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GLOBAL STABILITY OF A GENERALIZED MORISHIMA SYSTEM

Koji OKUGUCHI

Consider an economy with $n+1$ goods where goods 0 is a numeraire. The excess demand function for the i-th goods is denoted by $E_i=E_i(P)$, where $P=(p_0, p_1, \cdots, p_n)$ with $p_0=1$ is a price vector. Let $E_{ij}=\partial E_i/\partial p_j$, and partition all goods other than the numaraire into two separate groups, R and S. The original Morishima system (Morishima (1952)) satisfies the following condition for the signs of partial derivatives of the excess demand functions.

$$\text{(M)} \qquad \begin{aligned} &E_{hj}>0\,, \quad h\neq j\,, \quad h, j\in R\,; \quad E_{hk}<0\,, \quad h\in R, k\in S\,, \\ &E_{ij}<0\,, \quad i\in S, \quad j\in R\,; \quad E_{ik}>0\,, \quad i\neq k, \quad i, k\in S\,. \end{aligned}$$

It has been proven by Morishima (1970) that the competitive equilibrium is globally stable under the condition (N) below together with (M) and homogeneity of degree zero of the excess demand functions.

$$\text{(N)} \qquad \begin{aligned} &E_{h0}+2\sum_{k\in S} E_{hk}p_k>0\,, \qquad h\in R \\ &E_{i0}+2\sum_{i\in R} E_{ij}p_j+E_i>0\,, \qquad i\in S\,. \end{aligned}$$

The condition (N) evidently is asymmetric in that the second inequality in (N) is not independent of the value of the excess demand function, which, however, becomes zero in equilibrium. Ichioka (1979) has formulated a locally generalized version of (N) to prove the local stability of the competitive equilibrium. To recover symmetry which is lacking in (N), Okuguchi (1979) has considered the following set of conditions consisting of (N_1) and (N_2) which is more restrictive than (N).

$$(\mathrm{N}_1) \qquad \begin{aligned} &E_{h0}+2\sum_{k\in S} E_{hk}p_k>0\,, \qquad h\in R \\ &E_{i0}+2\sum_{j\in R} E_{ij}p_j>0\,, \qquad i\in S \end{aligned}$$

$$(\mathrm{N}_2) \qquad \begin{aligned} &E_{0j}+2\sum_{i\in S} E_{ij}p_j+E_j>0\,, \qquad j\in R \\ &E_{0k}+2\sum_{h\in R} E_{hk}p_h+E_k>0\,, \qquad k\in S \end{aligned}$$

In this note we shall prove that the condition (N_1) alone together with (M) and homogeneity of the excess demand functions is sufficient for the global stability of the competitive equilibrium for a general tâtonnement process

$$(1) \qquad dp_i/dt = f_i(E_i(P)), \qquad i = 1, 2, \cdots, n.$$

where

$$(2) \qquad f_i(0) = 0, \quad f_i' > 0, \qquad i = 1, 2, \cdots, n.$$

In all contributions referred to above, a simple tâtonnement process where $f_i(E_i(P)) = E_i(P)$ has been assumed.

Our proof relies on the following mathematical theorems.

THEOREM 1 (Namatame and Tse (1981)). *If an $n \times n$ real matrix with negative diagonal elements $A = [a_{ij}]$ satisfies*

$$c_i |a_{ii}| > \sum_{j \neq i} c_j |a_{ij}|, \qquad i = 1, 2, \cdots, n$$

for some positive c_i's, then $B = [b_{ij}] = [a_{ij} + a_{ji}]$ also satisfies

$$d_i |b_{ii}| > \sum_{j \neq i} d_j |b_{ij}|, \qquad i = 1, 2, \cdots, n$$

for some positive constants d_i's, which implies that B is negative definite and A is quasi-negative definite.

THEOREM 2 (Namatame and Tse (1981)). *If the Jacobian matrix of a system of differential equations*

$$dx_i/dt = f_i(x), \qquad i = 1, 2, \cdots, n$$

is always quasi-negative definite, the stationary point is globally stable.

