ON THEOREMS OF GENERAL COMPETITIVE EQUILIBRIUM OF PRODUCTION AND TRADE—A SURVEY OF SOME RECENT DEVELOPMENTS IN THE THEORY OF INTERNATIONAL TRADE—*

Akira TAKAYAMA

The purpose of this paper is to survey the theoretical structure of some recent developments in international trade such as the Heckscher-Ohlin-Samuelson model, the specific factor model, the general discussion of the three factor, two commodity model with an explicit recognition of the complementarity-substitutability relationship among three factors, etc., using the general perspective of an m factor, n commodity model obtained recently by Chang (1979), Jones-Scheinkman (1977), and others. The application of the $m \times n$ model which has attracted a great deal of attention recently is not limited to international trade, as Diewert-Woodland study (1977) on the Knight-Samuelson theorem demonstrates. In fact, it is a modern version of the well-known Walras-Cassel model of general competitive equilibrium of production. As such, it naturally has a very wide scope of applications which is not confined to international trade theory, while recent developments in international trade theory would naturally provide a stimulation to study such a general model.

In Section, 1, we shall briefly describe a *recent* development in the theory of comparative advantage, which should provide some background for the topics discussed in the present paper. This, in part, is also intended to motivate the discussions of the rest of the paper. Section 2 deals with the discussion of a general competitive model of production for the general $m \times n$ case. Section 3 obtains the basic results in the Heckscher-Ohlin-Samuelson 2×2 model as a special case of the $m \times n$ model. Section 4 treats the three factor, two commodity model (which includes the specific factor model) in the general $m \times n$ context. Section 5 discusses the theory of comparative advantage (including the Heckscher-Ohlin theory) in the context of a general framework developed earlier. This paper is complete with four appendices. Appendix A is concerned with the derivation of the properties of the substitution matrix of the $m \times n$ model of production. Appendix B discusses the relevance of such a model to an optimization scheme. The envelope theorem

* An earlier version of this paper was written for my lectures at Texas A & M University in February 1981, shortly after a very stimulating discussion with Katsuhiko Suzuki, to whom I am indebted for numerous subsequent exchanges on the related topic. I am also indebted to John Z. Drabicki, Winston Chang, Y. N. Shieh, Richard K. Anderson, Michihiro Ohyama, and two anonymous referees for very useful comments, and also to Kouichi Murayama, Roy Ruffin, Henry Thompson, and Ronald W. Jones for useful discussions. The discussion with Mr. Murayama was especially useful.

plays an important role in Appendices A and B. Appendix C is concerned with the "magnification effects," which reveals that such effects depend crucially on the assumption that the number of commodities is equal to two. Appendix D surveys empirical procedures used to determine the U.S. commodity structure of international trade.

1. THE HECKSCHER-OHLIN THEORY AND RELATED DEVELOPMENTS

One of the most exciting post-war developments in international trade is the so-called "Heckscher-Ohlin theorem" and related topics such as the Stolper-Samuelson theorem, the Rybczynski theorem, the factor price equalization theorem, etc. Since Paul Samuelson plays a key role in this development, the underlying model is often called the *Heckscher-Ohlin-Samuelson (H-O-S) Model.*

In a nutshell, the H-O-S model considers an economy consisting of two commodities, each of which is produced by employing two factors, in a competitive situation, under the neoclassical, convex and constant returns to scale technology. Using such a 2×2 framework, the Heckscher-Ohlin theorem states that a country exports use intensively the country's relatively abundant factor. The most appealing point of this theory is probably its simplicity and ability to derive a plausible conclusion out of an almost hopeless set of complex interactions of many elements in the real world. The popularity of other related results in the H-O-S model is based on similar reasons.

Leontief, as early as 1953, tested the empirical plausibility of the Heckscher-Ohlin theorem in which factors are aggregated into "capital" and "labor." Leontief's discovery was startling: he found that the U.S. (which is presumably capital abundant) exports labor intensive commodities and imports capital intensive commodities, contrary to the assertion of the Heckscher-Ohlin theorem.

The paradoxical finding, known as the *Leontief paradox*, casts some doubt on the strong appeal and the plausibility of the Heckscher-Ohlin theorem. Thanks to this challenge, a considerable amount of progress has been made, both analytically and empirically, during the last thirty years. On the theoretical level, the discussions of "demand bias" and "factor intensity reversals" are among the early and well-known contributions. There has been a huge amount of literature testing the validity of various assumptions which the Heckscher-Ohlin theory depends upon. This line of study was surveyed energetically by Stern (1975).

Although the discussion of the topic is, by no means, yet concluded, "there is · · · wide spread agreement that a simple two-factor (capital and labor) version of the Heckscher-Ohlin theory is inadequate" (Baldwin 1979, p. 40). Strong empirical evidence has been discovered that the U.S. exports skilled labor intensive (relative to unskilled labor) commodities and imports capital intensive (relative to unskilled labor) commodities. From this we may infer that the U.S. is more competitive in skilled labor ("human capital" or R & D) intensive commodities and less competitive in capital intensive commodities. Using such an implication, Ray

(1981a, b) recently successfully tested the hypothesis that the U.S. tariff rate tends to be lower in those industries which are intensive in the use of skilled labor, and higher in those industries which are capital intensive. Horiba-Kirkpatrik (1981), using the data of the U.S. *interregional* trade, confirmed the significance of human capital although they obtained a result opposite to the Leontief paradox with regard to (physical) capital intensity. In a sense, these empirical findings conform with the well-known discovery of the importance of R & D factor by Keesing (1967) and Gruber-Metha-Vernon (1967), since scientists and engineers are part of human capital or an example of Baldwin's classification of "skilled groups." (Baldwin 1971, 1979).

This then points to the importance of a *three* factor model of unskilled labor, (physical) capital, *and* skilled labor. In fact, as noted by Branson-Monoyios (1977, p. 112), the importance of the third factor was already hinted at in Leontief's conjecture, upon his discovery of the paradox (1953), that the paradox may be due to higher labor productivity in the U.S. In his 1956 paper, Leontief actually showed evidence (Table 2, p. 399) that U.S. exports employed more skilled labor than did production of import competing goods. About the same time, Kravis (1956) also published a paper which showed that leading U.S. exports industries paid, on average, higher wages than leading import-competing industries.

Granting that the third factor, skilled labor or human capital, is important, a question still remains. As Caves-Jones (1981, p. 48) aptly puts it, "It capital is cheap and abundant in the United States compared to other countries, why should American exports not utilize it heavily in both its human and physical forms?" Clearly, a satisfactory answer to such a question requires an explicit formulation of a three factor model, since the answer depends on (possibly) complicated substitutability and complementarity relationships of the three factors.

The general discussion of the three factor model is also useful to the so-called "specific factor model" of which interest has recently revived through the works of Samuelson (1971) and Jones (1971). Jones' illustration of this model, which sheds some light on the U.S. experience in the 19th century, has particularly been appealing to many economists.

On the theoretical level, the "synthesis" of the specific factor model and the three factor extension of the H-O-S 2×2 model has been attempted by Batra-Casas (1976). Unfortunately, their study contains unnecessarily cumbersome discussions and even contains an error (Suzuki, 1980 and 1981). Ruffin (1981, p. 177) also states, "Batra and Casas states a confusing array of propositions that lead one to believe the specific factor does not generalize." These defects are remedied by a recent article by Suzuki (1980) which studies the 3×2 model in a most comprehensive manner. Suzuki's study is followed by Ruffin (1981) and Jones-Easton (1981).

In parallel with these developments, a considerable amount of progress has been made with regard to the theoretical study of a general m factor, n commodity model. The pioneering study on this topic is seen in Samuelson's classical article

(1953) on the factor price equalization theorem, which stimulates the work by McKenzie (1955) and others. More recently, the general discussion on the $m \times n$ model of production under a competitive framework has virtually been completed by Woodland-Diewert (1977), Jones-Scheinkman (1977), and Chang (1979). This enables us to consider the 2×2 H-O-S model, the specific factor model, and the 3×2 model by Batra-Casas (1976), Suzuki (1980), and others in the general perspective of the $m \times n$ model. This paper intends to survey the basic theoretical structure of these models in a unified framework, obtaining some new results. Needless to say, the applications of the general results obtained in terms of the $m \times n$ model are not limited to the 2×2 and 3×2 cases. The reader may easily develop other interesting applications, once he understands the basic structure of the general $m \times n$ model. One such application may be to develop a four factor model which involves "energy" as well as unskilled labor, skilled labor, and capital. In this context, some features of specific factor models can be utilized to make the analysis manageable and interesting. We shall leave such tasks as "exercises" to the interested reader. In the next section, we shall survey the structure of the general $m \times n$ model.

2. THE STRUCTURE OF GENERAL COMPETITIVE EQUILIBRIUM OF PRODUCTION

Let $p' = (p_1, p_2, \cdots, p_n)$ and $w' = (w_1, \cdots, w_m)$, respectively, be the commodity price and the factor price vectors, and let $x' = (x_1, \cdots, x_n)$ and $v' = (v_1, \cdots, v_m)$, respectively, be the output vector and the factor endowment vector. The prime ($'$) is used to denote the transpose of a particular vector (or a matrix). Thus, for example, p is a column vector, whereas p' is a row vector: i.e., all the non-primed vectors are column vectors.[1] The endowment vector v is assumed to be constant.[2] Let $A = [a_{ij}]$ be the $m \times n$ input-output coefficient matrix, where a_{ij} signifies the amount of the i-th factor used to produce one unit of commodity j. The general competitive equilibrium of production, assuming full employment, can be described as:

(1-a) $$Ax = v\,,$$

(1-b) $$A'w = p\,,$$

where (1-a) and (1-b), respectively, signify the equilibrium condition for the factor market and the "profit condition" for competitive equilibrium. Here we assume

[1] To avoid clutter, we use 0 to denote both column and row vectors whose elements are all zero, as well as to denote a scalar zero.

[2] Needless to say, the constancy of v does not rule out parametric changes. The commodity price vector p is left unspecified (or treated as a constant vector) so long as the demand condition for commodities is unspecified. Again, this does not rule out the possibility of a parametric change in p. In fact, the parametric changes in p and v are a focal point of some important comparative statics results (e.g., the Stolper-Samuelson theorem and the Rybzynski theorem).

that x, v, p and w are strictly positive vectors. Assuming convex and constant returns to scale technology, we have $a_{ij}=a_{ij}(w)$, so that $A=A(w)$.[3] Eqs. (1-a) and (1-b) define the functional relations,

(2) $$v=v(w, x) \quad \text{and} \quad p=p(w)\,,$$

which we assume to be continuously differentiable. Differentiation of (1) yields[4]:

(3) $$\begin{bmatrix} dv \\ dp \end{bmatrix} = H \begin{bmatrix} dw \\ dx \end{bmatrix}, \qquad \text{where} \quad H \equiv \begin{bmatrix} S & A \\ A' & 0 \end{bmatrix}.$$

Here $S=[s_{ih}]$ and

(4) $$s_{ih} \equiv \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial w_h} x_j\,, \qquad i=1, 2, \cdots, m\,, \qquad h=1, 2, \cdots, m\,.$$

The matrix S may be termed as the *substitution matrix of the economy* (*as a whole*). The following properties of matrix S are fundamental, where Samuelson's condition for "regular minimimum" for cost minimization is used to obtain $R(S)=m-1$.[5] The detail of the proof of Theorem 1 will be exposited in Appendix A.

THEOREM 1. *S is symmetric and negative semidefinite with $Sw=w'S=0$, and $R(S)=m-1$ so that $s_{ii}<0$ for each i, where $R(S)$ denotes the rank of matrix S.*

[3] This is well-known. However, the reader who is interested in the exposition on this point and related issues is referred to Appendix A. Although the assumption of constant returns to scale technology may be unacceptable for each firm, it may be plausible to describe the aggregate production function of each *industry*.

[4] The proof of $A'dw=dp$ is exposited in Appendix A. The proof of the first part of (2), $Sdw+Adx=dv$, is simply mechanical and it may best be illustrated by taking a special case, say $m=3$ and $n=2$, $Ax=v$ for this case may be written as,

$$a_{i1}(w)x_1 + a_{i2}(w)x_2 = v_i\,, \qquad i=1, 2, 3\,.$$

Differentiation of this yields,

$$\begin{aligned} dv_i &= \left\{ a_{i1}dx_1 + x_1 \sum_{h=1}^{3} \frac{\partial a_{ih}}{\partial w_h} dw_h \right\} + \left\{ a_{i2}dx_2 + x_2 \sum_{h=1}^{3} \frac{\partial a_{ih}}{\partial w_h} dw_h \right\} \\ &= (a_{i1}dx_1 + a_{i2}dx_2) + \sum_{h=1}^{3} dw_h \sum_{j=1}^{2} \frac{\partial a_{ij}}{\partial w_h} x_j \\ &= (a_{i1}dx_1 + a_{i2}dx_2) + \sum_{h=1}^{3} s_{ih} dw_h\,. \end{aligned}$$

Thus, we have $dv=Adx+Sdw$, as desired. The reader should easily be able to generalize the above proof for the general $m \times n$ case.

[5] The notation $R(\cdot)$ to denote the rank of a particular matrix will be used throughout the paper. Since S is negative semidefinite with $R(S)=m-1$, the $(m-1)\times(m-1)$ matrix formed by deleting the m-th row and the m-th column of S is (by a well-known result in matrix algebra) is negative definite, from which $s_{ii}<0$, $i=1, 2, \cdots, m$, follows. To obtain $R(S)=m-1$, it is assumed that $z'Sz<0$ for all nonzero z not proportional to w, which is ensured under Samuelson's assumption of "regular minimum" for cost minimization (cf. Samuelson, 1947, p. 68). This, in essence, means that the production isoquant of each industry is strictly "bowed-in" toward the origin.

One unifying concept of the aggregate (national) economy would be "national income," which may be defined by $Y \equiv p'x$. Using (1) and noting $w'v = wAx = p'x$, we obtain:

$$(5) \qquad w'v = p'x\,,$$

i.e., the total factor income is equal to the total value of outputs. Using (5), we may also obtain:

$$(6\text{-a}) \qquad p'dx = w'dv\,,$$

$$(6\text{-b}) \qquad x'dp = v'dw\,,$$

which are "dual" for each other.[6]

Define the national income function by $Y^*(w, x) \equiv p(w)'x$. Then, $dY^* = x'dp + p'dx = v'dw + p'dx$ by using (6-b). Thus we obtain:

$$(7) \qquad Y_w^*(\equiv \partial Y^*/\partial w) = v' \quad \text{and} \quad Y_x^*(\equiv \partial Y^*/\partial x) = p\,,$$

$$(8) \qquad \begin{bmatrix} \partial v/\partial w & \partial v/\partial x \\ \partial p/\partial w & 0 \end{bmatrix} = \begin{bmatrix} Y^*_{ww} & Y^*_{wx} \\ Y^*_{xw} & Y^*_{xx} \end{bmatrix},$$

where $Y^*_{ww} \equiv \partial^2 Y^*/\partial w^2$, etc. The RHS of (8) is the Hessian matrix of $Y^*(w, x)$.[7] Recalling (3), we may observe:

$$\begin{bmatrix} \partial v/\partial w & \partial v/\partial x \\ dp/dw & 0 \end{bmatrix} = \begin{bmatrix} S & A \\ A' & 0 \end{bmatrix} \equiv H\,.$$

Namely, H is the Hessian matrix of the national income function $Y^*(x, x)$.

