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Abstract	The paper is an attempt to overcome some of the major difficulties inherent in the usual notion of the tatonnement process as applied to an economy with production. First, firms are not likely to adjust their production plans sensitively to price changes arising in the process which prohibits actual production and trade of commodities. Second, factors of production are not gross substitutes for each other under normal conditions so that the very basis of the Arrow-Block-Hurwicz theorem, fundamental in the stability analysis, is rather vacuous in the production model. Third, there is no place for money in the entire setting with the result that the issue of the stability of money equilibrium prices is made irrelevant. To get around these difficulties, the paper envisages an economic process extending over many inter-related periods with money functioning as a means of payment. A unit period of the short run economic activities, comparable to the Hicksian week, is supposed to consist of two sub-periods, the first devoted to production, and the second to consumption. The paper presents three distinct results. The first two results are concerned with the existence and global stability of the market equilibrium in each sub-period. In particular, the commodity market equilibrium is shown to be globally stable under conditions weaker than the gross substitutability with the indication that the explicit introduction of money as a means of payment adds to the stability. The last result is concerned with a simple process of firms' interperiodical price adjustment in terms of expectation. This process is shown, by the application of the Arrow-Block-Hurwicz theorem, to be globally stable under the weak gross substitutability only with respect to commodities.
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MONEY AS A MEANS OF PAYMENT AND STABILITY IN GENERAL EQUILIBRIUM

Michihiro OHYAMA

ABSTRACT: The paper is an attempt to overcome some of the major difficulties inherent in the usual notion of the tatonnement process as applied to an economy with production. First, firms are not likely to adjust their production plans sensitively to price changes arising in the process which prohibits actual production and trade of commodities. Second, factors of production are not gross substitutes for each other under normal conditions so that the very basis of the Arrow-Block-Hurwicz theorem, fundamental in the stability analysis, is rather vacuous in the production model. Third, there is no place for money in the entire setting with the result that the issue of the stability of money equilibrium prices is made irrelevant. To get around these difficulties, the paper envisages an economic process extending over many inter-related periods with money functioning as a means of payment. A unit period of the short run economic activities, comparable to the Hicksian week, is supposed to consist of two sub-periods, the first devoted to production, and the second to consumption. The paper presents three distinct results. The first two results are concerned with the existence and global stability of the market equilibrium in each sub-period. In particular, the commodity market equilibrium is shown to be globally stable under conditions weaker than the gross substitutability with the indication that the explicit introduction of money as a means of payment adds to the stability. The last result is concerned with a simple process of firms' interperiodical price adjustment in terms of expectation. This process is shown, by the application of the Arrow-Block-Hurwicz theorem, to be globally stable under the weak gross substitutability only with respect to commodities.

I. INTRODUCTION

Since the publication of the paper by Arrow and Hurwicz (1958), the stability of competitive economy has been systematically investigated within the framework of general equilibrium analysis. The most important result so far obtained along this line of inquiry is the proposition, attributable to Arrow, Block and Hurwicz (1959), that, under a tatonnement process of price adjustment, if all goods are "gross substitutes," then the competitive equilibrium is globally stable. There are, however, some annoying difficulties inherent in the common interpretation of this

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proposition as applied to an economy with production. First of all, a tatonnement mechanism, carried on by the help of a fictitious auctioneer, is often criticized as "unrealistic." As Negishi (1962) points out, there is, in fact, a serious question as to whose behaviour is really expressed by such a mechanism. It is also hard to imagine that firms' production plans are adjusted to price changes as swiftly as consumers' exchange plans throughout the process which prohibits actual production and trade until the equilibrium finally obtains. In the second place, the assumption of gross substitutability, if applied to an economy with production in the usual sense, must cover all goods including primary factors of production. It is shown by Rader (1968), however, that factors of production are not likely to be gross substitutes. The devastating effect of this after-thought is that it voids the basis of the Arrow-Block-Hurwicz theorem. Furthermore, as Rader (1971) argues, it also erodes the grounds of qualitative economics associated with the concept of general equilibrium. Thirdly, there is an interpretive difficulty concerning the stability of money (or absolute) prices in the equilibrium. The well-known argument of Patinkin (1965) runs as follows: suppose that an initial state of equilibrium is disturbed in such a way as to cause an equiproportionate departure of money prices. Because of the "homogeneity postulate," this does not generate excess-demand in any of the commodity markets. Money prices are, therefore, indeterminate or unstable. But this contradicts the implication of the quantity theory of money that once the aggregate supply of money is given, money prices are determinate and stable.[1] As a matter of fact, there is no place for money in the usual general equilibrium analysis. This convention is diplorable since it eliminates from consideration even the most genuine role of money as a means of payment. Finally, and perhaps most importantly, the static setting of the entire model is to be questioned. The stability analysis is only concerned with the behaviour of the short run market clearing process towards the temporary equilibrium within a self-contained period. There is little room in this setting for the extension of the analysis to a changing economy characterized by inter-periodical linkages such as expectations and capital accumulation.

In this paper, we intend to provide a provisional solution for the first three difficulties by considering an economic model striding over many inter-related periods. To start with, a unit period of the short run economic activities, comparable to the Hicksian week, is assumed to comprise two sub-periods, the first devoted to production, and the second to consumption. In the first sub-period, firms plan and carry out production. They are supposed to form a definite expectation of those prices which will clear the commodity markets in the second sub-period. All firms are competitive in the sense that they expect to sell as much as they wish at the expected prices. They hire factors of production so as to maximize the expected profits, and remunerate in cash the owners of factors towards the end of the sub-period. This will be referred to as the "day of

[1] See Patinkin (1965), Chapter 8, esp., pp. 177–188.

