\tfrac{1}{2}y))\notin Q$, which is a contradiction. Hence $y=0$ so that $z=1$. Then $(\tfrac{1}{2},\tfrac{1}{2})\in Y(p,x,c)$ & $u(x,y,z)=x<x+2=u(x,\tfrac{1}{2},\tfrac{1}{2})$ so that $((x,y,z),(x,\tfrac{1}{2},\tfrac{1}{2}))\notin Q$, a contradiction. Thus there is no $(y,z)\in Y(p,x,c)$ such that $((x,y,z),(x,y',z'))\in Q$ for every $(y',z')\in Y(p,x,c)$. Therefore, $((x,c),(x,c))\notin Q(p)$ so that $Q(p)$ is not reflexive.

The conclusion of Theorem 1 fails for the relation $Q(p)$. Note that the set $\{(x,y,z)\in X: ((x,y,z),(1,\tfrac{1}{2},\tfrac{1}{2}))\in Q\}$ is not closed in $X$ so that $Q$ is not upper semi-continuous. It is easy to see that $Q$ is lower semi-continuous.

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