Now suppose that (2) may be solved for w and x, i.e.,

$$(9) \qquad w = w(v, p)\,, \qquad \text{and} \quad x = x(v, p)\,,$$

and assume that the functions w and x are continuously differentiable. In terms of (9), the *national income function* Y may also be defined as,

$$(10) \qquad Y(v, p) \equiv p'x(v, p)[\equiv w(v, p)'v]\,.$$

Differentiation of this yields, $dY = p'dx + x'dp = w'dv + x'dp$, by virtue of (6-a). Thus we may conclude:

$$(11) \qquad Y_v(\equiv \partial Y/\partial v) = w'\,, \qquad \text{and} \quad Y_p(\equiv \partial Y/\partial p) = x'\,.$$

$$(12) \qquad \begin{bmatrix} \partial w/\partial v & \partial w/\partial p \\ \partial x/\partial v & \partial x/\partial p \end{bmatrix} = \begin{bmatrix} Y_{vv} & Y_{vp} \\ Y_{pv} & Y_{pp} \end{bmatrix},$$

where $Y_{vv} \equiv \partial^2 Y/\partial v^2$, etc. Note that the RHS of (12) is the Hessian matrix of

[6] To prove (6-a), simply observe from (3), $dv = Sdw + Adx$, which in turn implies $w'\,dv = w'Sdw + w'Adx = p'dx$ (as $w'S = 0$ and $w'A = p$). See Chang (1979, pp. 717–718), for example. To prove (6-b), differentiate (5) and use (6-a).

[7] $[\partial v/\partial w]$ denotes the Jacobian matrix of the function v with respect to w: thus it is an $m \times m$ matrix. Similarly, $[\partial v/\partial x]$ and $[dp/dw]$ are the relevant Jacobian matrices for $v(w, x)$ and $p(w)$.

$Y(v,p)$. By Young's theorem, we at once obtain[8]:

$$(13) \qquad [\partial w/\partial v]=[\partial w/\partial v]', \quad [\partial x/\partial p]=[\partial x/\partial p]', \quad [\partial w/\partial p]'=[\partial x/\partial v],$$

which is *Samuelson's reciprocity theorem* (see Samuelson, 1953, p. 10 and Chang, 1979, p. 718).

The $m\times n$ matrix $[\partial w/\partial p]$ is the Jacobian matrix of function w with respect to p, which signifies the effects of changes in commodity prices on factor prices when the factor endowments are fixed, and is known as the *Stolper-Samuelson matrix*. The matrix $[\partial x/\partial v]$ is known as the *Rybczynski matrix*, and it measures the effects of factor endowment changes on outputs where the commodity prices are fixed. The relation $[\partial w/\partial p]'=[\partial x/\partial v]$ in (13) states the important result that the *Rybczynski matrix is the transpose of the Stolper-Samuelson matrix*. Note that relations (11), (12), and (13) are valid for an arbitrary number of factors and commodities, so long as continuously differentiable functions $w(v,p)$ and $x(v,p)$ exist.

As is well-known, the general competitive equilibrium of production, (1), can also be regarded as a solution to the problem of choosing the resource allocation vector so as to maximize the value of national product subject to the resource constraints (cf. e.g., Samuelson 1953, p. 10). Our national income function, $Y(v,p)$, then becomes the maximum value function of such a constrained maximization problem, and (11) (from which (12) and (13) follow at once) is obtained as a simple application of the envelope theorem. Thus we may call (11)–(13) the *envelope results*. The exposition of this point will be left to Appendix B.

Once we understand that $Y(v,p)$ is the maximum value function of such a constrained maximization problem, it is not difficult to prove the following results (see Appendix B for the proof).[9]

THEOREM 2. *$Y(v, p)$ is concave in v and convex in p, so that*
(i) *$[\partial w/\partial v]$ is symmetric and negative semidefinite,*
(ii) *$[\partial x/\partial p]$ is symmetric and positive semidefinite.*
Needless to say, the symmetry of $[\partial w/\partial v]$ and $[\partial x/\partial p]$ is already obtained in (13).

Furthermore, for the functions $w(v,p)$ and $x(v,p)$, the following homogeneity properties hold.[10]

[8] That $[\partial w/\partial v]$ is symmetric and negative semidefinite can also be obtained directly by applying the "Caratheodory-Samuelson theorem" (on the inverse of bordered matrices) to matrix H, and by assuming $m\geqq n$. See Jones-Scheinkman (1977, p. 926), which also establishes $R[\partial w/\partial v]=m-n$.

[9] For an alternative proof of (i) and (ii) of Theorem 2 which does not utilize the envelope theorem formulation, see Chang (1979, p. 716), for example.

[10] Chang (1979, p. 715) obtained this result by explicitly assuming the nonsingularity of the coefficient matrix H (cf. his fn. 5). Our proof does not require this. Chang then noted that "a dual set of homogeneity properties hold for arbitrary m and n". Namely,

$$(14') \qquad \frac{\partial v}{\partial w}w=0, \quad \frac{\partial v}{\partial x}x=v, \quad \frac{\partial p}{\partial w}w=p, \quad \frac{\partial p}{\partial x}x=0,$$

which follows readily from (1), (3) and $Sw=0$ as Chang pointed out.

THEOREM 3. *The function $w(v, p)$ is homogeneous of degree zero in v and degree one in p, and the function $x(v, p)$ is homogeneous of degree one in v and degree zero in p, so that we have:*

$$\frac{\partial w}{\partial v} v=0\,, \quad \frac{\partial w}{\partial p} p=w\,, \quad \frac{\partial x}{\partial v} v=x\,, \quad \frac{\partial x}{\partial p} p=0\,. \tag{14}$$

Proof. Write $[\partial w/\partial v]\equiv w_v$, $[\partial w/\partial p]\equiv w_p$, $[\partial x/\partial v]\equiv x_v$, and $[\partial x/\partial p]\equiv x_p$. Differentiation of $Y(v, p)=v'w(v, p)$ in v yields, $Y_v=v'w_v+w'$. This, combined with $Y_v=w'$, yields $w_v v=0$, since $w_v{}'=w_v$. Also, differentiation of $Y(v, p)=p'x(v, p)$ in p yields, $Y_p=p'x_p+x'$. This, combined with $Y_p=x'$, yields $x_p p=0$, since $x_p{}'=x_p$. Next, differentiation of $p'x(v, p)=v'w(v, p)$ in v yields, $p'x_v=v'w_v+w'=w'$. Then recalling $x_v=w_p{}'$, we obtain $w_p p=w$. Similarly, differentiation of $v'w=p'x$ in p yields, $v'w_p=p'x_p+x'=x'$. Recalling $w_p=x_v{}'$, we obtain $x_v v=x$. Q.E.D.

Returning to our basic equation (3), the following result obtained by Egawa (1978, pp. 533–534) and Chang (1979, p. 711) is of obvious importance.

THOEREM 4. *The coefficient matrix of* (3),

$$H=\begin{bmatrix} S & A \\ A' & 0 \end{bmatrix},$$

is nonsingular if and only if $R(A)=n$.[11]

Recalling A is an $m\times n$ matrix, it follows from Theorem 4 that H is singular if $m<n$ (i.e., if the number of factor is less than the number of commodities): i.e., $m\geqq n$ is necessary for the nonsingularity of H. Although the case of $m<n$ is important in the theory of international trade (e.g., the Ricardian theory of comparative advantage in which $m=1$ and $n=2$), we confine our attention mostly to the case of $m\geqq n$. For a summary of important results for $m<n$, the reader is referred to Chang (1979), for example.

Assume $R(A)=n$ so that H is nonsingular. Then from (3), we obtain:

$$\begin{bmatrix} dw \\ dx \end{bmatrix}=H^{-1}\begin{bmatrix} dv \\ dp \end{bmatrix}=\begin{bmatrix} \partial w/\partial v & \partial w/\partial p \\ \partial x/\partial v & \partial x/\partial p \end{bmatrix}\begin{bmatrix} dv \\ dp \end{bmatrix}, \tag{15}$$

In view of (12), H^{-1} signifies the Hessian matrix of $Y(v, p)$. Relations (11), (12), (13), (14), and Theorem 2 offer some important properties of H^{-1}. For H^{-1}, we also have (e.g., Chang, 1979):

THEOREM 5. *Assume $R(A)=n$ (which requires $m\geqq n$). Then we have:*

$$R(\partial w/\partial v)=m-n\,, \quad R(\partial x/\partial p)=n-1\,, \quad R(\partial w/\partial p)=R(\partial x/\partial v)=n\,. \tag{16}$$

In particular, if $m=n$, $R(\partial w/\partial v)=0$, so that $[\partial w/\partial v]$ is a zero matrix. Thus,

[11] It is obvious that the nonsingularity of H implies $R(A)=m$. To show the converse, it is assumed (as in Theorem 1, cf. fn. 5) that $z'Sz<0$ for all nonzero z not proportional to w.

changes in factor endowments have no effect on factor prices when $m=n$ as long as commodity prices are held constant. This is a well-known result in the Heckscher-Ohlin-Samuelson theory in which $m=n=2$.

Also, when $m=n$, we have: $dw=[\partial w/\partial v]dv+[\partial w/\partial p]dp=[\partial w/\partial p]dp$, where $[\partial w/\partial p]$ is nonsingular by (16). Thus the *local* factor price equalization theorem holds. For the global factor price equalization theorem, we need to assert the *global* invertibility of the function $p=p(w)$ with respect to w so as to yield, $w=w^*(p)$ for all p: i.e., the equalization of commodity prices implies the equalization of factor prices. Note that the existence of such a function imposes a stricter restriction than the one required for the existence of $w(v, p)$ in (9). Here it is required $m=n$. In fact, we need much more. A sufficient condition for the invertibility of the function $p(w)$ is provided by the Gale-Nikaido theorem. Namely, a sufficient condition for such a global invertibility is that all the principal minors of $[\partial p_j/\partial w_i]$ are positive, where such a matrix is known as a "P matrix.").[12] Since $[\partial p_j/\partial w_i]=A$ by (3), such a condition is satisfied if all the principal minors of A are positive, i.e.,

$$(17)\qquad a_{ii}>0\,,\quad \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}>0\,,\quad \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}>0\,,\ \cdots,$$

for all $i, j, k \cdots$ [*and* for all w, where we may recall $a_{ij}=a_{ij}(w)$].

When $m=n$, we may obtain the following useful expression for H^{-1}, assuming H^{-1} exists (i.e., $R(A)=n$):

$$(18)\qquad H^{-1}=\begin{bmatrix} \partial w/\partial v & \partial w/\partial p \\ \partial x/\partial v & \partial x/\partial p \end{bmatrix}=\begin{bmatrix} 0 & (A')^{-1} \\ A^{-1} & -A^{-1}S(A')^{-1} \end{bmatrix}.$$

This, in particular, implies that the Rybczynski matrix $[\partial x/\partial v]$ is A^{-1} and that the Stolper-Samuelson matrix $[\partial w/\partial p]$ is $(A')^{-1}$.

Returning to a more general case in which $m\geqq n$, let $X=[x_{ij}]$, where $x_{ij}\equiv\partial x_i/\partial p_j$. Since X is positive semidefinite by Theorem 2, and since $R(X)=n-1$ by Theorem 5, every successive principal minor of X is *positive* up to the degree $n-1$. Thus, in particular, we obtain:

THEOREM 6. *Assume* $R(A)=n$. *Then* $\partial x_j/\partial p_j>0$ *for all* $j=1, \cdots, n$ *if* $n\geqq 2$. *If* $n=1$, $\partial x_1/\partial p_1=0$.

[12] For an exposition of the Gale-Nikaido theorem (Gale-Nikaido, 1965) and the factor price equalization theorem, see Takayama (1972, Chap. 18), for example. For a more recent development of the factor price equalization theorem, see Mas-Collel (1979). The global invertibility of $[\partial p/\partial w]$ is closely related to the Stolper-Samuelson theorem in the general $m\times n$ context, as is clarified by Chipman (1969), Kemp-Wegge (1969), and Uekawa (1971), for example. For further developments of the Stolper-Samuelson theorem in the multi-factor, multi-commodity context, see Ethier (1974), Jones-Scheinkman (1977), Egawa (1978), and Uekawa (1979).

Thus if $n=1$, for any $m \geqq 1$, "output cannot respond to price at constant factor endowments" (Chang, 1979, p. 717).[13] Needless to say, $n=1$ corresponds to the case discussed in most standard macroeconomics.

When $n=2$, the above corollary, together with $[\partial w/\partial p]p=0$, establishes at once the following sign pattern of X for any $m \geqq 2$.

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \quad \text{where} \quad x_{ij} \equiv \partial x_i/\partial p_j, \tag{19-a}$$

and

$$x_{11}p_1 + x_{12}p_2 = 0, \quad x_{21}p_1 + x_{22}p_2 = 0, \quad x_{12} = x_{21}. \tag{19-b}$$

For the two-sector economy, using the zero homogeneity of $x(v, p)$ in p, we have:

$$x_i = x_i(v, q, 1), \quad i=1, 2, \quad \partial x_1/\partial q > 0 \quad \text{and} \quad \partial x_2/\partial q < 0,$$

where $q \equiv p_1/p_2$. From this, we obtain: $q \equiv q(x_1, v)$, $\partial q/\partial x_1 > 0$. The *production possibility curve* is then defined by:

$$x_2 = \phi(x_1, v) \equiv x_2[v, q(x_1, v), 1]. \tag{20}$$

Then the following well-known properties can be shown readily from (19) and (20)[14]:

$$\partial \phi/\partial x_1 < 0 \quad \text{and} \quad \partial^2 \phi/\partial x_1{}^2 < 0. \tag{21}$$

Namely, the production possibility curve is negatively sloped and strictly concave. Note that (21) holds for *any* $m \geqq 2$.

Combining Theorems 2, 3, 5, and 6, we obtain the following theorem which summarizes all the basic results for the case of $m \geqq n$.