production." The day's interest, of course, centers around the factor market equilibrium.[2] In the second sub-period, firms bring their products as sellers, and consumers their money incomes as buyers, to the commodity markets. Consumers are competitive in the sense that, for a given set of prices, they design consumption plans so as to maximize their satisfaction subject to the constraint of their money incomes. This will be referred to as the "day of consumption." Here, our concern lies in the property of the commodity market equilibrium. Evidently, there is no guarantee that the commodity markets are *actually* cleared at the firms' expected prices. It is at this point that a crucial inter-periodical relationship arises, for, if betrayed by their expectation, firms will modify it in some way or another in the next period. The ensuing process of inter-periodical adjustment will continue until firms hit upon the correct expectation of the equilibrium prices.[3]

Our plan of the paper is as follows. In the next section, we shall argue that, in each period, the factor market is globally stable in the tatonnement sense. In Section III, it will be shown that, in each period, the commodity market equilibrium is globally stable under a condition weaker than the weak gross substitutability. Finally, in Section IV, we shall consider a simple process of inter-periodical adjustment of the firms' expected prices. This process, designed after the usual tatonnement process, will be shown, by an application of the Arrow-Block-Hurwicz theorem, to be globally stable under the weak gross substitutability only with respect to the final goods. Although the first two results are still dependent upon the possibility of recontract, they are not saddled with the second and third difficulties mentioned above. Moreover, our separate treatment of the factor and the commodity markets will lighten the burden of the recontract assumption. The last result may be considered to be immune from the first three difficulties, and in addition, to alleviate the fourth difficulty to some extent.

II. THE FACTOR MARKET EQUILIBRIUM

On the day of production, firms set out to employ the *mobile* primary factors of production in view of the *expected* prices of commodities. This section is concerned with the stability of the factor market equilibrium in which the aggregate demand for each factor is equated to the aggregate supply of that factor under some factor prices. Let there be a fixed list of mobile factors, numbered from 1 to m. All firms in industry h, say, are assumed to have the same positive expected price $\bar{p}_h$ for commodity h, and, in factor markets, to face factor prices $(w_1, \cdots, w_m)$

[2] A point of terminology: we distinguish between a commodity and a factor of production. The former refers to a produced good desired by consumers, whereas the latter signifies a natural good such as labour mainly desired by firms. A large class of intermediate and capital goods is not covered by these two categories.

[3] Note that a unit time period need not be isolated from another. Thus, the day of consumption of a period may coincide with the day of production of the next period. After all, firms are producing for the needs of tomorrow, and consumers are living on the products of yesterday even in the shortest run.

as given at each point of time. To begin with, consider a firm α producing a single commodity h.[4] For given prices, it exhibits its demands for m factors, $(v_1{}^{\alpha}, \cdots, v_m{}^{\alpha})$ so as to produce $\bar{x}_h{}^{\alpha}$ of commodity h. These are determined by the maximization of the firm's expected profit

$$\Pi(v^{\alpha}) = \bar{p}_h \bar{x}_h{}^{\alpha} - wv^{\alpha} \tag{1}$$

where w is the vector of given factor prices, and v^{α} the vector of the quantity of factors demanded. The firm is supposed to possess a twice continuously differentiable production function

$$x_h{}^{\alpha} = f^{\alpha}(v^{\alpha}) \tag{2}$$

defined over the non-negative orthant of the m-dimensional real space. The familiar first-order conditions for profit maximization are always satisfied:

$$\bar{p}_h \frac{\partial f^{\alpha}}{\partial v_i{}^{\alpha}} = w_i \qquad (i=1, \cdots, m)^5 . \tag{3}$$

The Hessian matrix of function f^{α} is assumed to be negative definite.[6] Conditions (3) form a set of m equations which determine factor demands v^{α} and, therefore, with the production function, determine output $\bar{x}_h{}^{\alpha}$ for a given w. Thus, $v_i{}^{\alpha}$ $(i=1, \cdots, m)$ are real-valued functions

$$v_i{}^{\alpha} = v_i{}^{\alpha}(w) \tag{4}$$

defined over a subset W of the m-dimensional real space. Now, let F^{α} denote the Hessian of f^{α} multiplied by scalar $\bar{p}_h$, i.e., $\bar{p}_h$ $(\partial^2 f^{\alpha}/\partial v_i \partial v_j)$, and V^{α} the matrix of partial changes in the firm's factor demands resulting from infinitesimal changes in factor prices, i.e., $(\partial v_i{}^{\alpha}/\partial w_j)$. Differentiation of (3) with respect to w_i $(i=1, \cdots, m)$ then yields

$$F^{\alpha} V^{\alpha} = I$$

where I is the identity matrix, or

$$V^{\alpha} = (F^{\alpha})^{-1} . \tag{5}$$

[4] This assumption is made merely to simplify the exposition. In fact, joint production of several commodities, as well as the use of commodities in the production of commodities, can be accommodated to the model without a change in the essential part of the following argument.

[5] Firm α may not desire some factors of production unless their prices are negative. In such a case, the number of relevant equations in (3) may be less than m. This possibility does not affect the gist of our discussion.

[6] This is, of course, the second-order condition for profit maximization. It implies that the firm's production function is strictly concave, or

$$f^{\alpha}(\xi v + (1-\xi)v') > \xi f^{\alpha}(v) + (1-\xi) f^{\alpha}(v')$$

for $0 < \xi < 1$ and $v \neq v'$. Thus, we flatly rule out the constant returns to scale. As long as there are some immobile factors of production suppressed in the production function, the law of diminishing returns will take sides with us. Strictly speaking, the constant returns to scale seem to be justified only in the long run where such factors as entrepreneurial resources are also completely mobile.

Since F^α is negative definite, V^α is also negative definite. Summing (4) over all firms and all industries, we obtain *aggregate* demand functions:

$$v_i = v_i(w) \qquad (i=1, \cdots, m) \tag{6}$$

Likewise, summing (5) over all firms and all industries, we obtain the matrix of partial changes in the *aggregate* factor demands resulting from infinitesimal changes in factor prices:

$$V = \left(\frac{\partial v_i}{\partial w_j}\right) \qquad (i, j=1, \cdots, m) \tag{7}$$

The matrix V is negative definite.[7]

With these preliminaries in mind, we make the following additional assumptions for the rest of this section.

A1. The matrix V is non-positive and continuous in the intersection of the factor-price space W and the non-negative orthant.

