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THE POLITICAL BUSINESS CYCLE UNDER THE MINIMUM TIME OBJECTIVE*

David CHAPPELL

Abstract A modified version of the Nordhaus model of the political business cycle is analysed under the assumption that the government's objective is to get re-elected with a given (increased) majority in the minimum time possible. The problem is modelled as a time-optimal control problem and some results from the time-optimal control literature are given. Applicability of this type of analysis to other policy problems is briefly discussed.

1. INTRODUCTION

In an interesting paper Nordhaus [4] developed a model of a political theory of the business cycle. The crucial feature of this model is that votes cast for the government in democratic elections are (inversely) related to such economic variables as the rate of inflation and the level of unemployment. The macro-economic framework within which the government is assumed to follow vote-maximising policies is described by an augmented Phillips curve and the 'adaptive' formation of the expected rate of inflation. Given this framework and the fact that democratically elected governments must generally seek re-election after, say, four or five years, the two most important implications of the model are that:

(i) The unemployment level will be (instantaneously) raised immediately after an election and thereafter steadily reduced until the next election.

(ii) The unemployment level immediately prior to an electon will be lower (and the inflation rate correspondingly higher) than the welfare optimum.

Recently, Chappell and Peel [1] have extended the Nordhaus model to allow for an endogenous inter-election period. This modification is pertinent to such economy's as the United Kingdom where the government is free to call an election at any time subject to an upper time limit. Chappell and Peel also examine the implications of alternative objective functions and show that the Nordhaus implications are a result of the particular form of objective function employed.

The purpose of the present paper is to derive the optimal policy for a government which is elected with a "small" majority. Such a government will be in a somewhat precarious position because, for example, of the possibility of sickness and/or death of its members which could result in it losing its majority. It is assumed that the government wishes to pursue a policy which will enable it to gain

* The author is grateful to D. A. Peel for helpful discussions but the usual caveat applies.

re-election with a more substantial majority *in the minimum possible time*. For a historical example the reader's attention is drawn to the election and re-election of the Labour government in the U.K. in October 1964 (majority 5) and March 1966 (majority 97) respectively.

Organisation of the paper is as follows. In the following section the Nordhaus model and its results are briefly re-stated. In section three the present problem is formally posed as a time-optimal control prolem and some results from the time-optimal control literature are stated. In section four we derive a solution to the problem for a particular (linear) vote function and in the final section (section five) we offer some concluding remarks.

2. THE NORDHAUS MODEL

Votes (w_T) cast for the government in an election at time T are given by

$$w(T) = -\int_0^T (u^2 + \beta p)\, e^{rt} dt \tag{1}$$

where u=unemployment level (%); p=rate of inflation (%); T=time between elections; β and r are positive constants.

The augmented Phillips curve is

$$p = \alpha_0 - \alpha_1 u + v \tag{2}$$

where v is the expected rate of inflation[1] and α_0 and α_1 are positive constants. The expected rate of inflation is the result of an adaptive process such that

$$\dot{v} = \phi(p - v) \tag{3}$$

where ϕ is a positive constant.

Thus the Nordhaus problem is to maximise (1) subject to (2) and (3). Mathematically this is a fairly simple optimal control problem and the optimal time path of the control variable (the level of unemployment) is

$$u(t) = \frac{\alpha_1 \beta}{2r} (r - \phi + \phi e^{r(T-t)}) \tag{4}$$

From this, it is clear that the level of unemployment falls monotonically throughout the government's period of office. Whilst the problem as posed is mathematically valid it seems somewhat unreal from an economists point of view that the government can *directly* manipulate the unemployment level (i.e. use this as a control variable); Nordhaus also recognised this point.[2] It seems more

[1] Nordhaus uses the slightly more general form $p = \alpha_0 - \alpha_1 u + \lambda v$ where $\lambda \in (0, 1)$, i.e. he allows for some money illusion. For simplicity however we focus on the case where $\lambda = 1$ and the long run trade-off is vertical.

[2] A further point here is that $v(t)$ must be treated as a state variable and thus we must assume that the government 'knows' the initial rate of expected inflation, $v(0)$, since this is needed to completely solve the problem posed.

reasonable to assume that the incoming (or re-elected) government 'inherits' both an initial level of unemployment and an initial inflation rate. In this way we can rule out instantaneous discrete jumps in these variables which, in any case, are not observable in practice.