THEOREM 7. *Assume $R(A)=n$. Then we have:*

(i) *$[\partial w/\partial v]$ is symmetric and negative semidefinite with $R(\partial w/\partial v)=m-n$.*
(ii) *$[\partial x/\partial p]$ is symmetric and positive semidefinite with $R(\partial x/\partial p)=n-1$.*
(iii) *$[\partial x/\partial v]=[\partial w/\partial p]'$.*
(iv) *$[\partial w/\partial v]v=0$, $[\partial w/\partial p]p=w$; $[\partial x/\partial v]v=x$, $[\partial x/\partial p]p=0$.*
(v) *$\partial w_i/\partial v_i \leqq 0$ for all i.*
(vi) *$\partial x_j/\partial p_j > 0$ for all j if $n \geqq 2$, and $\partial x_1/\partial p_1 = 0$ if $n=1$.*

[13] This result and Theorem 6 are also obtained in Jones-Scheinkman (1977, p. 928).

[14] Note that $dx_2/dx_1 = (\partial x_2/\partial q)(\partial q/\partial x_1) < 0$, i.e., the production possibility curve is negatively sloped. Also, by differentiating $x_i(v, p)$ in p and using (19-b), we obtain:

$$dx_1 = -p_2 x_{12}(\hat{p}_1 - \hat{p}_2), \qquad dx_2 = -p_1 x_{12}(\hat{p}_1 - \hat{p}_2),$$

whenever v is constant, where $\hat{p}_i \equiv dp_i/p_i$, $i=1, 2$. Namely, an increase in (p_1/p_2) increases the output of commodity 1 and reduces the output of commodity 2. Also, from the above two equations, we at once obtain, $dx_2/dx_1 = -p_1/p_2$: i.e., the production possibility curve is tangent to the price line at optimum. Recalling $p_1/p_2 = q(x_1, v)$, we obtain, $d^2x_2/dx_1^2 = -\partial q/\partial x_1 < 0$. Thus, the production possibility curve is strictly concave (i.e., "bowed-out").

Remark. Note that Theorems 5, 6, and 7 are concerned with the case of $m \geqq n$, while Theorems 1, 2, and 3 are obtained without such a restriction. On the other hand, Theorem 4, which requires $m \geqq n$ for the nonsingularity of H, plays an important role in connection with these two sets of results.

3. THE HECKSCHER-OHLIN (-SAMUELSON) MODEL

This model is concerned with the case in which $m = n = 2$. The basic results here are obviously well-known and can be seen in many of the textbooks on international trade theory.[15] The purpose of this section is to obtain these well-known results from the general persepctive of the $m \times n$ case, recognizing that the exposition of the 2×2 case is not typically done in such a general perspective. Not only will this increase our understanding of the general $m \times n$ case, but also the derivation of the results for the 2×2 case becomes simpler.

Letting $m = n = 2$, we may write (3) as:

$$(3') \qquad \begin{bmatrix} dv_1 \\ dv_2 \\ dp_1 \\ dp_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & a_{11} & a_{12} \\ s_{21} & s_{22} & a_{21} & a_{22} \\ a_{11} & a_{21} & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \\ dx_1 \\ dx_2 \end{bmatrix} = H \begin{bmatrix} dw_1 \\ dw_2 \\ dx_1 \\ dx_2 \end{bmatrix}$$

Assume that commodity j is relatively more intensive in the use of factor j, i.e.,

$$(22) \qquad a_{11}/a_{12} > a_{21}/a_{22} \, ,$$

where we assume $a_{ij} > 0$ for all i and j. In particular, (22) rules out the "factor intensity reversals," since the a_{ij}'s depend on w, $a_{ij}(w)$, and we require (22) to hold for all (revelant) w. Under (22), we obviously have $R(A) = 2$, so H is nonsingular by Theorem 4. As in (15), write H^{-1} as:

$$H^{-1} = \begin{bmatrix} \partial w/\partial v & \partial w/\partial p \\ \partial x/\partial v & \partial x/\partial p \end{bmatrix} .$$

Since $R(\partial w/\partial v) = 2 - 2 = 0$ by Theorem 5 (or Theorem 7-i), $[\partial w/\partial v]$ is a 2×2 zero matrix, i.e., factor prices are independent of changes in factor endowments. Also, by Theorem 7-ii, $[\partial x/\partial p]$ is symmetric and positive semidefinite. Furthermore, since $n = 2$, properties (19-a), (19-b) and (21) hold.

From (18), $[\partial x/\partial v] = A^{-1}$, so that we may at once obtain:

$$(23) \qquad \frac{\partial x_1}{\partial v_1} = \frac{a_{22}}{|A|} > 0 \, , \quad \frac{\partial x_1}{\partial v_2} = -\frac{a_{12}}{|A|} < 0 \, , \quad \frac{\partial x_2}{\partial v_1} = -\frac{a_{21}}{|A|} < 0 \, , \quad \frac{\partial x_2}{\partial v_2} = \frac{a_{11}}{|A|} > 0 \, ,$$

[15] For a useful exposition of this model, see Jones (1965), which follows Takayama (1964), etc. See also Takayama (1972), Chapters 2 and 3.

where $|A|=a_{11}a_{22}-a_{21}a_{12}$ which is positive by (22). The sign pattern of the $\partial x_j/\partial v_i$'s indicated in (23) corresponds to the *Rybczynski theorem.* This states that an increase in the endowment of one factor raises the output of the commodity which uses that factor relatively more intensively and lowers the output of the other commodity, provided that the commodity prices are kept constant. Note that (23) can also be obtained directly from (3′) by using Cramer's rule. Also, by virtue of the Samuelson reciprocity theorem, $[\partial w/\partial p]=[\partial x/\partial v]'$, we obtain:

$$\text{(24)}\qquad \begin{aligned} &\frac{\partial w_1}{\partial p_1}\left(=\frac{\partial x_1}{\partial v_1}\right)>0, \quad \frac{\partial w_1}{\partial p_2}\left(=\frac{\partial x_2}{\partial v_1}\right)<0, \\ &\frac{\partial w_2}{\partial p_1}\left(=\frac{\partial x_1}{\partial v_2}\right)<0, \quad \frac{\partial w_2}{\partial p_2}\left(=\frac{\partial x_2}{\partial v_2}\right)>0. \end{aligned}$$

The sign pattern of the $\partial w_i/\partial p_j$'s indicated in (24) corresponds to the *Stolper-Samuelson Theorem.* It states that an increase in the price of a commodity raises the price of the factor which is relatively more intensively used in the production of that commodity, and lowers the price of the other factor.

Next, recalling $[\partial w/\partial p]p=w$ in (14), we obtain,

$$\text{(25)}\qquad \partial \log w_i/\partial \log p_1+\partial \log w_i/\partial \log p_2=1\,, \qquad i=1,2\,.$$

Combining this with (24), we at once obtain:

$$\text{(26-a)}\qquad \partial \log w_j/\partial \log p_j\ (=\partial \log x_j/\partial \log v_j)>1\,, \qquad j=1,2\,.$$

$$\text{(26-b)}\qquad \partial \log w_i/\partial \log p_j\ (=\partial \log x_j/\partial \log v_i)<0\,, \qquad i\neq j,\ i,j=1,2\,.$$

Note that this can also be obtained by using $[\partial x/\partial v]v=x$ instead of $[\partial w/\partial p]p=w$.

Let $q\equiv p_1/p_2$ and $\omega\equiv w_1/w_2$. Then using the homogeneity of the function w in p, we define the function ω by,

$$\omega=\omega(v, q)\equiv w_1(v, q, 1)/w_2(v, q, 1)\,.$$

Using (24), we can easily assert: $\partial\omega/\partial q>0$.[16] Namely, an increase in the relative price of a commodity increases the real return to the factor used relatively intensively in its production, which is the usual statement of the Stolper-Samuelson theorem. Notice that we may write $\omega(v, q)=\omega^*(q)$, since $[\partial w/\partial v]$ is a zero matrix. That is, the factor price ratio ω $(\equiv w_1/w_2)$ for the 2×2 model depends only on the commodity price ratio q $(\equiv p_1/p_2)$.

By virtue of (22), we have:

$$a_{11}>0, \quad a_{22}>0 \quad \text{and} \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0\,.$$

Namely, condition (17) is satisfied. Thus, the *global factor price equalization* theorem holds under the present circumstances (in which both commodities are

[16] Since $[\partial w/\partial v]$ is zero matrix for the present case, we need not assume the constancy of the factor endowments to obtain the proposition stated here.

produced and there are no factor intensity reversals with $m=n=2$).

Noting that $[\partial w/\partial p]p=w$ implies (25) and (26), we may easily obtain,[17]

$$\hat{w}_1 > \hat{p}_1 > \hat{p}_2 > \hat{w}_2 , \qquad \text{if} \quad \hat{p}_1 > \hat{p}_2 , \tag{27-a}$$

while by utilizing $[\partial x/\partial v]v=x$, we obtain,

$$\hat{x}_1 > \hat{v}_1 > \hat{v}_2 > \hat{x}_2 , \qquad \text{if} \quad \hat{v}_1 > \hat{v}_2 \quad \text{and} \quad \hat{p}_1 = \hat{p}_2 = 0 . \tag{27-b}$$

where the circumflex (ˆ) represents the proportional change (i.e., $\hat{w}_1 = dw_1/w_1$, etc.). These relations are Jones' *magnification effects* (Jones, 1965). Verbally (27-b), for exmple, may be stated as follows (Jones 1965, p. 561):

> "If factor endowments expand at different rates, the commodity intensive in the use of the fastest growing factor expands at a greater rate than either factor, and the other commodity grows (if at all) at a slower rate than either factor."

Although the magnification results (27) are interesting, we have to note that they depend crucially on the two commodity assumption (i.e., $n=2$). This can be seen easily by reviewing the proof used to obtain (27) (cf. fn. 17). In particular, the proof and the results depend crucially on the fact there are only two terms in the RHS of (25). We shall discuss this further in Appendix C.

4. THE THREE-FACTOR, TWO-COMMODITY MODEL

In contrast with the above "Heckscher-Ohlin-Samuelson Model," there has been an alternative specification of the neoclassical model of production in the literature (e.g., Haberler 1936, Chapter 12, Harrod, 1957) in which commodities are produced by using "specific factors," employable only in some industries, as well as a non-specific factor (such as unskilled labor). The interest on such a "specific factor model" has been revived recently (e.g., Samuelson 1971, Jones 1971, Mayer 1974, Mussa 1974, Amano 1977, Falvey 1979). Batra and Casas (1976) attempted a "synthesis" of the specific factor model and the Heckscher-Ohlin-Samuelson model. It turns out this leads to a general discussion of the three factor, two commodity model. As mentioned earlier, Batra-Casas' discussion is

[17] To obtain (27-a), differentiate $w_i = w_i(v, p)$ to obtain,

$$\hat{w}_i = [(\partial w_i/\partial p_1)dp_1 + (\partial w_i/\partial p_2)dp_2]/w_i = \frac{\partial \log w_i}{\partial \log p_1}\hat{p}_1 + \frac{\partial \log w_i}{\partial \log p_2}\hat{p}_2 ,$$

where we may recall $[\partial w/\partial v]$ is a zero matrix for the present case of $m=n(=2)$. Substituting (25) into this, we obtain (27-a). To obtain (27-b), note that $[\partial x/\partial v]v=x$ yields,

$$\partial \log x_j/\partial \log v_1 + \partial \log x_j/\partial \log v_2 = x_j , \qquad j=1, 2 . \tag{25'}$$

Combine this equation to the following relation which is obtained by simply differentiating $x=x(v,p)$ and setting $p=$constant:

$$\hat{x}_j = (\partial \log x_j/\partial \log v_1)\hat{v}_1 + (\partial \log x_j/\partial \log v_2)\hat{v}_2 , \qquad j=1, 2 .$$

quite cumbersome and even contains an error (cf. Suzuki 1981). Suzuki's study (1980) on the other hand, contains a complete, correct, and elegant study on the topic. Not only did he extend Batra-Casas' scope of analysis to incorporate such new topics as comparative advantage and foreign investment, but also he removed the essentially cumbersome nature of the Batra-Casas analysis. Suzuki's study is followed by Ruffin (1981) and Jones-Easton (1981).[18] The purpose of this section is to study the 3 factor, 2 commodity model in the context of the general $m \times n$ perspective discussed in Section 2.

To this end, we first write out equation (3) for the 3×2 case as:

$$(3'') \quad \begin{bmatrix} dv_1 \\ dv_2 \\ dv_3 \\ dp_1 \\ dp_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & a_{11} & a_{12} \\ s_{21} & s_{22} & s_{23} & a_{21} & a_{22} \\ s_{31} & s_{32} & s_{33} & a_{31} & a_{32} \\ a_{11} & a_{21} & a_{31} & 0 & 0 \\ a_{12} & a_{22} & a_{32} & 0 & 0 \end{bmatrix} \begin{bmatrix} dw_1 \\ dw_2 \\ dw_3 \\ dx_1 \\ dx_2 \end{bmatrix} = H \begin{bmatrix} dw_1 \\ dw_2 \\ dw_3 \\ dx_1 \\ dx_2 \end{bmatrix}$$

Following Suzuki (1980) and Ruffin (1981), we assume that the following factor intensity relations hold:

$$(28) \qquad a_{31}a_{22} - a_{32}a_{21} > 0 \quad \text{and} \quad a_{11}a_{32} - a_{12}a_{31} > 0 \,.$$

If a_{12}, a_{22}, $a_{32} > 0$, then we may equivalently rewrite this as

$$(28') \qquad a_{11}/a_{12} > a_{31}/a_{32} > a_{21}/a_{22} \,.$$

Factors 1 and 2 may then be called *extreme factors*, and factor 3 may be called the *middle factor* (cf. Ruffin 1981).[19] This specification of the factor intensity condition is the key to the success in Suzuki (1980) and Ruffin (1981) in overcoming the cumbersome nature of the Batra-Casas study and in obtaining their elegant results.[20]

[18] The original manuscript of Ruffin (1981) is dated July 1980, and it is apparently independent of Suzuki's study (1980) which has been circulated since April 1980. However, Ruffin is concerned with only one aspect of Suzuki's comprehensive study, i.e., the sign pattern of $[\partial w/\partial v]$ in the 3×2 case. Jones-Easton (1981) is concerned with clarifying the underlying economic structure by ingenious graphic technique (which are different from the ones used by Suzuki (1980), or by Ruffin (1981)), and focused upon obtaining the extension of the Jones (1965) magnification effects from the 2×2 case to the 3×2 case.