A2. The aggregate factor supplies $\bar{v}_i$ $(i=1, \cdots, m)$ are positive, and fixed.

Define the aggregate excess demands $z_i(w)$ by

$$z_i(w) = v_i(w) - \bar{v}_i \qquad (i=1, \cdots, m) \tag{8}$$

A3. For $w \in W$ such that $w_i < 0$, $z_i(w) > 0$. $(i=1, \cdots, m)$.

A4. There are $w_i^* > 0$ such that $w_i > w_i^*$ and $w_j = 0$, for $\forall j \neq i$, imply $z_i(w) < 0$ $(i=1, \cdots, m)$.

A5. The dynamic behaviour of the competitive factor markets is governed by a *tatonnement* process.

The interpretation of A1 is that factors are used in production as complements for each other rather than as substitutes. This assumption will be fulfilled if, in the production of a typical firm α, factors are cooperative, or $\partial^2 f^\alpha / \partial v_i^\alpha \partial v_j^\alpha \geqq 0$, for $i \neq j$.[8] A2 serves to simplify our analysis.[9] A3 can be interpreted as stating that the negative price of a factor coaxes out a strong enough demand for that factor to supersede its fixed supply no matter what prices are assigned to other factors. If the disposal of unproductive factors were costless, an infinite amount would be demanded of any factor with a negative price. A3 is made, however, to avoid assuming free disposability. The interpretation of A4 is that none of mobile factors are strongly demanded at a high enough price when the free use of all other factors is available. As usual, A5 may be formulated as follows:

$$w_i'(t) = \lambda_i z_i(w(t)) \qquad (i=1, \cdots, m) \tag{9}$$

[7] The sum of negative definite matrices is negative definite.

[8] The notion of cooperative factors of production is time-honoured. Rader (1968) calls it "Wicksell's Law." A1 is, however, not indispensable for our result.

[9] This assumption may be justified in the short run. Nonetheless, some may wish to regard the supply of factors as variable at variable factor prices. This complication can be handled on the assumption that consumers (or factor owners) entertain the same definite expectation as firms regarding commodity prices.

where $w_i'(t)$ denote the time derivatives of factor prices, and λ_i are positive. This means that if excess demand for a factor is positive (resp. negative), the price of that factor is increased (resp. decreased).

To examine the stability of the factor price adjustment process governed by (9), we shall use the Liapounov function of the form:

$$D(w(t))=\frac{1}{2}\sum_{i=1}^{m}\lambda_i[z_i(w(t))]^2 \tag{10}$$

The crucial notion of the factor market equilibrium is substantiated by the existence of a vector $\hat{w}$ in W such that

$$z_i(\hat{w})=0 \qquad (i=1,\cdots,n) \tag{11}$$

That is to say, all excess demands are zero under this particular price vector.

PROPOSITION 1. *The differential equation system* (9) *has a solution* $w(t\colon w^0)$ *(abr.* $w(t)$*) for any initial price vector* $w^0 \geqq 0$. *Any solution* $w(t)$ *approaches a unique equilibrium price vector* $\hat{w}$ *as time tends to infinity. In other words, the factor market equilibrium is associated with a unique, globally stable price vector* $\hat{w}$.

In preparation for the proof of this proposition, it will be convenient to clarify a couple of points.

LEMMA 1. $D(w)>0$ *unless* $z_i(w)=0$ $(i=1,\cdots,m)$ *and* $D(w)=0$ *if and only if* $z_i(w)=0$ $(i=1,\cdots,m)$.

Proof. Straightforward. ■

LEMMA 2. $dD(w(t))/dt<0$ *for* $t\geqq 0$.

Proof. In view of (9) and (10), we find

$$dD(w(t))/dt=\sum_i\sum_j \lambda_i z_i\left(\frac{\partial z_i}{\partial w_j}\right)\lambda_j z_j \tag{12}$$

Because of A2, the matrix $(\partial z_i/\partial w_j)$ coincides with the matrix V in (7), which is negative definite. ■

LEMMA 3. *For* $w\geqq 0$ *such that* $w_i>w_i^*$, $z_i(w)<0$ $(i=1,\cdots,m)$.

Proof. Suppose to the contrary that $z_i(w)\geqq 0$ for $w\geqq 0$ such that $w_i>w_i^*$. Define a new price vector $\tilde{w}$ by setting $\tilde{w}_i=w_i$ and $\tilde{w}_j=0$ for $j\neq i$. Then, $\tilde{w}\leqq w$. Since, by A1, all factors are complements for one another, $\tilde{w}\leqq w$ implies $z_i(\tilde{w})\geqq z_i(w)\geqq 0$. This contradicts A4. ■

LEMMA 4. *If* $w^0\geqq 0$, *a local solution* $w(t\colon w^0)$ *(abr.* $w(t)$*) of* (9) *in the time interval* $[0,\sigma)$ *is contained in a compact subset of* W.

Proof. Since $z_i(w)$ are continuous in W, the Cauchy-Peano theorem[10] ensures

[10] See Coddington and Levinson (1955), pp. 6–15.

that, for any initial price vector $w^0 \in W$, there is a set of continuously differentiable functions of t

$$w_i(t) \qquad (i=1, \cdots, m)$$

defined on $[0, \sigma)$ $(\sigma > 0)$ with $w(t) \in W$ and satisfy (9) as well as the initial condition $w(0)=w^0$. For $w^0 \geqq 0$, define the set of factor prices:

$$\bar{W} = \{w \in W \mid 0 \leqq w_i \leqq \bar{w}_i \qquad (i=1, \cdots, m)\} \tag{13}$$

where $\bar{w}_i = \text{Max}\,(w_i^0, w_i^*)$ $(i=1, \cdots, m)$. It will be shown that $w(t)$ is contained in the compact set $\bar{W}$. Evidently, $w^0 \in W$. Suppose that, for some i, there is $t_1 \in [0, \sigma)$ such that $w_i(t_1) < 0$. Since $w_i^0 \geqq 0$, there is, by continuity of $w_i(t)$, $t_2 \in [0, t_1)$ such that $w_i(t_2) = 0$, and $w_i(t) < 0$ for $t \in (t_2, t_1)$. By the mean value theorem, for some $t_3 \in (t_2, t_1)$, $w_i'(t_3) = w_i(t_1)/(t_1 - t_2) < 0$. Hence, from (9), $z_i(w(t_3)) < 0$. Since $w_i(t_3) < 0$, this is a contradiction of A3. Thus, $w_i(t) \geqq 0$ for $t \in [0, \sigma)$ $(i=1, \cdots, m)$. A similar line of reasoning may be applied to the supposition that, for some k, there is $t_4 \in [0, \sigma)$ such that $w_k(t_4) > \bar{w}_k$. In fact, such a supposition can be shown to contradict the result of Lemma 3. ∎

We are prepared for

Proof of Proposition 1.

