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ON A COOPERATIVE SOLUTION FOR GENERALIZED N-PERSON GAMES

Hiroaki OSANA

1. INTRODUCTION

Scarf (1971) proved the existence of a cooperative solution for a general class of n-person games, in which each player has a fixed set of possible strategies and a utility function of joint strategies. In such games there may be some externalities in preferences but no externalities in technologies which determine the sets of possible strategies. In this note we consider a more general class of n-person games admitting externalities in technologies and provide a proof of the existence of a cooperative solution for the case $n \leqq 4$. Proof for the case $n > 4$ is left as an open problem.

2. COOPERATIVE SOLUTION: DEFINITION

Let $N = \{1, 2, \cdots, n\}$ be the *set of players*. A *strategy* of player $i \in N$ is denoted by a point x_i of R^m, which is called the *strategy space*. An n-tuple $x = (x_1, x_2, \cdots, x_n)$ of strategies of the players is called a *joint strategy*. The *set X of possible joint strategies* is a subset of R^{mn}. A *coalition S* is a nonempty subset of N. For each coalition S, we denote by x_S a collection of strategies of the members of S, i.e., $x_S = (x_i : i \in S)$. If $S = \{i\}$ we write x_i for $x_{\{i\}}$. For each coalition S, we denote by $)S($ the complementary coalition of S, i.e., $)S(= N - S$. If $S = \{i\}$ we write $)i($ for $)\{i\}($. Let

$$X_S = \{x_S \colon (x_S, x_{)S(}) \in X \text{ for some } x_{)S(}\},$$

$$F_S(x_{)S(}) = \{x_S : (x_S, x_{)S(}) \in X\},$$

$$X^S = \bigcap \{F_S(x_{)S(}) : x_{)S(} \in X_{)S(}\}.$$

X_S may be called the *set of conditionally possible collusive strategies* of coalition S and X^S may be called the *set of unconditionally possible collusive strategies* of coalition S. If $S = \{i\}$ we write X_i and X^i for $X_{\{i\}}$ and $X^{\{i\}}$, respectively. Let the preference relation of player $i \in N$ be represented by a utility function U_i defined on X. Then, following Scarf (1971), we can introduce a concept of cooperative solution.

DEFINITION. A joint strategy $x^* \in X$ is called a *cooperative solution* if for every $S \in 2^N - \{\phi\}$ and every $x_S \in X^S$ there is an $x_{)S(} \in F_{)S(}(x_S)$ such that

$U_i(x_S, x_{)S(}) \leqq U_i(x^*)$ for some $i \in S$.

It may be helpful to restate the definition as follows: A joint strategy $x^* \in X$ is called a *cooperative solution* if there is no $S \in 2^N - \{\phi\}$ such that there is $x_S \in X^S$ such that $U_i(x_S, x_{)S(}) > U_i(x^*)$ for every $i \in S$ and every $x_{)S(} \in F_{)S(}(x_S)$. That is, a cooperative solution is a joint strategy which can be improved upon by no coalition with unconditionally possible collusive strategies. We note that every joint strategy is trivially a cooperative solution if X^S is empty for every coalition S.

3. CHARACTERISTIC FUNCTION

In order to prove the existence of a cooperative solution, we begin by describing the game in characteristic-function form. Let the characteristic function V be defined for every $S \in 2^N - \{\phi\}$ by

$$V(S) = \{u \in R^n\text{: There is an } x_S \in X^S \text{ such that if either } S = N \text{ or } x_{)S(} \in F_{)S(}(x_S) \text{ then } U_i(x_S, x_{)S(}) \geqq u_i \text{ for every } i \in S\}.$$

In what follows, several properties of V will be studied.

We first note the following obvious facts.

PROPOSITION 1. *For every* $S \in 2^N - \{\phi\}$,

(i) $X_S = \bigcup\{F_S(x_{)S(}): x_{)S(} \in X_{)S(}\}$,

(ii) *if X is closed then the correspondence F_S is closed on $X_{)S(}$*,

(iii) *if X is closed then X^S is closed,*

(iv) *if X is bounded then X_S is bounded,*

(v) $X^S \subseteq X_S$,

(vi) *if X is compact then X^S is compact.*

PROPOSITION 2. *If X is compact and U_i is a continuous real-valued function on X for every $i \in S$ then $V(S)$ is closed in R^n.*