[19] Note that condition (28) means that industry j ($j = 1, 2$) uses the j-th extreme factor relatively more intensively when compared with the other industry. Incidentally, Suzuki (1980) called the first and the second factors in (28) the "intensive factors" and called the third factor the "intermediate factor". The choice of the third factor as the "middle" or the "intermediate" factor follows Suzuki (1980) rather than Ruffin (1981). This choice is not accidental, since, among others, it would make the discussion of the cases of specific factors and complements more natural. As noted by Suzuki (1980), such a factor intensity condition is also used by Kemp-Wegge (1969, p. 409).

[20] The key to Batra-Casas' discussion is their "strong factor intensity condition". (B-C, 1976, p. 26). B-C will say that factor 1 is *strongly intensive* if $a_{11}/a_{21} > a_{12}/a_{22}$ *and* $a_{11}/a_{31} > a_{12}/a_{32}$ (or

Note that the specific factor model amounts to assuming,

(29) $$a_{12}=a_{21}=0\,,\quad a_{11}>0\,,\quad a_{22}>0\,,\quad a_{31}>0\,,\quad a_{32}>0\,,$$

in which case condition (28) clearly holds. Namely, the specific factor specification may be considered as a special case of the factor intensity condition (28). To motivate (29), call commodities 1 and 2, respectively, "manufacturing goods" and "agricultural goods," and call the factors 1 and 2 "capital" and "land," respectively. The third factor, the mobile factor, is called "labor." Assume that no land is used in manufacturing activity and no capital is used in agriculture. Namely, capital is "specific" to the manufacturing industry, and land is "specific" to agriculture. This particular example of the specific factor model is used by Jones (1971, p. 12) "to shed light on some querries suggested by Peter Temin's... recent discussion of technology in Britain and America in the mid-nineteenth century." As mentioned earlier, such a specific factor model has been discussed extensively in the literature. Note that in the specific factor model, the specific factors are "extreme factors," and the mobile factor is the "middle factor," and that each industry uses only two factors, i.e., the factor which is specific to the industry and the mobile factor.

Under condition (28), $R(A)=2$ so that H is nonsingular. Again write H^{-1} as

$$H^{-1}=\begin{bmatrix}\partial w/\partial v & \partial w/\partial p\\ \partial x/\partial v & \partial x/\partial p\end{bmatrix}.$$

Then by Theorem 7, $[\partial w/\partial v]$ is symmetric and negative semidefinite with $R(\partial w/\partial v)=1$ and $\partial w_i/\partial v_i \leqq 0$ for all $i=1, 2, 3$. Also $[\partial x/\partial p]$ is symmetric and positive semidefinite in which the properties in (19) hold. Futhermore, we have:

$$[\partial w/\partial p]=[\partial x/\partial v]'\,.$$

By straightforward computation (e.g., use Cramer's rule), we may obtain the following results from (3′′) where $\Delta \equiv |H|$.[21]

(30) $$\partial w_i/\partial v_j=\alpha_i\alpha_j/\Delta\,,\quad i, j=1, 2, 3\,,$$

where

$$\alpha_1\equiv a_{31}a_{22}-a_{32}a_{21}\,,\quad \alpha_2\equiv a_{11}a_{32}-a_{12}a_{31}\,,\quad \alpha_3\equiv a_{12}a_{21}-a_{11}a_{22}\,.$$

equivalently, $a_{11}/a_{12}>a_{21}/a_{22}$ *and* $a_{11}/a_{12}>a_{31}/a_{32}$). Note that this definition is slightly weaker than the Suzuki-Ruffin condition, i.e., our (28), since either factor 2 *or* 3 can be the middle factor. However, the failure of the specification of the middle factor makes *B-C*'s discussion unnecessarily cumbersome.

[21] The computation to obtain (30) by Cramer's rule can be simplified if we recall the following rule in matrix algebra

$$\begin{vmatrix}A & B\\ C & D\end{vmatrix}=|A||D-CA^{-1}B|\,,$$

if A and D are square and if A is nonsingular. In the present application, D (which corresponds to the block of zero's at the southeast corner of H) is a zero matrix, and A corresponds to the S matrix in H.

Clearly $\alpha_1 > 0$ and $\alpha_2 > 0$ by condition (28). Also $\alpha_3 < 0$ follows from (28). In order to obtain (30), we need *not* compute all of the $\partial w_i/\partial v_j$'s since we may use the symmetry of $[\partial w/\partial v]$ as well as $[\partial w/\partial v]v=0$. It can easily be noted from (30) that $R(\partial w/\partial v)=1$, as it should be (cf. Theorem 7-i). Also, substituting (30) into $[\partial w/\partial v]v=0$, we may observe the following interesting relation:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \, .$$

Since $\partial w_i/\partial v_i \leqq 0$, we also obtain from (30) that[22]

$$\Delta < 0 \, . \tag{31}$$

Hence we may conclude[23]:

$$\begin{bmatrix} \partial w_1/\partial v_1 & \partial w_1/\partial v_2 & \partial w_1/\partial v_3 \\ \partial w_2/\partial v_1 & \partial w_2/\partial v_2 & \partial w_2/\partial v_3 \\ \partial w_3/\partial v_1 & \partial w_3/\partial v_2 & \partial w_3/\partial v_3 \end{bmatrix} = \begin{bmatrix} - & - & + \\ - & - & + \\ + & + & - \end{bmatrix} . \tag{32}$$

This means that an increase in the endowment of the middle factor will benefit the extreme factors, and an increase in the endowment of an extreme factor will benefit the middle factor and hurt the other extreme factor (cf. Ruffin, 1981). Following Ruffin (1981), we may say factors i and j are *friends* if $\partial w_i/\partial v_j$ $(=\partial w_j/\partial v_i) > 0$, and *enemies* if $\partial w_i/\partial v_j < 0$. Then (32) may also be interpreted as follows: The two extreme factors are always enemies, and the middle factor is the friend of either extreme factor. A remarkable feature of this proposition is that the signs of $\partial w_i/\partial v_j$'s are determined *independently* of the S matrix, i.e., the complementarity-substitutability relations among factors (cf. Ruffin 1981).

In terms of the specific factor model, we may conclude from the above that the two specific factors are enemies for each other, and that both of the specific factors are friends of the mobile factor (since the specific factors are extreme factors and the mobile factor is the middle factor as noted earlier).[24] In terms of the Jones' (1971) specific factor model which is mentioned earlier, labor is the friend of both capital and land. Namely, both capital and land favor the immigration of labor (if the country in question is small so that the commodity price vector is imposed by

[22] As a matter of fact, $|H|$ has the sign $(-1)^m \neq 0$, in general, where m is the number of factors. See Chang (1979, p. 711). Namely, $\Delta < 0$ is really a consequence of a more general result, and requires no specific proof for $m=3$.

[23] Batra-Casas (1976) obtained the expressions for, and the signs of, the $\partial w_i/\partial v_j$'s for all i and j [cf. their (19)–(21), and their Theorems 1, 2, and 2a], while Suzuki (1980) obtained the expression for the $\partial w_i/\partial v_j$'s for all i and j as recorded in (30) [cf. his eq. (9)], where his A_{13}, A_{23}, and A_{33}, respectively, correspond to our α_1, α_2, and α_3. Ruffin (1981) also obtained the sign pattern of $[\partial w/\partial v]$ indicated in (32). As mentioned earlier, Batra-Casas' discussion on this point is unfortunately unnecessarily tedious and cumbersome.

[24] In this context, Ruffin (1981, p. 177) remarked, "In the specific factor model, the two specific factors are natural enemies...; and both of the specific factors are natural friends of the mobile factor...."

the rest of the world). This may explain, at least partially, the influx of immigration to the U.S. during the 19th century (some of which were forced to immigrate through slavery).

In our condition (28), the third factor is designated as the middle factor. Since the basic results should be independent of numbering the factors, we may obtain the following result, depending on whether the second factor of the first factor is the middle factor, respectively,

$$\frac{\partial w}{\partial v}=\begin{bmatrix} - & + & - \\ + & - & + \\ - & + & - \end{bmatrix}, \quad \text{or} = \begin{bmatrix} - & + & + \\ + & - & - \\ + & - & - \end{bmatrix}. \tag{32'}$$

Namely, once again the extreme factors are enemies, and the middle factor is the friend of either extreme factor.

By straightforward (but somewhat tedious) computation, we may also obtain the following important result from (3'')[25]:

$$\begin{bmatrix} \partial x_1/\partial v_1 & \partial x_1/\partial v_2 & \partial x_1/\partial v_3 \\ \partial x_2/\partial v_1 & \partial x_2/\partial v_2 & \partial x_2 \partial v_3 \end{bmatrix} = \begin{bmatrix} \partial w_1/\partial p_1 & \partial w_2/\partial p_1 & \partial w_3/\partial p_1 \\ \partial w_1/\partial p_2 & \partial w_2/\partial p_2 & \partial w_3/\partial p_2 \end{bmatrix}$$
$$= \frac{1}{\Delta}\begin{bmatrix} (a_{32}\beta_2 - a_{22}\beta_3) & (a_{12}\beta_3 - a_{32}\beta_1) & (a_{22}\beta_1 - a_{12}\beta_2) \\ (a_{21}\beta_3 - a_{31}\beta_2) & (a_{31}\beta_1 - a_{12}\beta_3) & (a_{11}\beta_2 - a_{21}\beta_1) \end{bmatrix} \tag{33}$$

where $\Delta < 0$ and where the β_i's are defined by[26]:

$$\beta_i \equiv s_{i1}\alpha_1 + s_{i2}\alpha_2 + s_{i3}\alpha_3 , \qquad i = 1, 2, 3 . \tag{34}$$

The signs of the elements in the Rybczynski matrix $[\partial x/\partial v]$ (or the Stolper-Samuelson matrix $[\partial w/\partial p]$) cannot readily be established. They depend among other things, on the signs of the s_{ih}'s. Here it may be worthwhile to recall[27]:

(35-a) $\quad \partial a_{ij}/\partial w_h > 0$, if factors i and h are (*Allen*) *substitutes* (in industry j) , $\quad i \neq h$,

(35-b) $\quad \partial a_{ij}/\partial w_h < 0$, if factors i and h are (*Allen*) *complements* (in industry j) , $\quad i \neq h$,

[25] This is obtained by Suzuki (1980). Batra-Casas (1976, sec. 5) attempted to obtain some definite conclusions concerning $[\partial x/\partial v]$, which apparently contains an error (cf. Suzuki, 1980, fn. 15). In a recent note, Suzuki (1981) points out that B-C's definition of the factor intensity condition prevented them from drawing a clear distinction between extreme factors which are used relatively intensively in industries and the middle factor which is used relatively unintensively in both industries. He then goes on to state, "The lack of this distinction causes them to fail in revealing the essential properties of the model and leads them to such a tedious calculation in deriving the effects of factor endowment changes on commodity outputs that they very likely committed a miscalculation in their equation (28)." Suzuki then points out the specific error in B-C's eq. (28).

[26] In terms of matrix notation, $\beta = S\alpha$. Since $Sw = 0$, $\beta = 0$ if α is proportional to w. We assume away such a possibility.

[27] See Appendix A (especially, p. 26) for an exposition on this point.

Hence recalling (4), we have: $s_{ih}>0$ (resp. $s_{ih}<0$), if factors i and h are substitutes (resp. complements) in all industries. We may then define (for $i \neq h$),

(35′-a) Factor i and h are *aggregate substitutes* if $s_{ih}>0$,

(35′-b) Factor i and h are *aggregate complements* if $s_{ih}<0$.

Needless to say, in these cases, some $\partial a_{ij}/\partial w_h$ may have adverse signs. Recall that $s_{ii}<0$ for all i (cf. Theorem 1).

Although the sign pattern of $[\partial x/\partial v]=[\partial w/\partial p]'$ is, in general, indeterminate (crucially depending on the signs of the s_{ih}'s), we can make some useful assertions on the signs of the $\partial w_i/\partial p_j$'s (or the $\partial x_j/\partial v_i$'s) in special, but important cases.

(i) *Specific Factors*:

Suppose that the i-th factor ($i=1, 2$) is "specific" to industry i in the sense that it is used only in that industry, i.e., assume that condition (29) holds. With $a_{12}=a_{21}=0$, it would be natural to suppose $s_{12}=s_{21}=0$. Then, recallings $s_{ii}<0$, $i=1, 2$, and $Sw=0$ with $S=S'$, we may obtain the following sign specification of the S matrix.

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} - & 0 & + \\ 0 & - & + \\ + & + & - \end{bmatrix} \tag{36}$$

Then we have, by recalling (34),

$$\beta_1<0 , \quad \beta_2<0 \quad \text{and} \quad \beta_3>0 , \tag{37}$$

where we may recall $\alpha_1>0$, $\alpha_2>0$ and $\alpha_3<0$. Then recalling (33), we obtain:

$$\frac{\partial w_1}{\partial p_1}=(a_{32}\beta_2-a_{22}\beta_3)/\Delta>0 , \quad \frac{\partial w_2}{\partial p_1}=-a_{32}\beta_1/\Delta<0 , \quad \frac{\partial w_3}{\partial p_1}=a_{22}\beta_1/\Delta>0 . \tag{38-a}$$

$$\frac{\partial w_1}{\partial p_2}=-a_{31}\beta_2/\Delta<0 , \quad \frac{\partial w_2}{\partial p_2}=(a_{31}\beta_1-a_{11}\beta_3)/\Delta>0 , \quad \frac{\partial w_3}{\partial p_2}=a_{11}\beta_2/\Delta>0 . \tag{38-b}$$

Thus we may conclude:

$$\begin{bmatrix} \partial x_1/\partial v_1 & \partial x_1/\partial v_2 & \partial x_1/\partial v_3 \\ \partial x_2/\partial v_1 & \partial x_2/\partial v_2 & \partial x_2/\partial v_3 \end{bmatrix} = \begin{bmatrix} \partial w_1/\partial p_1 & \partial w_2/\partial p_1 & \partial w_3/\partial p_1 \\ \partial w_1/\partial p_2 & \partial w_2/\partial p_2 & \partial w_3/\partial p_2 \end{bmatrix} \tag{38'}$$
$$= \begin{bmatrix} + & - & + \\ - & + & + \end{bmatrix} .$$

The interpretation of the sign pattern of $[\partial x/\partial v]$ is straightforward. Namely, assuming constant commodity prices, an increase in the endowment of the i-th factor ($i=1, 2$) raises the output of the i-th industry (to which i-th factor is specific), and lowers the output of the other industry, while an increase in the endowment of the mobile factor (i.e., the third factor) increases the output of both industries.

The interpretation of the sign pattern of $[\partial w/\partial p]$ is analogous. Namely, an increase in the price of one commodity raises the price of the factor which is specific to that industry and lowers the price of the other specific factor, while increasing the price of the mobile (i.e., the middle) factor.