Uniqueness of the equilibrium price vector. By A3 and Lemma 3, $z_i(w) \neq 0$ for some i if $w \notin \bar{W}$ where $\bar{W}$ is a closed rectangular region in W as defined in (13). Hence, there can be no eqiulibrium vector outside the region $\bar{W}$. A1 and A2 imply that the Jacobian matrix $(\partial z_i/\partial w_j)$ of $z_i(w)$ $(i=1, \cdots, m)$ is continuous and negative definite in $\bar{W}$. By the Gale-Nikaidô theorem, the mapping $z_i(w)$ is, therefore, univalent in $\bar{W}$, and if $z_i(\hat{w})=0$ $(i=1, \cdots, m)$ for $\hat{w} \in \bar{W}$, $\hat{w}$ is the unique equilibrium price vector.

Existence and stability of the equilibrium. From Lemma 4, a local solution $w(t;\, w^0)$ of (9) can be continued to the time interval $[0, +\infty)$.[11] Evidently, $w(t;\, w^0) \in \bar{W}$ for all $t \geqq 0$. Lemma 2 ensures that the Liapounov function $D(w(t))$ declines through time. Suppose, however, that, for some $\varepsilon > 0$, $D(w(t)) \geqq \varepsilon$ for all $t \geqq 0$. Define the set $U = \{w \in \bar{W} \mid D(w) < \varepsilon\}$. Since $\bar{W}$ is compact and $w^0 \in \bar{W}$, $\bar{W} - U$ may be seen as a non-empty, compact set. As noted above, the Jacobian matrix of $z_i(w)$ is continuous on $\bar{W}$ so that the time derivative (12) of the Liapounov function is also continuous, and takes on the maximal value $-\varepsilon'(\varepsilon' > 0)$ on the compact set $\bar{W} - U$. Now, we can write

$$\begin{aligned} D(w(t)) &= D(w^0) + \int_0^t dD(w(t))/dt \cdot dt \\ &\leqq D(w^0) - \varepsilon' t \end{aligned}$$

For $t > D(w^0)/\varepsilon'$, we have $D(w(t)) < 0$ in contradiction to the property of

[11] See Nikaidô (1968), pp. 338–339.

$D(w(t))$ noted in Lemma 1. Therefore, there is t' such that $D(w(t'))<\varepsilon$. Since $dD(w(t))/dt<0$, and $D(w(t))-D(w(t'))=\int_{t'}^{t} dD(w(t))/dt\cdot dt$, $D(w(t))<\varepsilon$ for $t\geqq t'$. Let $\bar{U}=\{w\in\bar{W}\mid D(w)\leqq\varepsilon\}$. $\bar{U}$ is closed, and non-empty by the foregoing argument. Consider a sequence (ε^s) such that $\varepsilon^s\to 0^+$. If we defined $\bar{U}^s$ accordingly, the interesection $\bigcap_{s=1}^{\infty}\bar{U}^s$ of the collection of nested sets $\{\bar{U}^s\}$ is non-empty, and $w\in\bigcap_{s=1}^{\infty}\bar{U}^s$ implies $D(w)\leqq\varepsilon^s$ for all s, or $D(w)=0$. Thus, $z_i(w)=0$ $(i=1,\cdots,m)$, and w is an equilibrium vector. By uniqueness, $\bigcap_{s=1}^{\infty}\bar{U}^s$ is a singleton $\{\hat{w}\}$. Let (t^s) be a sequence of time such that $w(t^s)\in\bar{U}^s$. Since $\bar{U}^s$ are nested, and $w(t^s)\in\bar{U}^{s'}$ for $s\geqq s'$, $w(t^s)\to\hat{w}$ as $s\to\infty$. This establishes the stability and existence of the unique equilibrium price vector $\hat{w}$.[12] ∎

Given the expected prices of commodities, firms are supposed to enter into production after factor markets are brought into equilibrium. Firm α, for example, employs factor inputs $v_i^\alpha(\hat{w})$ and produces output $\bar{x}_n^\alpha$ according to its production function. Let there be a fixed list of n commodities numbered from 1 to n. Summing over all firms, we obtain the aggregate outputs $(\bar{x}_1,\cdots,\bar{x}_n)$. As we have just shown, there is a unique set of equilibrium factor prices for a given set of positive expected prices $(\bar{p}_1,\cdots,\bar{p}_n)$ of commodities. Therefore, we are able to write

$$\bar{x}_h=\bar{x}_h(\bar{p}) \qquad (h=1,\cdots,m)$$

where $\bar{p}$ is the positive vector of expected prices. In view of the basic relationship such as (3) holding for individual firms, $\bar{x}_h(\bar{p})$ is homogeneous of degree zero in the argument $\bar{p}$. This implies that expected prices are significant for the determination of outputs only up to scalar multiplication.