A more plausible specification, that we shall employ in what follows, may be obtained by differentiating (2) with respect to time and thus eliminating v and $\dot{v}$. By doing this we do not lose the 'spirit' of the Nordhaus macro-economy but our constraint set may now be written

$$\dot{u}=\phi(u_N-u)-\theta/\alpha_1;\ u(0)=u_0 \tag{5}$$

$$\dot{p}=\theta;\ p(0)=p_0 \tag{6}$$

where u_N (the 'natural rate' of unemployment) $=-\alpha_0/\alpha_1$. Thus we now have the more plausible hypothesis that the government may control the rate of change of the inflation rate perhaps by its monetary policy, and takes the initial values of u and p as historically given.

3. THE MINIMUM TIME PROBLEM

We commence this section with a brief statement of some results from the time-optimal control literature which are needed to solve the present problem. No proofs are included but the presentation here closely follows that of Lee and Markus [3] to which the interested reader is referred for formal proofs of all the statements made here.[3]

Consider the autonomous linear process in R^n represented by the vector differential equation

$$\dot{x}=Ax+bw \tag{7}$$

where x $(=x(t))$ is an n-vector, A is a constant $n\times n$ matrix, b is a constant n-vector and w $(=w(t))$ is a scalar which is bounded by c (a positive constant) in absolute value. Let $G(x(T))\leqslant 0$ be a given closed bounded convex set in R^n. Then the time-optimal control problem may be stated as follows. Given a vector of initial conditions $x(0)=x_0$ how should $w(t)$ be chosen with $|w(t)|\leqslant c$ on $t\in[0, T]$ such that the corresponding solution $x(t)$ of (7) defines a trajectory in R^n which reaches G at $t=T$ where T is a minimum?

Introduce the n-vector of co-state variables $\lambda(t)$ and the function $H(x, \lambda, w)$ defined by

$$H(x, \lambda, w)=\lambda\cdot(Ax+bw)$$

and consider the differential equation system

[3] See Lee and Markus [3] Chapter 2, particularly Theorems 2, 5, 8, 10, 19 and 20. A somewhat less formal, though more 'readable,' treatment together with some useful examples may be found in Harvey [2].

$$\dot{x}=\frac{\delta H}{\delta\lambda}=Ax+bw\,; \qquad x(0)=x_0 \tag{7^1}$$

$$\dot{\lambda}=-\frac{\delta H}{\delta x}=-\lambda A\,. \tag{8}$$

Let us assume that:

(a) All eigenvalues of A are real and non-positive.

(b) The matrix $(b, Ab, A^2b, \cdots, A^{n-1}b)$ has rank n.

(c) There exists a nowhere-zero, non-trivial solution to the vector differential equation (8) which also satisfies the boundary (transversality) conditions:

$$\lambda_i(T)=-\delta G(x(T))/\delta X_i(T)\,; \qquad i=1, 2, \cdots, n\,.$$

Then by some theorems of Lee and Markus (*op. cit.*):

(d) The process is controllable.

(e) There exists a unique optimal controller $w^*(t)$ steering $x(0)$ to the boundary of G in minimum time.

(f) $w^*(t)=c\cdot\text{sgn}\,(\lambda(t)\cdot b)$. Thus $w^*(t)$ is almost always piecewise constant with values $\pm c$ and only a finite number of discontinuities (called switches).

(g) $w^*(t)$ has at most $n-1$ switches on $t\in[0, \infty)$.

We are now ready to focus attention on the dynamic process given in (5) and (6). The reader will notice that this does not conform with the general class of problems treated above due to the presence of the constant term (ϕu_n) in equation (5). Making the simple transformation of coordinates, $y=u-u_n$, equation (5) becomes $\dot{y}=-\phi y-\theta/\alpha_1$, $y(0)=u_0-u_n$, and the constant is suppressed. Clearly apart from the change of origin in the u-direction the two system are equivalent and all the results given above are applicable to either process. Thus we will continue to work in (u, p) space for purposes of clarity.

In order to fit the present problem into the standard framework outlined above it remains to specify a suitable target (vote function) and to impose some restraints on the control variable, θ. For the former we suppose that votes cast for the government in an election at time T are an inverse function of the level of unemployment at time T and the *expected* rate of inflation at time T. The reasons for regarding the expected rather than the actual rate of inflation as the appropriate variable are two fold. Firstly, there is invariably a publication lag so that the official figure for the actual inflation rate at time t is not published until some time $(t+h)$ say. (Clearly, this is also true of the unemployment figures but here the publication lag tends to be smaller and the figures less volatile.) Secondly, whilst the official figures for the unemployment rate are widely accepted the same cannot be said of the published inflation figures. At least in the U.K. a bewildering plethora of official figures are published and it does not seem unreasonable to suppose that the electorate forms a "view" of the inflation rate at election time and votes on the basis of this view or expectation. However, the reader will see in what follows that inclusion of the actual rather than the expected rate of inflation in the vote function requires only a minor modification in the analysis and the results are not substantially altered. So as to enable us to derive a complete analytic solution to the problem we assume that the vote function may be represented by the linear form[4]:

$$S(T)=\bar{S}-\beta_0 u(T)-\beta_1 v(T).$$

where

$S(T)$ is the number of votes cast for (or seats gained by) the government in an election at time T.