The proof of our assertion proceeds as follows: From homogeneity of the excess demand functions,

$$(3) \qquad \sum_{j \in R} E_{hj} p_j + \sum_{k \in S} E_{hk} p_k + E_{h0} = 0, \qquad h \in R$$

$$(4) \qquad \sum_{j \in R} E_{ij} p_j + \sum_{k \in S} E_{ik} p_k + E_{i0} = 0, \qquad i \in S.$$

Taking into consideration of (N_1), we have from (3) and (4)

$$(5) \qquad \sum_{j \in R} E_{hj} p_j - \sum_{k \in S} E_{hk} p_k < 0, \qquad h \in R$$

$$(6) \qquad -\sum_{j \in R} E_{ij} p_j + \sum_{k \in S} E_{ik} p_k < 0, \qquad i \in S.$$

From (1),

$$(7) \quad \partial f_i/\partial p_i = f_i' \partial E_i/\partial p_i < 0, \quad \partial f_i/\partial p_j = f_i' \partial E_i/\partial p_j, \quad i \neq j, \qquad i, j = 1, 2, \cdots, n.$$

Hence (5) and (6) show that the Jacobian matrix of (1) has a dominant negative diagonal and is quasi-negative definite. Our assertion now follows from Theorems 1 and 2.

Inequalities (5) and (6) show also that an $n \times n$ Jacobian matrix $[E_{ij}]$ is quasi-negative definite, hence totally stable and Hicksian (see Quirk and Saposnik (1968, pp. 166–67) under (N_1), (M) and homogeneity of the excess demand functions. In (1) monotonicity of f_i in E_i is assumed. Hands (1981), without assuming it, has shown that if

(8) $$dp_i/dt = \alpha_i(P)E_i(P)\,, \quad \alpha_i(P) > 0\,, \qquad i = 1, 2, \cdots, n\,,$$

the competitive equilibrium is globally stable for some positive α_i's provided that $[E_{ij}]$ is Hicksian everywhere. We can therefore assert that the competitive equilibrium is globally stable for (8) under (N_1) together with (M) and homogeneity of the excess demand functions.

Finally, we should note that the same assertions as above can be made on the basis of (N_2), (M) and the Walras law.

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APPENDIX

We outline here the proof of Theorem 1. For detail, see Namatame and Tse [5]. Let $A = [a_{ij}]$ be a matrix with dominant negative diagonal, hence

(a.1) $$c_i |a_{ii}| > \sum_{j \neq i} c_j |a_{ij}|\,, \qquad i = 1, 2, \cdots, n$$

for positive constants c_i's. To show that $A + A'$ has a dominant negative diagonal, it is sufficient to show that a system of linear inequalities

(a.2) $$Bx > 0\,, \quad B'x > 0\,, \quad x > 0$$

has a solution, where $B = [b_{ij}]$ is defined by $b_{ii} = |a_{ii}|$ and $b_{ij} = -|a_{ij}|$, $i \neq j$. It is clear that (a.2) and

(a.3) $$Bx > 0\,, \quad B'x > 0, \quad x \geqq 0$$

are equivalent. Normalize properly. Then (a.2) is seen to have a solution if and only if there exists an optimal solution to a primal linear programming problem:

(a.4) $$\begin{cases} \text{Minimize} \quad 0'x \\ \quad \text{subject to} \\ \quad Bx \geqq e\,, \quad B'x \geqq e\,, \quad x \geqq 0 \end{cases}$$

where all elements of e are unities. The dual problem for (a.4) is

$$\text{(a.5)} \qquad \begin{cases} \text{Maximize} \quad e'y \\ \quad \text{subject to} \\ \qquad B'y \leqq 0, \quad By \leqq 0, \quad y \geqq 0 \end{cases}$$

By the duality theorem, $y=0$ is the optimal solution for (a.5).

Consider y such that

$$\text{(a.6)} \qquad B'y \leqq 0, \quad y \geqq 0.$$

In view of (a.1), (a.6) and the definition of B, we have

$$\text{(a.7)} \qquad \sum_i \sum_j c_i b_{ji} y_j \leq 0,$$

or

$$\text{(a.8)} \qquad \sum_i \left(c_i |a_{ii}| - \sum_{j \neq i} c_j |a_{ij}| \right) y_i \leq 0.$$

Hence $y=0$, and (a.6) is shown to have only a trivial solution $y=0$. Accordingly, $y=0$ is the only feasible solution for (a.5). The above arguments establish that $A+A'$ is a matrix with dominant negative diagonal if A is so. Under the same condition, $A+A'$ becomes negative definite as a symmetric matrix $A+A'$ has only negative characteristic roots.

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