Within the day of production, firms are supposed to remunerate *in cash* the owners of mobile *and* immobile factors. Naturally, these *advance* payments will add up to the expected value of the aggregate outputs, or

$$\sum_{h=1}^{n}\bar{p}_h\bar{x}_h(\bar{p})=M \tag{14}$$

where M is the amount of cash (outside money) required by firms for the purpose of factor remuneration. When there is no numéraire in the economy the supply of money is assumed to adapt itself to the needs of industries. That is to say, a typical firm α can obtain cash from banks, say, up to the expected value of its output $\bar{p}_h\bar{x}_h^\alpha$ if necessary, and will use it to pay out wages, rents, dividends and what not. Alternatively, if there is a unique numéraire like gold in the economy, the supply of money may be regarded as exogenously given. In this instance, firms form their price expectation only relative to the numéraire. Because of the zero-homogeneity

[12] The idea of this proof is adapted from McKenzie (1960) in which the global stability of a gross substitute economy is considered.

of functions $\bar{x}(\bar{p})$, this type of price expectation will suffice to determine industrial factor inputs and corresponding outputs. Let the first commodity be the numéraire so that we may set $\bar{p}_1 \equiv 1$. Now, it is necessary to distinguish expected prices relative to the numéraire and expected prices in money terms. We rewrite (14) as

$$(14') \qquad \sum_{h=1}^{n} \bar{q}_h \bar{x}_h(\bar{p}) = \bar{\rho} \sum^{n} \bar{p}_h \bar{x}_h(\bar{p}) = M$$

where $\bar{q}_h$ $(h=1, \cdots, n)$ denote expected *money* prices, and, in particular, $\bar{\rho}$ is the expected money price of the first commodity. Needless to say, firms are responsible only for the choice of expected *relative* prices. Upon the attainment of equilibrium in factor markets, the given supply of money is allocated among firms to achieve the equation (14′) and determine the expected money prices.

III. THE COMMODITY MARKET EQUILIBRIUM

At the end of the day of production, firms acquire the *real* outcome of the day's work, or commodities, and the owners of mobile and immobile factors of production receive the *financial* outcome, or cash in remuneration. The next day is the day of consumption in which the owners of factors (or consumers) meet with firms in the commodity markets. Consumers are interested in buying commodities, and firms in selling them. In the markets, money is supposed to serve as the means of transactions between the two parties. In this section, we investigate the stability of the commodity market equilibrium in which, for each commodity, the aggregate demand balances the aggregate supply under some commodity prices. Firms are assumed to act as auctioneers, and, as usual, consumers as price takers. Consider a consumer β facing market prices $(p_1, \cdots, p_n)$ with money income M^β. His demands for n commodities, $(x_1{}^\beta, \cdots, x_n{}^\beta)$ are determined uniquely by the maximization of a twice continuously differentiable utility function subject to the budget constraint

$$(15) \qquad \sum_{h=1}^{n} p_h x_h{}^\beta = M^\beta$$

Let p denote the vector of commodity prices. Resulting demand functions

$$(16) \qquad x_h{}^\beta = x_h{}^\beta(p, M^\beta) \qquad (h=1, \cdots, n)$$

are defined over the product of a subset P of the n-dimensional real space and the positive real line. Summing (16) over all consumers, we obtain aggregate demand functions

$$(17) \qquad x_h = \sum_\beta x_h{}^\beta(p, M^\beta) \qquad (h=1, \cdots, n)$$

Similarly, aggregation of (15) over all consumers yields

$$\sum_{h=1}^{n} p_h x_h = \sum_{\beta} M^\beta = M \tag{18}$$

We shall refer to (18) as the aggregate budget constraint. In light of the discussion in the preceding section, the doubling of the total money supply will merely double the money income of each consumer as long as the distribution of factor ownerships is unchanged. This implies that consumer β's money income is related to the total money supply by

$$M^\beta = \theta^\beta M \tag{19}$$

where θ^β is a constant fraction. From (17) and (19), we can express aggregate demand functions as

$$x_h = x_h(p, M) \qquad (h=1, \cdots, n) \tag{20}$$

As is well known, individual demand functions such as (16) are homogeneous of degree zero in the argument (p, M^β). Thereupon, it is easily seen that aggregate demand functions (20) are homogeneous of degree zero in (p, M). Given M, the Jacobian matrix of (20) with respect to p,

$$X = (\partial x_h / \partial p_k) \qquad (h, k=1, \cdots, n) \tag{21}$$

gives the array of partial changes in the aggregate demands resulting from infinitesimal changes in commodity prices.

To proceed further, we need to introduce a few more concepts. The price elasticities of demand are defined as

$$\eta_{hk} = \frac{p_k}{x_h} \cdot \frac{\partial x_h}{\partial p_k} \qquad (h, k=1, \cdots, n) \tag{22}$$

where $x_h > 0$. If $\eta_{hk} \geqq 0$ and $\eta_{kh} \geqq 0$, for $h \neq k$, commodities h and k are said to be weak gross substitutes for each other. The relative values of demand are defined as

$$\gamma_{hk} = \frac{p_h x_h}{p_k x_k} \qquad (h, k=1, \cdots, n) \tag{23}$$

where $p_k x_k > 0$. The average and marginal propensities to demand are defined as

$$C_h = \frac{p_h x_h}{M} \qquad (h=1, \cdots, n) \tag{24}$$

and

$$c_h = p_h \frac{\partial x_h}{\partial M} \qquad (h=1, \cdots, n) \tag{25}$$

The ratios c_h / C_h are usually referred to as the *income* elasticities of demand for commodity h.

The additional assumptions of this section are as follows.

A6. Given M, the matrix X is continuous in the intersection of commodity price space P and the positive orthant.

A7. The aggregate commodity supplies $\bar{x}_h$ $(h=1, \cdots, n)$ and the aggregate money supply M are positive and fixed.

Now, define the aggregate excess demands $e_h(p, M)$ by

(26) $$e_h(p, M)=x_h(p, M)-\bar{x}_h \qquad (h=1, \cdots, n)$$

Let us single out a price vector $p^*=(p_h{}^*)$ in P such that

(27) $$p_h{}^*=\frac{M}{\bar{x}_h} \qquad (h=1, \cdots, n)$$

A8. Given M and $\bar{x}_h$, there are $\delta_h>0$ such that $p_h<\delta_h$ and $p_k \leqq p_k{}^*$, for $k \neq h$, imply $e_h(p, M)>0$ $(h=1, \cdots, n)$.