$\bar{S}$, β_0 and β_1 are positive constants.

Let us suppose that the government requires a particular minimum number of votes (or seats), S_T say, so as to attain its required majority. Then substituting for $v(T)$ from (2) above the target set is given by

$$\alpha_1(u_N-Bu(T))-p(T)+\bar{W}=0 \tag{9}$$

where

$$\bar{W}\equiv\frac{\bar{S}-S_T}{\beta_1} \quad \text{and} \quad B\equiv\frac{\beta_0+\alpha_1\beta_1}{\alpha_1\beta_1}.$$

Finally let us consider permissible magnitudes for the control variable θ. Clearly, restrictions are required here for both economic and mathematical reasons. An unrestricted control set would result in θ taking on only the values $\pm\infty$ which would in turn result in the economy exhibiting some highly implausible behaviour. Thus, for simplicity, we shall assume that the control set is given by

$$|\theta(t)|\leqslant\theta; \qquad Z \quad \text{a finite positive constant}. \tag{10}$$

The complete problem may now be simply stated as follows:

Given initial conditions (u_0, p_0) we wish to find a control $\theta(t)$ satisfying (10) on $t\in[0, T]$ such that the solutions $(u(t), p(t))$ of (5) and (6) define a trajectory in R^2 which satisfies (9) at $t=T$ where T is a minimum.

4. DERIVATION OF THE SOLUTION

Applying the results of the previous section and noting that

$$A=\begin{bmatrix}-\phi & 0\\ 0 & 0\end{bmatrix}; \qquad b=\begin{bmatrix}-1/\alpha_1\\ 1\end{bmatrix}$$

[4] For a non-linear target numerical methods are usually required to derive a solution. The problems associated with non-linear target boundary's and numerical solution procedures are described in Lee and Markus [3] p. 136, et seq.

we see that:

(i) The eigenvalues of A are 0 and $-\phi$ which are real and non-positive.

(ii) $(b; Ab; \cdots; A^{n-1}b)=\frac{1}{\alpha_1}\begin{bmatrix}-1 & \phi \\ \alpha_1 & 0\end{bmatrix}$ has rank 2.

(iii) $H(u, p, \lambda_1, \lambda_2, \theta)=\lambda_1\,(\phi(u_N-u)-\theta/\alpha_1)+\lambda_2\theta$ from which

$$\dot{\lambda}_1=\phi\lambda_1\,; \qquad \lambda_1(T)=\alpha_1\beta_1 B\,,$$

$$\dot{\lambda}_2=0\,; \qquad \lambda_2(T)=\beta_1\,.$$

The complete solutions to these diffential equations are

$$\lambda_1(t)=\alpha_1\beta_1\, Be^{-\phi(T-t)}\,,$$

$$\lambda_2(t)=\beta_1\,.$$

Thus, statements (d) to (g) of the previous section are valid. Also, the optimal path for the control variable is given by

$$\theta(t)=Z\,\mathrm{sgn}\,(-\lambda_1/\alpha_1+\lambda_2)\,,$$

Therefore

$$\theta(t)=\begin{Bmatrix}-Z\\ Z\end{Bmatrix} \text{ as } t\begin{Bmatrix}<\\ \geqslant\end{Bmatrix}T-\gamma \tag{11}$$

where

$$\gamma=\frac{1}{\phi}\log_e(B)\,.$$

This gives us the information that, even though T is a so far unknown function of initial conditions and parameters, for the last γ units of time (providing of course that $T>\gamma$) $\theta(t)=Z$. This will enable us to construct the switching locus i.e. the locus of points in (u, p) space where $\theta(t)$ switches in value from $-Z$ to Z.