The use of the Jones specific factor model (1971) may be helpful in shedding some light on the well-known struggle between the North and the South in the U.S. concerning tariffs during the ante-bellum period. As can be seen from (38′), an increase in the price of manufacturing goods (p_1) and a fall in the price of agricultural goods, "cotton," (p_2) by protective tariffs will unequivocally benefit capital and hurt land ($\partial w_1/\partial p_1 > 0$, $\partial w_1/\partial p_2 < 0$, $\partial w_2/\partial p_1 < 0$, $\partial w_2/\partial p_2 > 0$). It may not then be surprising that the tariff issue lead to a bitter fight between the Northern industrialists and the Southern landlords.[28]

The sign pattern of $[\partial w/\partial p]$ indicated in (38′) may also shed some light on the famous controversy concerning the Repeal of the Corn Laws in Great Britain. Suppose that the Repeal lowers the price of agricultural goods (p_2), but keeps the price of manufacturing goods (p_1) constant in Britain. Then, by (38′), this increases the return on capital ($\partial w_1/\partial p_2 < 0$), lowers the return on land ($\partial w_2/\partial p_2 > 0$), and reduces the return on labor ($\partial w_3/\partial p_2 > 0$). Thus the owners of capital would favor the Repeal, while landlords and labor would oppose the Repeal. If, however, the price of manufacturing goods increases by the Repeal, its effect on labor becomes indeterminate since $\partial w_3/\partial p_1 > 0$ and $\partial w_3/\partial p_2 > 0$, while capitalists would still favor the Repeal and landlords still oppose it. If a rise in the price of manufacturing goods is sufficiently high, then labor joins the capitalists to favor the Repeal. In the case of the Corn Laws, both industrialists and workers favored the Repeal.

(ii) *The Case of Complements (for the Extreme Factors)*:

Suppose that the extreme factors are aggregate complements for each other in every industry: i.e., $s_{ij} < 0$, $i \neq j$, $i, j = 1, 2$.[29] Recalling $Sw = 0$ and $S = S'$, we then have:

$$(39) \qquad \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} - & - & + \\ - & - & + \\ + & + & - \end{bmatrix}$$

Assume further that the factor intensity condition (28) holds, and that $a_{ij} > 0$ for

[28] For a brief but very colorful description of this struggle as well as scholarly work on this topic, see Pope (1972). This topic, at the analytical level, is often discussed in terms of the Stolper-Samuelson theorem. However, the S-S theorem is concerned with the two (mobile) factor model, while we are concerned with the three factor model.

[29] Such a case is investigated by Jones (1977) and Suzuki (1980).

all i and j. Then we again have: $\alpha_1>0$, $\alpha_2>0$ and $\alpha_3<0$. Using this with (39) and recalling (34), we may easily determine the signs of the β_i's again as,

$$\beta_1<0\,, \quad \beta_2<0\,, \quad \text{and} \quad \beta_3>0\,. \tag{37}$$

Then recalling (33), we obtain:

(40-a) $\partial x_1/\partial v_1=\partial w_1/\partial p_1>0\,,\quad \partial x_1/\partial v_2=\partial w_2/\partial p_1<0\,,\quad \partial x_1/\partial v_3=\partial w_3/\partial p_1\gtrless 0\,;$

(40-b) $\partial x_2/\partial v_1=\partial w_1/\partial p_2<0\,,\quad \partial x_2/\partial v_2=\partial w_2/\partial p_2>0\,,\quad \partial x_2/\partial v_3=\partial w_3/\partial p_2\gtrless 0\,.$

We may summarize the result in (40) as follows:

$$[\partial x/\partial v]=[\partial w/\partial p]'=\begin{bmatrix} + & - & ? \\ - & + & ? \end{bmatrix}. \tag{40'}$$

Namely, assuming constant commodity prices, an increase in the endowment of one extreme factor increases the output of the industry which is intensive in the use of that factor and lowers the output of the other industry (which is intensive in the use of the other extreme factor). The effect of an increase in the endowment of the middle factor (with constant commodity prices) upon the output of either of the two industries is indeterminate. Also, assuming constant factor endowments, an increase in the price of one commodity increases the price of the extreme factor which is relatively intensitvely used in that industry and lowers the price of the other extreme factor, while its effect on the price of the middle factor is indeterminate.

The sign pattern of $[\partial w/\partial p]$ indicated in (40) may shed some light in the current trade problem that the U.S. faces. To this end, call factors 1, 2, and 3, respectively, "skilled labor," "(physical) capital," and "unskilled labor" (raw labor). Also, let industries 1 and 2, respectively, signify "exportables" and "importables." As mentioned in Section 1, there seems to be strong evidence that the current U.S. commodity structure of trade is such that her exports are relatively skilled labor (or R & D) intensive vis à vis unskilled labor, and that her imports are relatively capital intensive vis à vis unskilled labor (e.g., Baldwin, 1971, 1979): i.e., in symbols,

$$a_{11}/a_{31}>a_{12}/a_{32} \quad \text{and} \quad a_{22}/a_{32}>a_{21}/a_{31}\,.$$

This implies that our factor intensity condition (28) is satisfied, where skilled labor and (physical) capital are the extreme factors, and unskilled labor is the middle factor. Furthermore, there is some evidence that skilled labor and capital are (aggregate) complements (e.g., Branson-Monoyios, 1977). This indicates that our assumption of complements for extreme factors are satisfied. Then we may utilize our (40). From (40), we may conclude that an import restriction which raises the domestic price of importables (say, automobiles from Japan) in the U.S. increases the return on capital and lowers the return on skilled labor (or R & D) in the U.S. Similarly, the reduction of import restrictions if it decreases the domestic price of importables, reduces the return on capital but increases the return on skilled labor.

5. COMPARATIVE ADVANTAGE

We consider a (unsual) two country world. For the sake of simplicity, we only investigate a two-commodity model where the number of factors (m) is assumed to be greater than or equal to two ($m \geqq 2$).[30]

First differentiating $x_j = x_j(v, p)$ and recalling (19-b), we obtain,

$$\hat{x}_j = \sum_{k=1}^{m} \varepsilon_{jk}\hat{v}_k + (p_2 x_{j2}/x_j)(\hat{p}_2 - \hat{p}_1), \qquad j=1, 2, \tag{41}$$

where $x_{j2} \equiv \partial x_j/\partial p_2$, $j=1$, 2, and

$$\varepsilon_{jk} \equiv (\partial x_j/\partial v_k)(v_k/x_j), \quad j=1, 2, \quad k=1, 2, \cdots, m, \tag{42}$$

which signifies the partial elasticity of the j-th output change with respect to a change in the k-th factor endowment. We may call them the *Rybczynski elasticities*. Since $[\partial x/\partial v]v = x$, we have,

$$\varepsilon_{j1} + \varepsilon_{j2} + \cdots + \varepsilon_{jm} = 1, \qquad j=1, 2. \tag{43}$$

Using (41) and (43), we obtain,

$$\hat{x}_1 - \hat{x}_2 = \sum_{k=1}^{m-1} (\varepsilon_{1k} - \varepsilon_{2k})(\hat{v}_k - \hat{v}_m) + p_2[(x_{12}/x_1) - (x_{22}/x_2)](\hat{p}_2 - \hat{p}_1). \tag{44}$$

In order to focus our attention on the role of the supply side of the economy and rule out the possibility of "demand bias," we (heroically) assume that both countries have identical homothetic tastes, following a popular assumption in many empirical studies of comparative advantage. Let D_j denote the demand for the j-th commodity. Then the homothetic demand function can be specified by,

$$D_1/D_2 = h(q), \qquad h' < 0, \qquad q \equiv p_1/p_2. \tag{45}$$

Differentiating (45), we have,

$$\hat{D}_1 - \hat{D}_2 = -\sigma_D \hat{q}, \qquad \text{where} \quad \sigma_D \equiv -h'q/h > 0. \tag{46}$$

Here σ_D measures the elasticity of homothetic demand. Under autarkic equilibrium, we have $X_1/X_2 = D_1/D_2$, so that we obtain from (44) and (46),

$$-\gamma(\hat{p}_1 - \hat{p}_2) = \sum_{k=1}^{m-1} (\varepsilon_{1k} - \varepsilon_{2k})(\hat{v}_k - \hat{v}_m), \tag{47}$$

where $\gamma \equiv \sigma_D - p_2[(x_{12}/x_1) - (x_{22}/x_2)] > 0$. From this we at once obtain,

[30] For the restrictive nature of the two commodity assumptions in the theory of comparative advantage, see Drabicki-Takayama (1979).

$$\hat{p}_1 - \hat{p}_2 \lesseqgtr 0\,, \quad \text{according as} \quad \sum_{k=1}^{m-1} (\varepsilon_{1k} - \varepsilon_{2k})(\hat{v}_k - \hat{v}_m) \gtreqless 0\,. \tag{48}$$

To illustrate the meaning of (48), we consider the case of $m = 2$ (two factors, say, capital and labor). Recalling (23), we have $(\varepsilon_{11} - \varepsilon_{21}) > 0$. Hence, (48) is simplified as:

$$\hat{p}_1 - \hat{p}_2 \lesseqgtr 0 \quad \text{according as} \quad \hat{v}_1 - \hat{v}_2 \gtreqless 0\,. \tag{49}$$

This means that a country which is relatively heavily endowed in the i-th factor exports the commodity which is intensive in the use of the i-th factor. This is nothing but the well-known statement of the *Heckscher-Ohlin Theorem.*

In the three-factor model, we obtain from (48),

$$\hat{p}_1 - \hat{p}_2 \lesseqgtr 0\,, \qquad \text{according as} \quad (\varepsilon_{11} - \varepsilon_{21})(\hat{v}_1 - \hat{v}_3) + (\varepsilon_{12} - \varepsilon_{22})(\hat{v}_2 - \hat{v}_3) \gtreqless 0\,. \tag{50}$$

We now consider the two special cases which were analyzed in the previous sections.

(i) *The Specific Factor Case*:

Assume that the i-th factor ($i = 1, 2$) is specific to the i-th industry and the third factor is the mobile factor. Assume that the sign pattern of the S matrix is given by (36). Then from (38′) it is evident that

$$\varepsilon_{11} > 0\,, \quad \varepsilon_{21} < 0\,, \quad \varepsilon_{12} < 0\,, \quad \varepsilon_{22} > 0\,. \tag{51}$$

Hence we have:

$$\hat{p}_1 - \hat{p}_2 < 0\,, \qquad \text{if} \quad \hat{v}_1 - \hat{v}_3 > 0 \quad \text{and} \quad \hat{v}_2 - \hat{v}_3 < 0\,, \tag{52-a}$$

$$\hat{p}_1 - \hat{p}_2 > 0\,, \qquad \text{if} \quad \hat{v}_1 - \hat{v}_3 < 0 \quad \text{and} \quad \hat{v}_2 - \hat{v}_3 > 0\,. \tag{52-b}$$

Namely, if a country is heavily endowed in one specific factor relative to the mobile factor (i.e., factor 3) and if she is scarcely endowed in the other specific factor (again relative to the mobile factor), then she will export the commodity which uses the relatively heavily endowed specific factor. This result corresponds to the one obtained by Amano (1977).

To illustrate the above results, call commodities 1 and 2, respectively, "manufacturing goods" and "agricultural goods," and call factors 1 and 2 "capital" and "land." The third factor is called "labor." This specific factor model is used, as mentioned before, by Jones (1971) on the discussion of technology in Britain and America in the mid 19th century. Call the two countries in question A and B, and let us suppose:

$$(v_1/v_3)^A < (v_1/v_3)^B \quad \text{and} \quad (v_2/v_3)^A > (v_2/v_3)^B\,. \tag{53}$$

Namely, country B is relatively capital abundant, while country A is relatively land abundant. Then from (52-b), we may at once conclude:

$$(54) \qquad (p_1/p_2)^A > (p_1/p_2)^B ,$$

so that country A ("America") exports agricultural goods ("cotton") to country B ("Britain"), and imports manufacturing goods from B.

(ii) *The Case of Complements (for the Extreme Factors)*:

This is the case in which the extreme factors are aggregate complements of each other in every industry; i.e., $s_{ij} < 0$, $i, j = 1, 2$, and we obtain the sign pattern of the S matrix as indicated by (39). Assume again that the factor intensity condition (28) holds and that $a_{ij} > 0$ for all i and j. Under such circumstances, we obtain the signs of $[\partial w_k/\partial p_j] = [\partial x_j/\partial v_k]'$ as indicated by (40'). From (40') we may again assert:

$$(51) \qquad \varepsilon_{11} > 0 , \quad \varepsilon_{21} < 0 , \quad \varepsilon_{12} < 0 , \quad \varepsilon_{22} > 0 .$$

Hence, we once again have (52). The interpretation of (52) is, on the other hand, to be modified as follows. If a country is heavily endowed in one extreme factor relative to the middle factor (i.e., factor 3) and if she is scarcely endowed in the other extreme factor (again relative to the middle factor), then she will export the commodity which uses the relatively heavily endowed extreme factor.

We may provide an alternative interpretation of (52) which may shed some light on the curent U.S. pattern of trade. To this end, call factors 1, 2, and 3, respectively, "skilled labor," "capital," and "unskilled labor" (raw labor). Industries 1 and 2, respectively, signify exportables and importables. As mentioned earlier, we may assume that U.S. exports are skilled labor intensive (relative to raw labor), and her imports are capital intensive (relative to raw labor), i.e.,

$$a_{11}/a_{31} > a_{12}/a_{32} \quad \text{and} \quad a_{22}/a_{32} > a_{21}/a_{31} ,$$

so that our factor intensity conditions (28) is satisfied. Also, we may assume that the extreme factors, skilled labor and capital, are complements. Under such circumstances, we have (51) and we may utilize (52). Call two countries A and B, where A denotes the U.S. Then we have $(p_1/p_2)^A < (p_1/p_2)^B$ as the U.S. exports commodity 1 and imports commodity 2. Suppose that the U.S. is relatively abundant both in skilled labor and capital, i.e.,

$$(55\text{-a}) \qquad (v_1/v_3)^A > (v_1/v_3)^B ,$$

$$(55\text{-b}) \qquad (v_2/v_3)^A > (v_2/v_3)^B .$$

Then, from (52), we may infer that the fact that country A (the U.S.) exports commodity 1 and imports commodity 2 must mean that her relative abundance in skilled labor, in the sense of (55-a), "dominates" the adverse effect of her relative abundance in capital, in the sense of (55-b). If such a domination is indeed the case, the "Leontief paradox," that the U.S. imports capital intensive commodities is no longer a paradox, as it can be explained in the context of a three factor model.