A9. Given M and $\bar{x}_h$, the aggregate demands $x_h(p, M)$ $(h=1, \cdots, n)$ are positive, and bounded away from zero for $p \leqq p^*$.

Let I_h be a subset of indices, $I=\{1, \cdots, n\}$, such that $\eta_{hk}<0$ for $k \in I_h$ and $h \notin I_h$.

A10. Conditions

(28) $$2\sum_{k \in I_h} \eta_{hk} > -\frac{c_h}{C_h} \qquad (h=1, \cdots, n)$$

hold for positive $p \leqq p^*$.

Let I_k be a subset of I such that $\eta_{hk}<0$ for $h \in I_k$ and $k \notin I_k$.

A11. Conditions

(29) $$2\sum_{h \in I_k} \gamma_{hk}\eta_{hk} > -1 \qquad (k=1, \cdots, n)$$

hold for positive $p \leqq p^*$.

A12. The dynamic behaviour of the competitive commodity markets is governed by a tatonnement process.

If there are no stocks of commodities, A7 will be satisfied, for the supply of a commodity is equal to its output realized on the preceding day of production. The interpretation of A8 is that once the aggregate supplies are given, a sufficiently low, yet positive price of a commodity generates a positive excess demand for that commodity if other prices are not greater than certain critical values. The values of δ_h depend on the aggregate fixed supplies of commodities as well as the supply of money. A9 means that all commodities are desired unless prices are too high. A10 and A11 constitute the key assumption for the proposition of this section. A10 means that when there are gross complements for a commodity, the sum of all *negative* cross elasticities of demand for that commodity is smaller, in the absolute value, than 1/2 times the concurrent value of the income elasticity of demand. Note that this assumption excludes, by implication, the possibility of inferior goods. All can be paraphrased that when a commodity is a gross complement for other commodities, the sum of all the negative cross elasticities of demand for those commodities, each weighted by the concurrent relative value of demand, falls short of 1/2 in the absolute value. Combined together, A10 and A11 allow for some degree of gross complementarily between commodities in the aggregate consumption of the economy. In particular, they will be completely satisfied in the economy with only weak gross substitutes. Finally, we may specify A12 as

$$p_h'(t)=\mu_h e_h(p(t), M) \qquad (h=1, \cdots, n) \tag{30}$$

where $p_h'(t)$ denote the time derivatives of commodity prices, and μ_h are positive. The interpretation of this formulation is that if excess demand for a commodity is positive (resp. negative), the price of that commodity is increased (resp. decreased).

We shall use the Liapounov function of the same form as before, i.e.

$$G(p(t))\leqq 1/2 \sum_{h=1}^{n} \mu_h[e_h(p(t), M)]^2 . \tag{31}$$

Again, the concept of the commodity market equilibrium is substantiated by the existence of a vector $\hat{p}$ in P such that

$$e_h(\hat{p}, M)=0 \qquad (h=1, \cdots, n) . \tag{32}$$

When this particular price vector prevails in the market, all excess demands are zero.

PROPOSITION 2. *The differential equation system* (30) *has a solution* $p(t; p^0)$ (*abr.* $p(t)$) *for any initial price vector* $p^0>0$ *such that* $p^0\leqq p^*$. *Any solution* $p(t)$ *approaches a unique equilibrium price vector* $\hat{p}$ *as time tends to infinity. In other words, the commodity market equilibrium is associated with a unique, globally stable price vector* $\hat{p}$.

The proof will follow a route essentially similar to the one by which we established Proposition 1.

LEMMA 5. $G(p)>0$ *unless* $e_h(p, M)=0$ $(h=1, \cdots, n)$ *and* $G(p)=0$ *if and only if* $e_h(p, M)=0$ $(h=1, \cdots, n)$.

Proof. Straightforward. ■

LEMMA 6. *For* $p\in P$ *such that* $p_h>p_h^*$, $e_h(p, M)<0$ $(h=1, \cdots, n)$.

Proof. By the aggregate budget constraint (18) and the definition (27) of the price vector p^*, $p_h>p_n^*$ implies $p_h\bar{x}_h>M\geqq p_h x_h(p, M)$ $(h=1, \cdots, n)$. Since $p_h>p_h^*>0$, $e_h(p, M)=x_h(p, M)-\bar{x}_h<0$ $(h=1, \cdots, n)$. ■

LEMMA 7. *The equalities*

$$\sum_{k=1}^{n} \eta_{hk}=-\frac{c_h}{C_h} \qquad (h=1, \cdots, n) \tag{33}$$

and

(34) $$\sum_{h=1}^{n} \gamma_{hk}\eta_{hk} = -1 \qquad (k=1, \cdots, n)$$

hold wherever η_{hk}, γ_{hk}, and C_h are defined.

Proof. As noted above, aggregate demand functions (20) are positively homogeneous of degree zero in (p, M). Therefore, by the Euler theorem,

$$\sum_{k=1}^{n} p_k(\partial x_h/\partial p_k) = -M(\partial x_h/\partial M) \qquad (h=1, \cdots, n)$$

From definitions (22), (24), and (25), this can be rewritten as (33). Next, differentiate the aggregate budget constraint (18) with respect p_k to obtain

$$\sum_{h=1}^{n} p_h(\partial x_h/\partial p_k) = -x_k \qquad (k=1, \cdots, n)$$

Using definitions (22) and (23), we can express this as (34). ■

LEMMA 8. *If $p^0 \leqq p^*$, a local solution $p(t; p^0)$ (abr. $p(t)$) of (30) in the time interval $[0, \tau]$ is contained in a compact subset of P.*

Proof. Since $e_h(p, M)$ are continuous, the existence of a local solution $p(t; p^0)$ is ensured. For $p^0 \leqq p^*$, define the set of prices