Now consider some arbitrary point on the vote function (9), i.e. Let

$$u(T)=u_\tau\,,$$

$$p(T)=p_\tau=\bar{W}+\alpha_1(u_N-Bu_\tau)\,.$$

It is convenient at this point to temporarily shift the time-origin so that $T-\gamma=0$. Thus, using the foregoing and setting $\theta(t)=Z$ the solutions of (5) and (6) are given by, (where $\bar{u}\equiv u_N-Z/\phi\alpha_1$),

$$u(t)=\bar{u}+B(u_\tau-\bar{u})e^{-\phi t} \tag{12}$$

$$p(t)=Z(t-\gamma)+\bar{W}+\alpha_1(u_N-Bu_\tau)\,. \tag{13}$$

From which

$$p = \frac{Z}{\phi} \log_e \left\{ \frac{u_\tau - \bar{u}}{u - \bar{u}} \right\} + \bar{W} + \alpha_1 (u_N - Bu_\tau) \tag{14}$$

and

$$\frac{dp}{du} = \frac{-Z}{\phi(u - \bar{u})} \gtrless 0 \quad \text{as} \quad u \lessgtr \bar{u} . \tag{15}$$

Setting $t = 0$ in (12) and solving for u_τ in terms of $u(0)$ we derive

$$u_\tau = \frac{u(0) + (B-1)\bar{u}}{B} .$$

Substituting this into (13) and setting $t = 0$ gives the equation of the switching locus, i.e.

$$p = \bar{W} - Z\gamma + \alpha_1 (u_N - u - (B-1)\bar{u}) . \tag{16}$$

We note that, from (2), this is simply a particular short-run Phillips curve.

Let us now revert to the normal time origin and consider the solutions of (5) and (6) for $\theta(t) = -Z$ and the given initial conditions. These solutions are

$$\begin{aligned} u(t) &= 2u_N - \bar{u} + (u_0 - 2u_N + \bar{u}) e^{-\phi t} , \\ p(t) &= p_0 - Zt . \end{aligned} \tag{17}$$

From which

$$p = p_0 - \frac{Z}{\phi} \log_e \left\{ \frac{u_0 - 2u_N + \bar{u}}{u - 2u_N + \bar{u}} \right\} ,$$

and

$$\frac{dp}{du} = \frac{Z}{\phi(u - 2u_N + \bar{u})} \gtrless 0 \qquad \text{as} \quad u \gtrless 2u_N - \bar{u} . \tag{18}$$

It remains to solve for the time taken to reach the switch locus from the initial conditions. Let us denote this time $\hat{T}$ and then total time will be given by $T = \hat{T} + \gamma$. To solve for $\hat{T}$ we substitute (17) into (14) setting $t = T$ and re-arrange to derive

$$\begin{aligned} F(\hat{T}) \equiv \alpha_1 (u_0 - 2u_N + \bar{u}) e^{-\phi \hat{T}} - Z\hat{T} + \cdots \\ + \{p_0 - \bar{W} + Z\gamma + \alpha_1 u_N + \alpha_1 (B-2)\bar{u}\} = 0 . \end{aligned} \tag{19}$$

It is not difficult to see that this possesses a unique real positive solution if and only if the initial point (u_0, p_0) lies 'above' the switching locus. Henceforth we shall assume that this is the case; the implications of the initial point lying on or below the switch locus are obvious.

Finally, using (9), (15), (16), and (18) we construct a Nyquist diagram (Fig. 1, below) and illustrate some typical optimal unemployment/inflation trajectories.

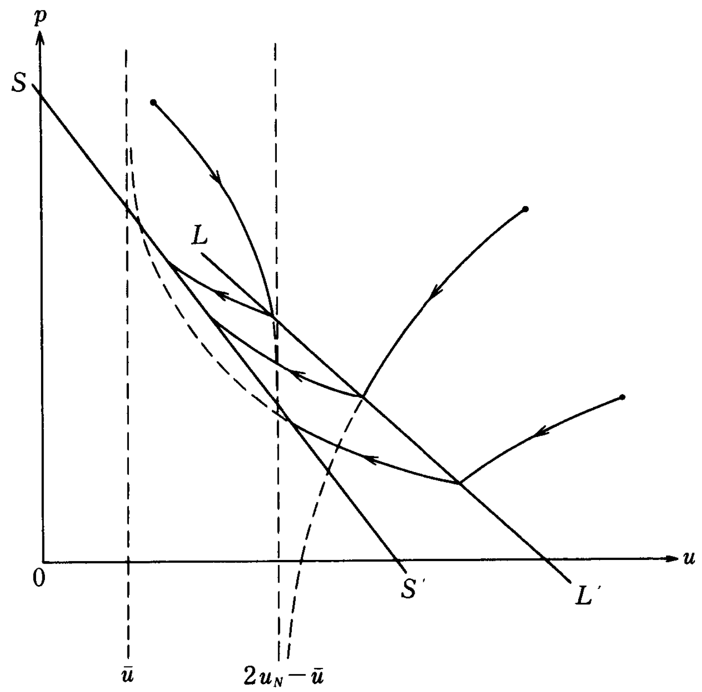

Fig. 1. Typical trajectories for various initial conditions and target SS'.