Proof. Let $\{u^q\}_{q=1}^{\infty}$ be any sequence in $V(S)$ converging to some $u^0 \in R^n$. Then for every q there is an $x_S^q \in X^S$ such that if $S = N$ or $x_{)S(} \in F_{)S(}(x_S^q)$ then $U_i(x_S^q, x_{)S(}) \geqq u_i^q$ for every $i \in S$. By (vi) of Proposition 1, X^S is compact and hence we may, without loss of generality, suppose that the sequence $\{x_S^q\}_{q=1}^{\infty}$ converges to some $x_S^0 \in X^S$. If $S = N$ then $U_i(x^0) \geqq u_i^0$ for every $i \in N$ by the continuity of U_i, so that $u^0 \in V(N)$. Next suppose $S \neq N$. Suppose further that $U_i(x_S^0, x_{)S(}^0) < u_i^0$ for some $i \in S$ and some $x_{)S(}^0 \in F_{)S(}(x_S^0)$. Since $(x_S^q, x_{)S(}) \in X$ for every q and every $x_{)S(} \in X_{)S(}$, it follows that $(x_S^q, x_{)S(}^0) \in X$, i.e., $x_{)S(}^0 \in F_{)S(}(x_S^q)$ for every q. Therefore $U_i(x_S^q, x_{)S(}^0) \geqq u_i^q$ for every q and every $i \in S$, so that, by continuity, $U_i(x_S^0, x_{)S(}^0) \geqq u_i^0$ for every $i \in S$, a contradiction. Hence $U_i(x_S^0, x_{)S(}) \geqq u_i^0$ for every $i \in S$ and every $x_{)S(} \in F_{)S(}(x_S^0)$. Thus $u^0 \in V(S)$, completing the proof of the closedness of $V(S)$.

PROPOSITION 3. *If X is compact, X^S is nonempty, and U_i is a continuous real-*

valued function on X for every $i \in S$ then $V(S)$ is nonempty.

Proof. Since, for every $i \in S$, U_i is a continuous real-valued function on a compact set X, there is a $u_i = \min U_i(X)$ for every $i \in S$. Let $u = (u_S, u_{)S(})$, where $u_{)S(}$ may be arbitrary. Take some $x_S \in X^S$. By (v) of Proposition 1, $x_S \in X_S$, so that $F_{)S(}(x_S)$ is nonempty. Let $x_{)S(}$ be any element of $F_{)S(}(x_S)$. Then $(x_S, x_{)S(}) \in X$, so that $U_i(x_S, x_{)S(}) \geqq u_i$ for every $i \in S$. Thus $u \in V(S)$, proving the nonemptiness of $V(S)$.

For every $i \in N$, let

$$V'(i) = \{u_i : (u_i, u_{)i(}) \in V(i) \text{ for some } u_{)i(}\},$$

where $V(i) = V(\{i\})$.

PROPOSITION 4. *If X is compact, X^i is nonempty, and U_i is a continuous real-valued function on X then $V'(i)$ has a maximum.*

Proof. By proposition 3, $V(i)$ is nonempty, and so is $V'(i)$. Let $u_i \in V'(i)$. Then $(u_i, u_{)i(}) \in V(i)$ for some $u_{)i(}$. Hence there is an $x_i \in X^i$ such that $U_i(x_i, x_{)i(}) \geqq u_i$ for every $x_{)i(} \in F_{)i(}(x_i)$. Since $(x_i, x_{)i(}) \in X$ for every $x_{)i(} \in F_{)i(}(x_i)$, it follows that $U_i(x_i, x_{)i(}) \leqq \max U_i(X)$. Thus $\max U_i(X)$ is an upper bound of $V'(i)$. Since $V'(i)$ is nonempty, it has a least upper bound. Let $\bar{u}_i = \sup V'(i)$. Then there is a sequence $\{u_i^q\}_{q=1}^{\infty}$ in $V'(i)$ converging to $\bar{u}_i$. Let $u_{)i(}$ be any element of R^{n-1}. Then $(u_i^q, u_{)i(}) \in V(i)$ for every q. Since $V(i)$ is closed by Proposition 2, $(\bar{u}_i, u_{)i(}) \in V(i)$, so that $\bar{u}_i \in V'(i)$. Thus $\bar{u}_i = \max V'(i)$, completing the proof of the proposition.