(35) $$\bar{P} = \{p \in P \mid \bar{P}_h \leqq p_h \leqq p_h{}^* \ (h=1, \cdots, n)\}$$

where $\bar{P}_h = \text{Min}\ (p_h{}^0, \delta_h)$ $(h=1, \cdots, n)$. Evidently, $p^0 \in \bar{P}$. A reasoning similar to the proof of Lemma 4 will show that $p(t)$ is contained in the compact set $\bar{P}$. In fact, if, for some h, there were $t_1 \in [0, \tau)$ such that $p_h(t) > p_h{}^*$, the result of Lemma would be contradicted. If $p(t) \leqq p^*$, and if, for some h, there were $t_2 \in [0, \sigma)$ such that $p_h(t) < \bar{P}_h$, A8 would be violated. ■

LEMMA 9. *$dG(p(t))/dt < 0$ for $t \geqq 0$.*

Proof. From (30) and (31), we find

(36) $$\frac{dG(p(t))}{dt} = \sum_h \sum_k \mu_h e_h(\partial e_h/\partial p_k)\mu_k e_k$$

We shall show that the matrix $(\partial e_h/\partial p_k)$ is quasinegative definite.[13] In view of Lemma 8, a local solution $p(t; p^0)$ can be continued, and $p(t; p^0) \in \bar{P}$ for all $t \geqq 0$ for any positive initial price vector $p^0 \leqq p^*$. Thus, by A9, the aggregate demands $x_h(p(t))$ are positive, and η_{hk}, γ_{hk} and C_{hk} are all definable for $t \geqq 0$. From (28) (A10), (29) (A11), (33) and (34) (Lemma 7), we obtain

$$\sum_{k \notin I_h} \eta_{hk} - \sum_{k \notin I_h} \eta_{hk} < 0 \qquad (h=1, \cdots, n)$$

[13] A square matrix A is said to be quasinegative definite if the sum $(A + A^T)$ is negative definite.

and

$$\sum_{h \notin I_k} \gamma_{hk}\eta_{hk} - \sum_{h \notin I_k} \gamma_{hk}\eta_{hk} < 0 \qquad (k=1, \cdots, n)$$

These, together with A7, imply

$$\sum_{k \notin I_h} p_k \frac{\partial e_h}{\partial p_k} - \sum_{k \in I_h} p_k \frac{\partial e_h}{\partial p_k} < 0 \qquad (h=1, \cdots, n) \tag{37}$$

and

$$\sum_{h \notin I_k} p_h \frac{\partial e_h}{\partial p_k} - \sum_{h \in I_k} p_k \frac{\partial e_h}{\partial p_k} < 0 \qquad (k=1, \cdots, n). \tag{38}$$

Now, by the choice of sets of indices I_h and I_k, (37) means that the matrix $(\partial e_h/\partial p_k)$ has a negative row dominant diagonal with a strictly positive price vector serving as the coefficient, and (38) means that the matrix also has a negative column dominant diagonal with the same coefficient.[14] Thus, the sum symmetric matrix $(\partial e_h/\partial p_k)+(\partial e_h/\partial p_k)^T$ has a neagative dominant diagonal. By the McKenzie theorem, the real part of every characteristic root of the matrix $(\partial e_h/\partial p_k)+(\partial e_h/\partial p_k)^T$ is negative, or the matrix is negative definite.[15] This result implies the quasinegative-definiteness of $(\partial e_h/\partial p_k)$ as desired. ∎

Finally, we can complete

Proof of Proposition 2.

Uniqueness of the equilibrium price vector. By A8 and Lemma 6, $e_h(p, M) \neq 0$ for some h if $p \notin \bar{P}$ where $\bar{P}$ is a closed rectangular region in P as defined in (34). Thus, there is no equilibrium vector outside the region $\bar{P}$. From A6, A7 and the proof of Lemma 9, we note that the Jacobian matrix $(\partial e_h/\partial p_k)$ of $e_h(p, M)$ $(h=1, \cdots, n)$ is continuous and quasinegative definite in $\bar{P}$. Therefore, the Gale-Nikaidô theorem ensures the univalence of mapping $e_h(p, M)$ in $\bar{P}$. Consequently, if $e_h(\hat{p}, M)=0$ $(h=1, \cdots, n)$ for $p \in \bar{P}$, $\hat{p}$ is the unique equilibrium price vector.

Existence and stability of the equilibrium. By virtue of Lemma 4 and Lemma 9, the continuation of a local solution $p(t; p^0)$ of (30) is possible, and the Liapounov function $G(p(t))$ declines through time if the process of price adjustment starts from any positive $p^0 \leqq p^*$. The rest of proof can be accomplished by making use of the fact that $p(t; p^0)$ is contained in the compact subset $\bar{P}$ of P for $t \geqq 0$. ∎

[14] A square matrix $A=[a_{hk}]$ is said to have a negative row dominant diagonal if a_{hh} is negative for all h, and there are positive numbers d_k such that

$$d_h|a_{hh}| > \sum_{k \neq h} d_k|a_{kk}| \qquad \text{for all } h.$$

Similarly, A is said to have a negative column dominant diagonal if a_{kk} is negative for all k, and there are positive numbers d_k such that

$$d_k|a_{kk}| > \sum_{h \neq k} d_h|a_{hk}| \qquad \text{for all } k.$$

[15] See McKenzie (1956), p. 49.

As noted above, the strategic assumptions A10 and A11 are implied by the weak gross substitutability of commodities. The explicit introduction of money as a means of payment enabled us to prove the stability of the commodity market equilibrium under these conditions which are indeed weaker than the assumption of gross substitutability.[16]

IV. TOWARDS THE OVERALL EQUILIBRIUM

We have thus far established the stability of the usual tatonnement processes for the factor market equilibrium as well as for the commodity market equilibrium, each envisaged within a short-run unit time period of economic activity. Needless to say, however, our result does not imply the stability of the *overall* equilibrium in which all economic agents of the model are able to enjoy their subjective equilibrium. In fact, there is no assurance that the expected prices entertained on the day of production *actually* clear the commodity markets on the succeeding day of consumption. As long as firms are confronted with shortages or surpluses of their products when they quote their expected prices, they will naturally try to modify their price expectation for the next day of production. In other words, firms are not in their *expectation* equilibrium under such a circumstance. In this section, we shall briefly consider the stability of an inter-period adjustment process of firms' price expectation. It will turn out that the Arrow-Block-Hurwicz theorem is conveniently applicable to our problem.