APPENDIX A: COST MINIMIZATION AND THEOREM 1

The purpose of this appendix is to summarize some of the basic results in the theory of cost minimization, and provide the proof of Theorem 1 and $A'dw=dp$ found in the text.

We first focus our attention on a particular industry, and to ease the notation, we omit the subscript j which is used to denote a particular (the j-th) industry. Write its (aggregate) production function as $f(v_1, \cdots, v_m)=f(v)$, where v_i denotes the i-th factor input (*not* the factor *endowment*). Letting $x>0$ be the target level output of this commodity, and letting w be a given vector of input prices, we may write the cost minimization problem (in the usual way) as:

$$\underset{v}{\text{Minimize}}\ w'v$$
$$\text{subject to}\quad f(v)\geqq x\,,\quad v\geqq 0\,.$$

We assume all input prices are positive (i.e., $w>0$). Assuming $\partial f(v)/\partial v_i>0$ for all i at optimum and also $v_i>0$ for all i at optimum (an interior solution), the first order condition is written as:

$$\text{(A-1)}\qquad w_i=\lambda\partial f(v)/\partial v_i\,,\quad i=1, 2, \cdots, m\,,\quad f(v)=x\,,$$

where λ is the Lagrangian multiplier. Assuming that f is strictly quasiconcave, (A-1) provides a *necessary and sufficient* condition for unique global optimum.[31] The $(n+1)$ equations in (A-1) are assumed to yield $v_i(w, x)$, $i=1, 2, \cdots, m$, and $\lambda(w, x)$, where $v_i(w, x)$ signifies the demand function of the i-th factor. From (A-1), it is easy to see $\lambda>0$.

The minimum total cost function C is then defined by $C(w, x)\equiv w'v(w, x)$. By applying the Envelope Theorem (e.g., Takayama 1974, pp. 160–161, Takayama 1977, p. 20) we may readily obtain:

$$\text{(A-2)}\qquad \partial C/\partial w_i=v_i\,,\quad i=1, 2, \cdots, m\,,\quad \partial C/\partial x=\lambda\,,$$

where $\partial C/\partial w_i=v_i$ is known as *Shephard's lemma*. As is well-known, C is a concave function in w.[32] Since $v(w, x)$ is homogeneous of degree zero in w, $C(w, x)$ is homogeneous of degree one in w. Assume that C is twice continuously differentiable. Let $S^*\equiv[\partial v_i/\partial w_h]$, an $m\times m$ matrix.[33] Then, as is well-known, we have:

[31] Since $-w'x$ is concave and since $\{v\in R^n\colon v\geqq 0, f(v)\geqq x\}$ is convex due to the quasi-concavity of f, (A-1) is *sufficient* for global optimum by the Arrow-Enthoven theorem (cf. e.g., Takayama 1974, p. 111, especially the second remark). The *uniqueness* follows from the strict quasi-concavity of $f(v)$. Let $g(v)\equiv f(v)-x$. Then (A-1) is *necessary* for optimum by the Arrow-Hurwicz-Uzawa theorem, if the rank of the gradient vector of $g(v)$, i.e., $f_v\equiv[\partial f/\partial v_i\cdots, \partial f/\partial v_m]$, at optimum, is equal to one (the number of effective constraints), i.e., if f_v is non-zero vector (cf. e.g., Takayama 1974, pp. 93–94, especially condition (v)]. This is obviously satisfied since $\partial f(v)/\partial v_i>0$ for all i at optimum by assumption.

[32] See, for example, Takayama (1977, p. 21, fn. 14), Dixit-Norman (1980, p. 321). The proof used by them is, in essence, due to McKenzie (1957).

[33] The asterisk (*) is added to distinguish $[\partial w_i/\partial w_h]$ from S in the text.

(A-3a) S^* is symmetric, i.e., $\partial v_i/\partial w_h = \partial v_h/\partial w_i$ for all i and h,

(A-3b) S^* is negative semidefinite, i.e., $z'S^*z \leqq 0$ for all z,

(A-3c) $$S^*w = 0 \quad \text{and} \quad w'S^* = 0\,,$$

where (A-3b) follows from the concavity of C in w, and $S^*w = 0$ follows from the homogeneity of C in w. Shephard's lemma is used to obtain both (A-3a) and (A-3b).[34] $w'S^* = 0$ obviously follows from $S^*w = 0$ and (A-3a). From (A-3b), it also follows

$$\partial v_i/\partial w_i \leqq 0\,, \qquad i = 1, 2, \cdots, n\,.$$

Assume $z'S^*z < 0$ for all nonzero z not proportional to w.[35] Let $N^* \equiv \{z : S^*z = 0\}$, i.e., N^* is the null space of S^*. Since $S^*w = 0$ and since $z'S^*z < 0$ for all z which are not proportional to w, $N^* = \{z : z = tw,\ t \in R\}$. Thus the rank of N^* is equal to one, which in turn implies,

(A-4) $$R(S^*) = m - 1\,,$$

where $R(S^*)$ signifies the rank of S^*.[36] From this and (A-3b), it follows that the rank of the $(m-1) \times (m-1)$ matrix formed by deleting the m-th row and the m-th column of S^* is negative definite (cf. footnote 5). Thus we obtain:

(A-5) $$\partial v_i/\partial w_i < 0\,, \qquad i = 1, 2, \cdots, m\,.$$

Namely, an increase in the i-th input price reduces the usage of i-th input. Properties (A-3), (A-4) and (A-5) exhaust all the important properties of the substitution matrix S^*.

Note that in obtaining the above properties, we have *not* utilized the assumption that $f(v)$ is homogeneous of degree one (i.e., the industry production function f exhibits constant returns to scale). Now suppose further that $f(v)$ is homogeneous of degree one. In this case, the following results (known as the *Shephard-Samuelson Theorem*) hold (cf. e.g., Takayama 1977, pp. 21–22, for an exposition):

(A-6a) $$C/x = \partial C/\partial x \ (= \lambda)\,,$$

(A-6b) $$\lambda(w, x) = c(w)\,.$$

[34] By Young's theorem, $\partial^2 C/\partial w_i \partial w_h = \partial^2 C/\partial w_h \partial w_i$. Applying Shephard's lemma to this, we obtain (A-3a). Also, recalling that a function is concave if and only if its Hessian matrix is negative semidefinite, (A-3b) follows.

[35] Samuelson (1948, p. 68) imposed this condition as a condition for a "regular minimum". It may be conjectured that this condition may be replaced by the strict quasi-concavity of $f(v)$ to obtain (A-4) and (A-5) below.

[36] In general, let A be an $m \times n$ matrix. Then the set N defined by $N \equiv \{x : z \in R^n,\ Az = 0\}$ forms a subspace of R^n, and N is called the *null space* (associated with A). It is known that

$$R(N) + R(A) = n\,.$$

Thus for example, if $R(N) = 1$, then $R(A) = n - 1$. For a further discussion, see any textbook of matrix algebra.

Namely, the unit cost is equal to the marginal cost, and is independent of x (output). This, in particular, implies that *if* $f(v)$ is homogeneous of degree one, then we have,

$$C(w, x) = c(w)x \,. \tag{A-7}$$

Clearly, $c(w)$ signifies the *unit cost*. Applying Shephard's lemma to this, we at once obtain,

$$\partial c(w)/\partial w_i = v_i/x \,, \qquad i = 1, 2, \cdots, m \,. \tag{A-8}$$

Let $a_i \equiv v_i/x$, i.e., a_i is the i-th input coefficient. From (A-8), it is clear that a_i depends only on w. Namely, if the production function exhibits constant returns to scale, each input coefficient is a function of w alone, i.e., $v_i/x = a_i(w)$. Also, we may rewrite (A-8) as,

$$\partial c(w)/\partial w_i = a_i(w)(\equiv v_i/x) \,. \tag{A-9}$$

Furthermore, observing

$$\frac{x\partial^2 c(w)}{\partial w_h \partial w_i} = \frac{\partial^2 C(w, x)}{\partial w_h \partial w_i} = \frac{\partial v_i}{\partial w_h} \,,$$

from Shephard's lemma, and noting $\partial^2 c(w)/\partial w_h \partial w_i = \partial a_i(w)/\partial w_h$ from (A-9), we have:

$$\frac{\partial v_i}{\partial w_h} = x\frac{\partial a_i}{\partial w_h} \,. \tag{A-10}$$

Therefore $[x\partial a_i/\partial w_h]$ is equal to the substitution matrix S^*, whose properties are summarized in (A-3)–(A-5).

Here it may be worthwhile to recall the defitions of (Allen) substitutes and complements, which are not necessarily well understood in the literature. Factors i and h are said to be *(Allen) substitutes* (resp. *complements*) for each other, if an increase in the h-th factor price raises (resp. lowers) the use of the i-th factor, i.e., $\partial v_i(w, x)/\partial w_h > 0$ (resp. < 0). Since S^* is symmetric, i.e., $\partial v_i/\partial w_h = \partial v_h/\partial w_i$, this definition is perfectly symmetric in the sense that i and h in the above definition is interchangeable. Furthermore, if $f(v)$ is homogeneous of degree one, we have $\operatorname{sgn} \partial v_i/\partial w_h = \operatorname{sgn} \partial a_i/\partial w_h$ by (A-10). Our definitions in (35) in the text follows from the above discussion.

Now as in the text, let us suppose that there are n industries in the economy and x_j signifies the output of the i-th industry. Also, as in the text, $f_j(v_{1j}, \cdots, v_{mj})$ signifies the production function of the j-th industry which is assumed to be homogeneous of degree one ($j = 1, 2, \cdots, n$). Let c_j be the unit cost of the j-th industry, and let $a_{ij} \equiv v_{ij}/x_j$ be the i-th input coefficient of the j-th industry (i.e., the amount of the i-th factor used to produce one unit of the j-th commodity). Then from the above consideration, the c_j's and a_{ij}'s are the functions of w (input price vector) alone; i.e., $c_j = c_j(w)$ and $a_{ij} = a_{ij}(w)$.

From (A-10), we know that $[x_j\partial a_{ij}/\partial w_h]$ is the substitution matrix of the j-th industry, $[\partial v_{ij}/\partial w_h]$. Then defining s_{ih} by

$$s_{ih}\equiv \sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial w_h} x_j\,, \qquad i=1, 2, \cdots, m\,, \qquad h=1, 2, \cdots, m\,. \tag{A-11}$$

and recalling (A-3)–(A-5), the properties of the substitution matrix, we obtain the following Theorem 1 recorded in the text.

THEOREM 1. *Let* $S\equiv[s_{ih}]$. *Then* S *is symmetric and negative semidefinite with* $Sw=w'S=0$, *and* $R(S)=m-1$, *so that* $s_{ii}<0$ *for all* $i=1, 2, \cdots, m$.

To obtain $A'dw=dp$ in the text, note that the profit condition,

$$\sum_{i=1}^{m} w_i a_{ij}=p_j\,, \qquad j=1, 2, \cdots, n\,, \tag{A-12}$$

can also be written as,

$$c_j(w)=p_j\,, \qquad j=1, 2, \cdots, n\,. \tag{A-13}$$

Totally differentiating this and noting $\partial c_j/\partial w_i=a_{ij}$ from (A-9), we obtain,

$$\sum_{i=1}^{m} a_{ij}dw_i=dp_j\,, \qquad j=1, 2, \cdots, n\,. \tag{A-14}$$

Let $A\equiv[a_{ij}(w)]$ be the $m\times n$ input-output coefficient matrix. Then (A-12) may also be written as,

$$A'dw=dp\,, \tag{A-14$'$}$$

as written in the text.

APPENDIX B: ENVELOPE RESULTS

As in the text, we consider an economy producing m commodities x_j ($j=1, 2, \cdots, m$) with m factors. Letting v_{ij} denote the input of the i-th factor in the j-th industry, we write the production function constraint in each commodity as,

$$f_j(v_{1j}, v_{2j}, \cdots, v_{mj})\geqq x_j\,, \qquad j=1, 2, \cdots, n\,. \tag{B-1}$$

where each f_j is assumed to be continuously differentiable and strictly quasiconcave with decreasing and positive marginal productivity in each factor. Assuming that the economy is organized in a competitive way. Then, as is well-known (e.g., Samuelson 1953–1954, p. 10, 1967, p. 291), such an economy chooses the v_{ij}'s and x_j's so as to

$$\text{maximize} \quad y\equiv \sum_{j=1}^{n} p_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} v_{ij}\leqq v_i\,, \qquad i=1, \cdots, m\,, \quad f_j(\cdot)\geqq x_j\,, \qquad j=1, \cdots, m\,,$$

$$\text{and} \quad v_{ij} \geqq 0\,, \qquad \text{for all} \quad i \quad \text{and} \quad j\,,$$

where v_i signifies the endowment of the i-th factor. The Lagrangian of the above problem may be written as,

$$L \equiv \sum_{j=1}^{m} p_j x_j + \sum_{i=1}^{m} w_i\{(v_i - (v_{i1} + \cdots + v_{im})\} + \sum_{j=1}^{m} \mu_j[f_j(\cdot) - x_j]\,,$$

where the w_i's and μ_j's signify the Lagrangian multipliers. Write the optimal values of v_{ij}, x_j, and w_i, respectively, as $v_{ij}(v, p)$, $x_i(v, p)$, and $w_i(v, p)$, which are assumed to be continuously differentiable. We also assume an interior sulution, $v_{ij}(v, p) > 0$ for all i and j, and $x_j(v, p) > 0$ for all j. This then implies:

$$x_j(v, p) = f_j[v_{1j}(v, p), \cdots, v_{mj}(v, p)]\,, \qquad \text{for all} \quad j\,,$$

and $w_i(v, p) > 0$ for all i. Define the *national income function* $Y(v, p)$ by,

$$Y(v, p) \equiv \sum_{j=1}^{n} p_j f_j[v_{1j}(v, p), \cdots, v_{mj}(v, p)]\;.$$

Our trick here in obtaining some basic results in a remarkably simple way, is to utilize the Envelope Theorem (e.g., Afriat 1971, Takayama 1974, pp. 160–161). Applying this theorem, we at once obtain,

(B-3) $\partial Y/\partial v_i = w_i(v, p)\,, \quad i = 1, \cdots, m\,, \quad \partial Y/\partial p_j = x_j(v, p)\,, \qquad j = 1, \cdots, n\,.$

Note that this corresponds precisely to (12) in the text, where the national income function $Y(v, p)$ signifies the maximum value function of the given constrained maximization problem. Since the $w_i(v, p)$'s and $x_j(v, p)$'s are all continuously differentiable, $Y(v, p)$ is twice continuously differentiable. Hence its Hessian matrix is symmetric, i.e.,

(B-4)
$$\begin{bmatrix} Y_{vv} & Y_{vp} \\ Y_{pv} & Y_{pp} \end{bmatrix} = \begin{bmatrix} \partial w/\partial v & \partial w/\partial p \\ \partial x/\partial v & \partial x/\partial p \end{bmatrix},$$

is symmetric. Namely, the following *Samuelson reciprocity theorem* holds:

(B-5) $[\partial w/\partial p] = [\partial w/\partial p]'\,, \quad [\partial x/\partial p] = [\partial x/\partial p]'\,, \quad [\partial w/\partial p]' = [\partial x/\partial v]\,.$

(B-4) and (B-5), respectively, correspond precisely to (13) and (14) in the text.