For every nonempty $S \in 2^N$, let

$$W(S) = \{u \in V(S) : u_i \geqq \max V'(i) \text{ for every } i \in S\}.$$

PROPOSITION 5. *If X is compact and, for every $i \in S$, X^i is nonempty and U_i is a continuous real-valued function on X then $W(S)$ is nonempty.*

Proof. By Proposition 4, we may define $\bar{u}_i = \max V'(i)$ for each $i \in S$. Let $u = (\bar{u}_S, u_{)S(})$, where $u_{)S(}$ may be arbitrary. If we can show that $u \in V(S)$ then $u \in W(S)$ and the proposition will be proved. Since $\bar{u}_i \in V'(i)$ for every $i \in S$, $u \in V(i)$ for every $i \in S$. Hence for every $i \in S$ there is an $x_i' \in X^i$ such that $U_i(x_i', x_{)i(}) \geqq \bar{u}_i$ for every every $x_{)i(} \in F_{)i(}(x_i')$, i.e., for every $x_{)i(}$ such that $(x_i', x_{)i(}) \in X$. Clearly $x_S' \in X^S$. Let $x_{)S(} \in F_{)S(}(x_S')$. Then $(x_S', x_{)S(}) \in X$, so that $U_i(x_S', x_{)S(}) \geqq \bar{u}_i$ for every $i \in S$. Thus $u \in V(S)$.

PROPOSITION 6. *If X is compact and, for every $i \in N$, X^i is nonempty and U_i is a continuous real-valued function on X then $W(N)$ is compact.*

Proof. By Proposition 2, $V(N)$ is closed and so is $W(N)$. Let $u \in W(N)$. Then $u \in V(N)$, so that there is an $x \in X$ such that $U_i(x) \geqq u_i$ for every $i \in N$. Since $x \in X$, $U_i(x) \leqq \max U_i(X)$ for every $i \in N$ and therefore $u_i \leqq \max U_i(X)$ for every $i \in N$. Thus $W(N)$ is bounded from above. Evidently $W(N)$ is bounded from below. Hence $W(N)$ is compact.

PROPOSITION 7. *If* $u \in V(S)$ *and* $u' \leqq u$ *then* $u' \in V(S)$.

Proof. Obvious by the definition of V.

4. CORE

In this section, we establish a relation between a cooperative solution of the game in normal form and the core of the game in characteristic-function form. We say that a utility vector $u \in V(N)$ can be *improved upon* by coalition S if $u'_S > u_S$ for some $u' \in V(S)$. The *core* $C(V, N)$ of a game (V, N) in characteristic-function form is defined by

$$C(V, N) = \{u \in V(N) : u \text{ can be improved upon by no coalition}\} .$$

PROPOSITION 8. *Suppose that* X *is compact and* U_i *is a continuous real-valued function on* X *for every* $i \in N$. *If* $C(V, N)$ *is nonempty then there is a cooperative solution.*

Proof. Let $u^* \in C(V, N)$. Then $u^* \in V(N)$, so that there is an $x^* \in X$ such that $U_i(x^*) \geqq u_i^*$ for every $i \in N$. Suppose x^* were not a cooperative solution. Then there would exist an $S \in 2^N$, and an $x_S \in X^S$ such that $U_i(x_S, x_{)S(}) > U_i(x^*)$ for every $i \in S$ and every $x_{)S(} \in F_{)S(}(x_S)$. By (i), (ii), and (iv) of Proposition 1, $F_{)S(}(x_S)$ is compact. Hence for every $i \in S$ there exists a $\bar{u}_i = \min\{U_i(x_S, x_{)S(}) : x_{)S(} \in F_{)S(}(x_S)\}$. Then $U_i(x_S, x_{)S(}) \geqq \bar{u}_i > U_i(x^*)$ for every $i \in S$ and every $x_{)S(} \in F_{)S(}(x_S)$. Let $\bar{u} = (\bar{u}_S, u^*_{)S(})$. Then $\bar{u}_S > u^*_S$ and $\bar{u}_S \in V(S)$, so that u^* can be improved upon by S. Hence $u^* \notin C(V, N)$, a contradiction. Thus x^* is a cooperative solution.

For proving the existence of a cooperative solution, it is therefore sufficient to show the nonemptiness of the core $C(V, N)$. For that matter, Scarf (1967) proved the theorem stated below. We need two definitions. A collection T of coalitions is said to be *balanced* if there is a $p = (p_S : S \in T)$ such that $p_S \gneqq 0$ for every $S \in T$ and $\sum_{S \in T, i \in S} p_S = 1$ for every $i \in N$. An n-person game (V, N) is said to be *balanced* if $\bigcap\{V(S) : S \in T\} \subseteq V(N)$ for every balanced collection T of coalitions.