Suppose that, on the day of consumption, firms start out by quoting their expected prices formed on the preceding day of production. This is a natural supposition in the sense that the quotation of any other prices contradicts firms' expectation from the outset. The overall equilibrium is said to be attained if it so happens that all the markets are cleared under these expected prices. Unless the overall equilibrium is not attained, firms are supposed to modify their price expectation for the next period in view of the existing shortages or surpluses of their products.

For simplicity, suppose that there exists a continuum of time periods. If there is no numéraire in the economy, then firms' price adjustment process may be formalized as

$$\bar{p}_h'(s) = \kappa_h e_h(\bar{p}(s)) \qquad (h = 1, \cdots, n) \tag{39}$$

where s denotes time period, $\bar{p}_h'(s)$ the derivative of expected prices with respect to time period, and κ_h positive adjustment speeds.[17] This formalization means that if, under the expected prices of period s, there is a positive (resp. negative) excess

[16] Even in a pure exchange model in which money has no role to play, traces of gross complements are compatible with the stability of an equilibrium. For some recent results on complementarity and local stability, see Mukherji (1970) and Ohyama (1969).

[17] Excess demands e_h are now considered as functions only of expected prices since money income is a function of the latter.

demand for a commodity, the expected price of that commodity is increased (resp. decreased). From the discussion of the foregoing sections, it follows that the aggregate excess demands $e_h(\bar{p}(s))$ are positively homogeneous of degree zero in the expected prices $\bar{p}(s)$. We also have the relationship

$$\sum_{h=1}^{n} \bar{p}_h(s)\bar{x}_h(\bar{p}(s)) = M(\bar{p}(s)) = \sum_{h=1}^{n} \bar{p}_h(s)x_h(\bar{p}(s))$$

or

$$\sum_{h=1}^{n} \bar{p}_h(s)e_h(\bar{p}(s)) = 0 \tag{40}$$

which may be considered as Walras Law in this case. The concept of overall equilibrium is substantiated by the existence of a price vector $\hat{\bar{p}}$ in P sich that

$$e_h(\hat{\bar{p}}) = 0 \qquad (h=1, \cdots, n)\,. \tag{41}$$

Since $e_h(\bar{p})$ are homogeneous of degree zero in $\bar{p}$, a solution of (41) is unique only up to scalar multiplication. From (40), however, one of equations in (41) can be regarded as redundant. Thus, we are able to consider the economy in which there is a unique numéraire, say the first commodity. In this case, we have an alternative price adjustment system

$$\bar{p}_h{}'(s) = \kappa_h e_h(\bar{p}(s)) \qquad (h=2, \cdots, n) \tag{42}$$

with $p_1 = 1$.

Suppose that functions $e_h(\bar{p})$ are continuously differentiable and that all commodities are weak gross substitutes for one another in terms of *excess* demand (and with respect to expected prices). Then, under some additional, standard assumptions, we can establish

PROPOSITION 3 (Arrow, Block and Hurwicz (1959)). *The differential equation system* (39) (*or* (42)) *has a solution* $\bar{p}(t; \bar{p}^0)$ *for any initial expected price* $\bar{p}^0 > 0$. *Any solution* $\bar{p}(t; \bar{p}^0)$ *approaches a unique equilibrium price vector* $\hat{\bar{p}}$ *as time tends to infinity. In other words, the overall equilibrium is associated with a unique, globally stable price vector* $\hat{\bar{p}}$.

This result is, of course, well-known. But, in the framework of the present study, it has several distinct advantages over the conventional story. First, it is exempt both from the fiction of mysterious price shouting auctioneers in the commodity markets, and from the charge against a typical tatonnement process that "the grouping for equilibrium is prior to the conclusion of any trades.[18]" Secondly, the assumption of weak gross substitutes needs to be made only with respect to commodities. Normally, factors of production may never be gross substitutes. But the recognition of this fact does not undermine the stability of overall equilibrium

[18] McKenzie (1966), pp. 606–607.

nor the possibility of qualitative economics.[19] Thirdly, the money (or absolute) prices are determined in each equilibrium regardless of the presence of a unique numéraire in the economy. Consequently, the model is free from the kind of internal contradiction (or obscurity) discussed by Patinkin. Evidently, this is again the result of our explicit recognition of money as a means of payment. Finally, the model takes account of the firms' price expectation, though in an *ad hoc* manner, as a linkage of succeeding periods of economic activities.[20]

Needless to say, the present study suffers from various limitations especially in the analysis of inter-periodical linkages. First of all, our formulation of price adjustment process may be considered more convenient than natural. Alternative processes are also to be considered. Secondly, the storability of some commodities, or the possibility of capital accumulation, ruled out in this paper, will complicate the analysis of intra-period markets as well as inter-periodical adjustment processes. As soon as we take these factors into consideration, we shall have to concern ourselves with savings, consumer's demand for money, and portfolio selection. This vast area of controversy is completely left untouched. Thirdly, the supply of primary factors may change from period to period both endogenously and exogenously. We have not investigated the consequence of this complication. Finally, those factors of production which are immobile and embedded in the firms' production functions in the short run will move from one firm to another in the long run provided that there are persistent rent-differentials. This is also an important consideration we have set aside in this paper.

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[19] Note, however, that a change in the expected prices of commodities affects the factor market equilibrium thereby changing factor prices. Therefore, the concept of gross substitutes adopted here is actually quite different from the traditional one.

[20] In connection with this point, it may be interesting to consider an alternative interpretation of the model. Instead of regarding $\bar{p}$ as the expected price vector, we may regard it as the price vector decreed by the government on the day of production. The result of this paper can be applied to a central pricing economy in which firms are forced to play the game of competition. In fact, this interpretation is more natural than the previous one since it is *actually* possible to make all firms face a unique and definite price vector in a central pricing economy.

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