Next, noting that the maximum function y is linear (and hence convex) in p, we can at once establish $Y(v, p)$ is convex in p. Also noting that constraint functions are linear (and hence concave) in v, we can at once establish $Y(v, p)$ is concave in v, where we may note that the set,

$$\left\{(v_{11}, \cdots, v_{m1}, v_{12}, \cdots, v_{mn}; v_1, \cdots, v_m)\colon \sum_{j=1}^{m} v_{ij} \leqq v_i\,, f_j(\cdot) \geqq x_j\,, \text{for all } i \text{ and } j\right\},$$

is convex since the f_j's are quasi-concave.[37] Thus $Y_{pp} = [\partial x_i/\partial p_j]$ is positive

[37] See Dixit-Norman (1980), pp. 321–322, for example.

semidefinite as well as symmetric, and $Y_{vv}=[\partial w_i/\partial v_j]$ is negative semidefinite (as well as symmetric). From this we may conclude:

(B-6) $\quad \partial x_j/\partial p_j \geqq 0 \quad$ for all j, and $\partial w_i/\partial v_i \leqq 0 \quad$ for all i.

Also, from the above maximization problem it is evident that $Y(v, p)$ is homogeneous of degree one in p, so that $Y_p = x(v, p)$ is homogeneous of degree zero in p. Then $[\partial x/\partial p]p = 0$.

APPENDIX C: ON THE MAGNIFICATION EFFECT[38]

In a recent paper, Jones-Easton (1981) extended the Jones (1965) magnification effect for the 2×2 case to the 3×2 case. The purpose of this appendix is to exposit the basic structure of the magnification effect being stimulated by Jones-Easton (1981).

As is illustrated in (27) for the 2×2 case, the magnification effects have two aspects, the $(p-w)$ price effect and the $(x-v)$ quantity effect, where the former is obtained from $[\partial w/\partial p]p = w$ and the latter is obtained from $[\partial x/\partial v]v = x$. Since $[\partial x/\partial v] = [\partial w/\partial p]'$, it suffices to consider either one of the $(p-w)$ and the $(x-v)$ effects. Here we shall focus our attention on the $(p-w)$ price effect.

Assume $n=2$, we may obtain, from $[\partial w/\partial p]p = w$, the following relation:

(C-1) $\quad \partial \log w_i/\partial \log p_1 + \partial \log w_i/\partial \log p_2 = 1\,, \qquad i = 1, 2, \cdots, m.$

As can be seen from the subsequent discussion, this relation forms the foundation of the magnification results.

Assume $m=3$ as in Jones-Easton (1981). In the specific factor model, we may easily obtain, from (C-1) and (38′) of the text.

(C-2) $$\partial \log w_i/\partial \log p_i > 1\,, \qquad i=1, 2\,,$$

(C-3) $$0 < \partial \log w_3/\partial \log p_i < 1\,, \qquad i=1, 2\,.$$

From this, we may easily conclude:

(C-4) $$\hat{w}_1 > \hat{p}_1 > \hat{w}_3 > \hat{p}_2 > \hat{w}_2\,, \qquad \text{if} \quad \hat{p}_1 > \hat{p}_2 \quad \text{and} \quad \hat{v}_1 = \hat{v}_2 = \hat{v}_3 = 0\,,$$

which corresponds to (3.2) of Jones-Easton (1981).

In the case of complements for the extreme factors, we again obtain (C-2) from (C-1) and (40′), while we do not obtain (C-3). Using (C-2), we may conclude:

(C-5) $$\hat{w}_1 > \hat{p}_1 > \hat{p}_2 > \hat{w}_2\,, \qquad \text{if} \quad \hat{p}_1 > \hat{p}_2 \quad \text{and} \quad \hat{v}_1 = \hat{v}_2 = \hat{v}_3 = 0\,,$$

while nothing can be said about the signs of $(\hat{w}_3 - \hat{p}_1)$ and $(\hat{w}_3 - \hat{p}_2)$. Relations (C-4) and (C-5) are comparable to (27-a), the magnification effect for the two factor case.

Another interesting case is:

[38] The discussion here is based on Takayama (1981c).

(C-6) $$\partial \log w_i/\partial \log p_i > \partial \log w_3/\partial \log p_i\,, \qquad i=1, 2\,.$$

For $\hat{p}_1 > \hat{p}_2$ and $\hat{v}_1 = \hat{v}_2 = \hat{v}_3 = 0$, we may conclude from (C-1) that

(C-7) $$\hat{w}_1 > \hat{w}_3 > \hat{w}_2\,,$$

holds if and only if (C-6) holds. This case corresponds to the *better substitute* case of Jones-Easton (1981).

The above discussion should reveal that the magnification results depends crucially on the assumption of $n=2$ (two commodities) and the specification of extreme factors. In (C-4)–(C-7), facrors 1 and 2 are designated as the extreme factors.

To probe more deeply into the key nature of the two commodity assumption, we define ε_i by,

(C-8) $$\varepsilon_i \equiv \partial \log w_i/\partial \log p_1\,, \qquad i=1, 2, \cdots, m\,,$$

so that $1-\varepsilon_i \equiv \partial \log w_i/\partial \log p_2$, $i=1, 2, \cdots, m$, by (C-1). Combining this with (C-1) and assuming that v is a fixed vector, we obtain:

(C-9a) $$\hat{w}_i - \hat{p}_1 = (\varepsilon_i - 1)(\hat{p}_1 - \hat{p}_2)\,, \qquad i=1, 2, \cdots, m\,,$$

(C-9b) $$\hat{w}_i - \hat{p}_2 = \varepsilon_i(\hat{p}_1 - \hat{p}_2)\,, \qquad i=1, 2, \cdots, m\,,$$

(C-9c) $$\hat{w}_i - \hat{w}_k = (\varepsilon_i - \varepsilon_k)(\hat{p}_1 - \hat{p}_2)\,, \qquad i, k=1, 2, \cdots, m\,,$$

where $\hat{w}_i \equiv dw_i/w_i$, etc., and where (C-9c) follows at once from (C-9a) and (C-9b).

One important case is the one in which $\varepsilon_1 > 0$ and $\varepsilon_2 < 0$. This assumption is satisfied when $m=2$ as in the H-O-S model (cf. (26) of the text): For the three factor case, the assumption of $\varepsilon_1 > 0$ and $\varepsilon_2 < 0$ is satisfied in the specific factor model or in the case of complements for the extreme factors, since we have (C-2) in these cases. Assuming $\varepsilon_1 > 0$ and $\varepsilon_2 < 0$, we may easily conclude from (C-9) that, for $\hat{p}_1 > \hat{p}_2$ and $\hat{v}_1 = \cdots = \hat{v}_m = 0$,

(C-10a) $$\hat{w}_1 > \hat{p}_1 > \hat{p}_2 > \hat{w}_2\,;$$

(C-10b) $$\hat{w}_1 > \hat{w}_i \ \text{ iff } \ \varepsilon_1 > \varepsilon_i\,; \quad \hat{w}_i > \hat{w}_2 \ \text{ iff } \ \varepsilon_i > \varepsilon_2\,, \qquad i=1, 2, \cdots, m\,,$$

which holds for any $m \geqq 2$. This then generalizes (C-4) and (C-5) for the three factor case (as well as (27-a) in the text for the two factor case) to the m factor case.

The natural extension of condition (C-6) would be:

(C-11) $$\partial \log w_i/\partial \log p_i > \partial \log w_k/\partial \log p_i\,, \qquad i=1, 2\,, \quad k=3, 4, \cdots, m\,.$$

This is equivalent to assuming,

(C-11′) $$\varepsilon_1 > \varepsilon_k > \varepsilon_2\,, \quad k=3, 4, \cdots, n\,.$$

Substituting this into (C-9), we may conclude that this holds *if and only if*,

(C-12) $$\hat{w}_1 > \hat{w}_k > \hat{w}_2\,, \qquad \text{for } \ \hat{p}_1 > \hat{p}_2 \ \text{ and } \ \hat{v} = 0\,, \quad k=3, 4, \cdots, m\,,$$

which generalizes Jones-Easton's "better substitute case," (C-7), to the general m factor case in which m is *not* restricted to three.

APPENDIX D: EMPIRICAL PROCEDURES[39]

Here we shall survey some of the procedures used to determine the comparative advantage structure of the U.S. trade. Various factors are aggregated into a single factor (say, "capital" or "labor") by taking a value sum. Similarly, various commodities are aggregated into a single commodity again taking a value sum. Although this simple aggregation procedure can be criticized in terms of more sophisticated indexes and aggregation techniques which has become available recently (e.g., Kendrick-Vaccara 1980, Diewert 1980),[40] none of such techniques have yet been used in determining the commodity structure of international trade.[41]

In any case, as a result of aggregation, the same commodity is typically exported *and* imported. Let X_j and M_j, respectively, denote the exports and the imports of the j-th commodity. If $X_j - M_j > 0$ (resp. < 0), a particular country or region is a net exporter (resp. net importer) of commodity j.

Let b_{kj} be the amount of the k-th commodity used to produce one unit of the j-th commodity ($k, j = 1, 2, \cdots, n$), and let $B \equiv [b_{kj}]$, an $n \times n$ matrix. Let X and M be (column) vectors whose j-th elements are X_j and M_j, respectively. Then the "*net* export" and the "*net* import" vectors may be defined as:

$$X^* \equiv [I - B]^{-1} X, \qquad M^* \equiv [I - B]^{-1} M .$$

Let a_{ij} be the amount of the i-th factor used to produce one unit of the j-th commodity ($i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$). Then the net i-th factor content of the export and the import of the j-th commodity may, respectively, be expressed as:

$$x_{ij} \equiv a_{ij} X_j^* \quad \text{and} \quad m_{ij} \equiv a_{ij} M_j^* .$$

Using these concepts, there are three important methods which are used to determine the relation between factor intensities and the trade pattern.

(a) *The Leontief Method*:

The method which is used in Leontief (1953) to reveal the Leontief paradox and followed by many others (e.g., Baldwin, 1971, 1979), is to compute the ratio of the factors contained in exports and imports. To compute such a ratio, define x_i and

[39] This appendix is based on a joint study by Mr. Kouichi Murayama and myself undertaken during the spring of 1981.

[40] The recent development in index number theory and aggregation techniques have been applied to the measurement of productivity growth by Christensen-Cummings-Jorgenson (1980), Gollop-Jorgenson (1980), and others. For a survey of recent developments of index number theory, see Takayama (1981b).

[41] An interesting research project would be to replicate the previous studies in determining the comparative advantage structure of trade (as summarized below) using more sophisticated aggregation techniques such as the Törnquivst index.

m_i by,

$$x_i \equiv \sum_{j=1}^{n} x_{ij} \quad \text{and} \quad m_i \equiv \sum_{j=1}^{n} m_{ij},$$

which, respectively, signify the i-th factor content of a particular country's (or region's) exports and imports.

Let $i = K, L$, respectively, signify capital and labor. Leontief (1953) observed for the 1947 U.S. data:

$$m_K/m_L = 1.30\ x_K/x_L \text{ (so that } m_K/m_L > x_K/x_L).$$

Thus, U.S. exports require *less* capital per worker than do U.S. import-competing goods. In terms of the two-factor Heckscher-Ohlin theory, the U.S. trade pattern is that of a labor rich country (the "Leontief paradox"). Concerned with possible distortions in the U.S. trade pattern just after the war, Leontief (1956) repeated the same computation using the 1951 data to obtain:

$$m_K/m_L = 1.06\ x_K/x_L.$$

A repeated test for 1962 by Baldwin (1971, p. 134) again yielded the paradoxical result, almost as strong as ever, i.e.,

$$m_K/m_L = 1.27\ x_K/x_L.$$

Furthermore, Bowen-Aho-Rousslang (1980) produced a rather dramatic figure (their Figure 1) depicting the time paths of (m_K/m_L) and (x_K/x_L) for the period of 1961–1977, in which the former is always larger than the latter, during this period.

Baldwin (1971) further computed similar intensity ratios for other "factors." Namely, for factor i, he computed the following ratio:

$$(m_i/m_L)/(x_i/x_L).$$

He found that these ratios are less than unity when i represents various measures of education or proportions of engineers and scientists (Baldwin 1971, p. 134). This means that U.S. exports require *more* "human capital" per worker than do U.S. import-competing industries.

(b) *The Correlation Method*:

Another interesting method is to compute the correlation between the export-import factor content ratio and the endowment ratio. Namely, letting $\tau_r \equiv (m_K/m_K)^r/(x_K/x_L)^r$ be the import-export ratio of capital intensity in country (or region) r, and letting $(K/L)_r$ denote the capital-labor endowment ratio of country r, we may compute the correlation ρ between these, i.e.,

$$\rho[\tau_r, (K/L)_r],$$

where each r (country) represents a single observation. If such a correlation ρ is positive, then it would confirm the Leontief paradox. However, Baldwin (1979, p. 42) reported that ρ is significantly (at 1% level) negative when we use the U.S.

and the EEC coefficients, which in turn seem to refute the Leontief paradox and substantiate the H-O theory. On the other hand, Baldwin continues to state (1979, p. 42): "When, however, Hufbauer's list, which contains only 6 developing countries is expanded to 34 countries including 15 developing nations.... The Heckscher-Ohlin theory is substantiated only with Japanese coefficients." Namely, the attempt to determine the validity or invalidity of the Leontief paradox via the correlations method turns out to be inconclusive.