The line SS' is the vote function and LL' is the switching locus. Note that the behaviour is highly dependent on the magnitude of the initial level of unemployment, u_0. If $u_0 < 2u_N - \bar{u}$ then initially unemployment rises while the inflation rate falls whereas if $u_0 > 2u_N - \bar{u}$ both unemployment and inflation decline. However, in both cases, in the period immediately prior to the election unemployment falls while inflation is allowed to increase.

5. CONCLUDING REMARKS

It is of some interest to compare our results here with those of Nordhaus (*op. cit.*). In both cases, cyclical behaviour is observed but whereas the Nordhaus cycles are a result of instantaneous jumps in the unemployment level at election dates the cycles here are the result of a 'mid-term' policy switch. We would also point out that our analysis has wider relevance than the particular problem studied above. It is relevant whenever a government wishes to attain a particular unemployment/inflation configuration in the minimum time, whatever its reasons. Two examples will illustrate this.

Example (i). Suppose it is desired to attain some particular terminal state (u_τ, p_τ). The solution to a typical problem of this type is depicted in Fig. 2 below. The switching locus is the heavy curve LL' passing through the terminal point and we have made use of the fact that there may be at most one policy switch (from (*g*) in Section 2 above).

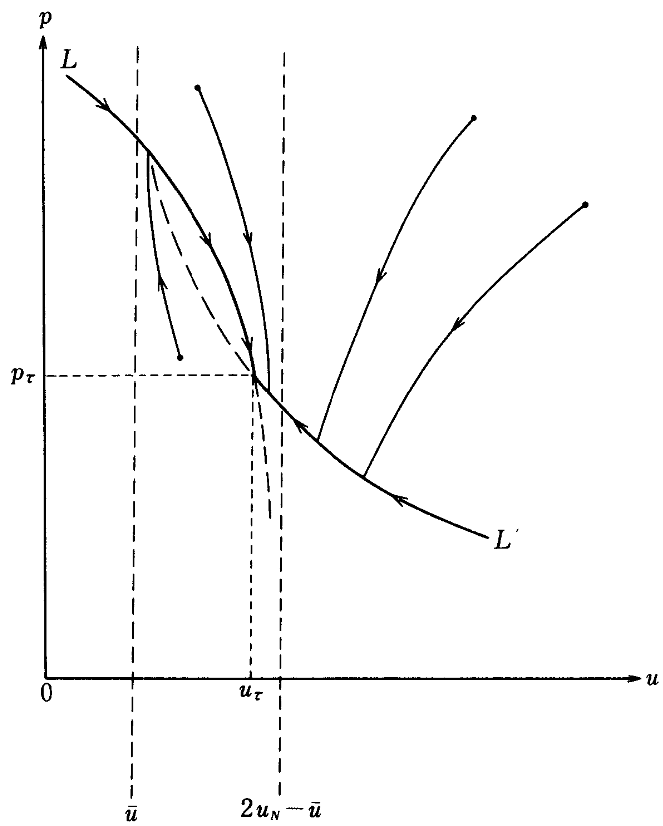

Fig. 2. Optimal trajectories for alternative initial conditions and target set a point in R^2.

Example (ii). Suppose it is required to drive the unemployment rate to the 'natural rate' in minimum time. Here, the optimal policy is to set $\theta(t) = Z \; \forall t \in [0, T]$ since, in some sense, inflation is "costless." Thus

$$T = \frac{1}{\phi} \log_e \left\{ \frac{\phi \alpha_1 (u_0 - \bar{u})}{Z} \right\}.$$

There examples illustrate fairly well the wide applicability of this type of analysis.

Finally we should like to point out that this paper is intended to be illustrative rather than exhaustive. Interesting results may be obtainable from alternative forms of the vote function but numerical methods may very well be required in order to obtain solutions.

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REFERENCES

[1] Chappell, D. and D. A. Peel, On the political theory of the business cycle, *Economics Letters*, 1979.

[2] Harvey, C. A., Determining the switching criterion for time-optimal control, *Journal of Mathematical Analysis and Applications*, 1962.

[3] Lee, E. B. and L. Markus, *Foundations of Optimal Control Theory*, John Wiley & Sons, New York, 1967.

[4] Nordhaus, W. D., The political business cycle, *The Review of Economic Studies*, 1975.