(c) *The Linear Regression Method*:

An important drawback of the previous two methods arises when there are more than two factors of productions. Under such circumstances there are obviously more than two factor intensity ratios. In this case, it would be desirable to capture the impact of multi factor-intensity ratios *simultaneously*. To cope with this problem, Baldwin (1971, 1979) used a multiple regression method in which he estimated a linear regression between the (normalized) net trade flow and the set of intensity ratios by industry. More specifically, let v_{ij} denote the i-th factor used in the j-th industry. Namely, if Y_j denotes the output of the j-th industry, then we have,

$$v_{ij} \equiv a_{ij} Y_j \, .$$

Let θ_{ij} denote the intensity ratio of the i-th factor (relative to labor) in the j-th industry: i.e.,

$$\theta_{ij} \equiv v_{ij}/v_{Lj} \, .$$

Let $(X_j - M_j)$ signify the ("normalized") net trade flows of the j-th industry. Then the multiple regression equation may be specified as:

$$X_j - M_j = \beta_0 + \beta_1 \theta_{ij} + \beta_2 \theta_{2j} + \beta_3 \theta_{3j} + \cdots ,$$

where each j represents one observation, Baldwin (1971, p. 137) discovered that there is a significant negative relationship between the net trade flow and the capital-labor intensity ratio (which conforms with Leontief's paradoxical discovery), while he also discovered a significant positive relationship between the percentage of engineers and scientists, craftsman, workers with more than a high school education. Baldwin's 1979 regression study confirms his earlier study: i.e., "the trade balance by industry is negatively related to the capital ratio and positively correlated with the proportion of craftsman and foremen..." (p. 44). A similar multiple regression test is also used to study the interregional trade pattern of the U.S. by Horiba-Kirkpatrick (1981).

Kyoto University and Southern Illinois University at Carbondale

REFERENCES

Afriat, S. N. (1971), "Theory of Maxima and the Method of Lagrange," *SIAM Journal of Applied Mathematics*, 20, May, 343–357.

Amano, A. (1977), "Specific Factors, Comparative Advantage, and International Investment," *Economica*, 44, May, 131–144.

Baldwin, R. E. (1971), "Determinants of the Commodity Structure of U. S. Trade," *American Economic Review*, 61, March, 126–146.

Baldwin, R. E. (1979), "Determinants of Trade and Foreign Investment: Further Evidence," *Review of Economics and Statistics*, 51, February, 40–48.

Batra, R. N. and Casas, R. R. (1976), "A Synthesis of the Heckscher-Ohlin and the Neoclassical Model of International Trade," *Journal of International Economics*, 6, February, 21–38.

Berndt, E. R. and L. R. Christensen (1974), "Testing for the Existence of a Constant Aggregate Index of Labor Inputs," *American Economic Review*, 64, June, 391–404.

Berndt, E. R. and D. O. Wood (1975), "Technology Prices, and the Demand for Energy," *Review of Economics and Statistics*, 57, August, 259–268.

Bowen, H. P., C. M., Aho, and D. Roussang, (1980), "U.S. Trade in Factor Services, Revealed Resource Abundance and the Impact of Changing Patterns of International Trade on Domestic Resource Inputs," Presented at the San Diego Meeting of the Western Economic Association, June.

Branson, W. H. and N. Monoyios (1977), "Factor Inputs in U.S. Trade," *Journal of International Economics*, 7, May, 111–131.

Caves, R. E. and R. W. Jones (1981), *World Trade and Payments: An Introduction*, Boston, Little, Brown & Co., 3rd Edition.

Chang, W. W. (1979), "Some Theorems on Trade and General Equilibrium with Many Goods and Factors," *Econometrica*, 47, May, 709–726 (presented at the Atlantic City Meeting of the Econometric Society, September 1976).

Chipman, J. S. (1969), "Factor Price Equalization and the Stolper-Samuelson Theorem," *International Economic Review*, 10, October, 399–406.

Christensen, L. R., D. Cummings, and D. W. Jorgenson (1980), "Economic Growth, 1947–73: An International Comparison," in Kendrick and Vaccara (eds.), *New Development in Productivity Measurement and Analysis*, 595–691.

Diewert, W. E. (1974), "Applications of Duality Theory," in M. Intrilligator and D. Kendrick (eds.), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam, North-Holland, 106–171.

Diewert, W. E. and D. Woodland (1977), "Frank Knight's Theorem in Linear Programming Revisited," *Econometrica*, 45, March, 375–398.

Diewert, W. E. (1980), "Aggregation Problems in the Measurement of Capital," in Usher (ed.), *The Measurement of Capital*, 433–528.

Dixit, A. K. and V. Norman (1980), *Theory of International Trade*, Cambridge, Cambridge University Press.

Drabicki, J. Z. and A. Takayama (1979), "An Antinomy in the Theory of Comparative Advantage," *Journal of International Economics*, 9, May, 211–223.

Egawa, I. (1978), "Some Remarks on the Stolper-Samuelson and Rybczynski Theorems," *Journal of International Economics*, 8, November 1978, 525–536.

Ethier, W. (1974), "Some of the Theorems of International Trade with Many Goods and Factors," *Journal of International Economics*, 4, May, 199–206.

Falvey, R. E. (1979), "Specific Factors, Comparative Advantage and International Investment: An Extension," *Economica*, 46, February, 77–82.

Fuss, M. and D. McFadden (ed.) (1978), *Production Economics: A Dual Approach to Theory and Applications*, Amsterdam, North-Holland, Vols. 1 and 2.

Gale, P. and H. Nikaido (1965), "The Jacobian Matrix and Global Univalence of Mappings," *Mathematische Annalen*, 159, 1965, 89–93.

Gollop, F. M. and D. W. Jorgenson (1980), "U.S. Productivity Growth by Industry, 1947–73," in Kendrick and Vaccara (eds.), *New Developments in Productivity Measurement and Analysis*, 17–124.

Griliches, Z. (1980), "Return to Research and Development Expenditures in the Private Sector," in Kendrick and Vaccara (eds.), *New Developments in Productivity Measurement and Analysis*, 419–454.

Gruber, W., Metha, D. and Vernon, R. (1967), "The R & D Factors in International Trade and International Investment of United States Industries," *Journal of Political Economy*, 45, February, 20–37.

Haberler, G. (1936), *The Theory of International Trade*, London, William Hodge (translated from German, *Internationale Handel*, Berlin, 1933).

Harkness, J. and J. Kyle (1975), "Factors Influencing United States Comparative Advantage," *Journal of International Economics*, 5, May 153–165.

Harrod, R. F. (1957), *International Economics*, London, Nisbet, 4th ed. (1st ed. 1933).

Hillman, A. L. and C. W. Ballard III (1978), "Energy, the Heckscher-Ohlin Theorem, and U.S. International Trade," *American Economic Review*, 68, March, 96–106.

Horiba, Y. (1974), "General Equilibrium and the Heckscher-Ohlin Theory of Trade: The Multi-Country Case," *International Economic Review*, 15, June, 440–449.

Horiba, Y. and R. C. Kirkpatrick (1981), "Factor Endoments, Factor Proportions, and the Allocative Efficiency of U.S. Interregional Trade," *Review of Economics and Statistics*, LXII, May, 178–187.

Hufbauer, G. C. (1970), "The Impact of National Characteristics and Technology on the Commodity Composition of Trade in Manufactured Goods," in R. Vernon (ed.), *The Technology Factor in International Trade*, New York, National Bureau of Economic Research, 145–272.

Jones, R. W. (1965), "The Structure of Simple General Equilibrium Models," *Journal of Political Economy*, 73, December, 557–572.

Jones, R. W. (1971), "A Three-Factor Model in Theory, Trade and History," in *Trade, Balance of Payments, and Growth: Papers in International Economics in Honor of Charles P. Kindleberger*, ed. by J. N. Bhagwati, R. W. Jones, R. A. Mundell, and J. Vanek, Amsterdam, North-Holland, 49–65.

Jones, R. W. and S. Easton (1981), "Factor Intensities and Fractor Substitution in general equilibrium," presented at the New Orleans Meeting of the Southern Economic Association, November, 1981.

Jones, R. W. and J. A. Scheinkman (1977), "The Relevance of the Two-Sector Production Model in Trade Theory," *Journal oj Political Economy*, 85, October, 909–935.

Keesing, D. B. (1967), "The Impact of Research and Development on United States Trade," *Journal of Political Economy*, 75, February, 38–48.

Keesing, D. B. (1968), "Labor Skills and the Structure of Trade in Manufacturing." in P. Kenen and R. Lawrence (eds.), *The Open Economy: Essays in International Trade and Finance*, New York, 3–18.

Kemp, M. C. and L. L. F. Wegge (1969), "On the Relation between Commodity Prices and Factor Rewards," *International Economic Review*, 10, October 1969, 407–413.

Kendrick, J. W. and B. N. Vaccara, eds. (1980), *New Developments in Productivity Measurement and Analysis*, Chicago, University of Chicago Press.

Kenen, P. B. (1965), "Nature, Capital, and Trade," *Journal of Political Economy*, 73, October, 437–460.

Kravis, I.B. (1956), "Wages and Foreign Trade," *Review of Economics and Statistics*, 38, February, 14–30.

Kravis, I. B. (1976), "A Survey of International Comparisons of Productivity," *Economic Journal*, 86, March, 1–44.

Leamer, E. E. (1980), "The Leontief Paradox Reconsidered," *Journal of Political Economy*, 88, June, 495–503.

Leontief, W. W. (1953), "Domestic Production and Foreign Trade: The American Capital Position Reexamined," *Proceedings of the American Philosophical Society*, 97, September, 332–349.

Leontief, W. W. (1956), "Factor Proportions and the Structure of American Trade: Further Theoretical and Empirical Analysis," *Review of Economics and Statistics*, 38, November, 386–407.

Linder, S. B. (1961), *An Essay on Trade and Transformation*, New York, John Wiley & Sons.

Mas-Collel, A. (1979), "Two Propositions on the Global Univalence of System of Cost Function," in *General Equilibrium, Growth, and Trade, Essays in Honor of Lionel McKenzie*, ed. by J. R. Green and J. A. Scheinkman, New York, Academic Press, 323–331.

Mayer, W. (1974), "Short-Run and Long-Run Equilibrium for a Small Open Economy," *Journal of Political Economy*, 82, September/October, 955–967.

McKenzie, L. W. (1955), "Equality of Factor Prices in World Trade," *Econometrica*, 23 July.

McKenzie, L. W. (1957), "Demand Theory without a Utility Index," *Review of Economic Studies*, 24, June 1957, 185–189.

McKenzie, L. W. (1969), "The Theory of International Trade: II, Mathematical Theory," in *The International Encyclopedia of the Social Sciences*, New York, Free Press.

Mussa, M. (1974), "Tariffs and the Distribution of Income: The Importance of Factor Specificity, Substitutability and Intensity in the Short and Long-Run," *Journal of Political Economy*, 82, November/December, 1191–1203.

Ray, E. J. (1981a), "The Determinants of Tariff and Nontariff Restrictions in the U.S.," *Journal of Political Economy*, 89, February, 105–121.

Ray, E. J. (1981b), "Tariff and Nontariff Barriers to Trade in the U.S. and Abroad," *Review of Economics and Statistics*, May, 161–177.

Pope, C. (1972), "The Impact of the Ante-Bellum Tariff on Income Distribution," *Explorations in Economic History*, Summer, 375–421.

Rybczynski, T. M. (1955), "Factor Endowments and Relative Commodity Prices," *Economica*, XXII, November.

Ruffin, R. J. (1981), "Trade and Factor Movements with Three Factors and Two Goods," *Economics Letters*, 7, 177–182 (the original version is dated July 1980).

Samuelson, P. A. (1947), *Foundations of Economic Analysis*, Cambridge, Mass., Harvard University Press.

Samuelson, P. A. (1953–54), "Prices of Factors and Goods in General Equilibrium," *Review of Economic Studies*, 21, 1–20.

Samuelson, P. A. (1967), "Summary on Factor Price Equilization," *International Economic Review*, 7, October 1967, 34–41.

Samuelson, P. A. (1971), "Ohlin Was Right," *Swedish Journal of Economics*, 73, December, 365–384.

Stern, R. M. (1974), "The U.S. Tariff and the Efficiency of the U.S. Economy," *American Economic Review*, 64, May, 459–470.

Stern, R. M. (1975), "Testing Trade Theories," in P. B. Kenen (ed.), *International Trade and Finance: Frontiers for Research*, Cambridge, Cambridge University Press, 3–49.

Stern, R. M. (1976), "Some Evidence on the Factor Content of West Germany's Foreign Trade," *Journal of Political Economy*, 84, February, 131–141.

Stolper, W. F. and Samuelson, P. A. (1941), "Protection and Real Wages," *Review of Economic Studies*, IX, November, 58–73.

Suzuki, K. (1980), "The Pattern of International Trade and Factor Movements," presented at the annual meeting of the Japan Association of Economics and Econometrics, October.

Suzuki, K. (1981), "A Synthesis of the Heckscher-Ohlin and the Neoclassical Models of International Trade: A Comment," to appear in *Journal of International Economics*.

Takayama, A. (1964), "Economic Growth and International Trade," *Review of Economic Studies*, XXI, July, 206–220 (taken from a chapter of a Ph.D. thesis submitted to the University of Rochester, March 1962).

Takayama, A. (1972), *International Trade*, New York, Holt, Rinehart and Winston.

Takayama, A. (1974), *Mathematical Economics*, Hinsdale, Ill., Dryden Press. 2nd ed. is to appear from Cambridge University Press.

Takayama, A. (1977), "Sensitivity Analysis in Economic Theory," *Metroeconomica*, XXIX, Gennaio-Dicembre, 9–37.

Takayama, A. (1981a), "On Some Theorems of General Competitive Equilibrium of Production and International Trade," (unpublished manuscript), February.

Takayama, A. (1981b), "On Index Numbers," (unpublished manuscript), July.

Takayama, A. (1981c), "Comments on 'Factor Intensities and Factor Substitution in General Equilibrium,' by R. W. Jones and S. Easton," presented at the New Orleans Meeting of the Southern Economic Association, November, 1981.

Terleckyj, N. (1974), *Effects of R & D on the Productivity Growth of Industries: An Explanatory Study*, Washington, National Planning Association.

Travis, W. P. (1964), *The Theory of Trade and Protection*, Cambridge, Mass., Harvard University Press.

Uekawa, Y. (1971), "On the Generalization of the Stolper-Samuelson Theorem," *Econometrica*, 39, March, 197–217.

Uekawa, Y. (1979), "On the Concepts of Factor Intensities and the Relation between Commodity Prices and Factor Rewards," in *General Equilibrium, Growth, and Trade, Essays in Honor of Lionel McKenzie*, ed. by J. R. Green and J. A. Scheinkman, New York, Academic Press, 333–346.

Usher, D. (ed.) (1980), *The Measurement of Capital*, Chicago, University of Chicago Press.

Vanek, J. (1959), "The Natural Resource Content of Foreign Trade, 1870–1955, and the Relative Abundance of National Resources in the United States," *Review of Economics and Statistics*, May, 146–153.

Vanek, J. (1963), *The Natural Resource Content of United States Foreign Trade*, Cambridge, Mass., MIT Press.

Vanek, J. (1968), "The Factor Proportions Theory; the *N*-Factor Case," *Kyklos*, 21, October, 749–756.