THEOREM (Scarf). *If* (i) $V(S)$ *is closed for every nonempty* $S \in 2^N$, (ii) $W(S)$ *is nonempty for every nonempty* $S \in 2^N$, (iii) $W(N)$ *is compact, and* (iv) $u \in V(S)$ *and* $u' \leqq u$ *imply* $u' \in V(S)$, *then the balanced n-person game* (V, N) *has a nonempty core* $C(V, N)$.

This theorem, together with the results in the previous section, in particular, Propositions 2, 5, 6, and 7, implies that in order to show the nonemptiness of $C(V, N)$ we have only to show that the game (V, N) is balanced. In the succeeding sections, we show that the game (V, N) is balanced if $n \leqq 4$.

5. PRELIMINARY OBSERVATIONS

Before proceeding to the special cases with $n \leqq 4$, we find it useful to prove a few preliminary results. A collection T of coalitions is said to be *structural* if there is a subcollection T' such that $N=\bigcup\{S: S\in T'\}$ and either $S=S'$ or $S\cap S'=\varnothing$ for every S, $S'\in T'$. If a collection T of coalitions is structural then it is balanced. For we may let $p_S=1$ for every $S\in T'$ and $p_S=0$ for every $S\in T-T'$.

PROPOSITION 9. *If T is a structural collection then $\bigcap\{V(S): S\in T\}\subseteq V(N)$.*

Proof. Since T is structural there is a subcollection T' such that $N=\bigcup\{S: S\in T'\}$ and either $S=S'$ or $S\cap S'=\varnothing$ for every S, $S'\in T'$. Let $u\in\bigcap\{V(S): S\in T\}$. Then $u\in V(S)$ for every $S\in T'$. Hence for every $S\in T'$ there is an $x_S(S)\in X^S$ such that if $S=N$ or $x_{)S(}\in F_{)S(}(x_S(S))$ then $U_i(x_S(S), x_{)S(})\geqq u_i$ for every $i\in S$. Let $x=(x_S(S)$: $S\in T')$ and $T'=\{S_1, S_2, \cdots, S_m\}$. Let $\bar{x}=(\bar{x}_{S_1}, \bar{x}_{)S_1(})\in X$. Then $\bar{x}_{)S_1(}\in X_{)S_1(}$. Since $x_{S_1}(S_1)\in X^{S_1}$, $(x_{S_1}(S_1), \bar{x}_{S_2}, \bar{x}_{)S_1\cup S_2(})=(x_{S_1}(S_1), \bar{x}_{)S_1(})\in X$, so that $(x_{S_1}(S_1), \bar{x}_{)S_1\cup S_2(})\in X_{)S_2(}$. Since $x_{S_2}(S_2)\in X^{S_2}$, it follows that $(x_{S_1}(S_1), x_{S_2}(S_2), \bar{x}_{)S_1\cup S_2(})\in X$. Continuing this process, we arrive at $x=(x_{S_1}(S_1), x_{S_2}(S_2), \cdots, x_{S_m}(S_m))=(x_S(S)$: $S\in T')\in X$. For each $S'\in T'$, let $x_{)S'(}(\,)S'(\,)=(x_S(S)$: $S\in T'-\{S'\})$. Then $x_{)S'(}(\,)S'(\,)\in X_{)S'(}$, so that $U_i(x)=U_i(x_{S'}(S'), x_{)S'(}(\,)S'(\,))\geqq u_i$ for every $i\in S'$. Since S' may be arbitrary in T' and $N=\bigcup\{S$: $S\in T'\}$, it follows that $U_i(x)\geqq u_i$ for every $i\in N$. Thus $u\in V(N)$.

PROPOSITION 10. *Suppose that* (i) *X is convex and* (ii) *for every $i\in N$, U_i is a real-valued function on X such that $U_i((1-t)x+tx')\geqq U_i(x')$ whenever $U_i(x)\geqq U_i(x')$ and $0\leqq t\leqq 1$. If T contains all the $(n-1)$-person coalitions then T is balanced and $\bigcap\{V(S)$: $S\in T\}\subseteq V(N)$.*

Proof. Let T' be the collection of all the $(n-1)$-person coalitions. We may write $T'=\{S_1, S_2, \cdots, S_n\}=\{N-\{1\}, N-\{2\}, \cdots, N-\{n\}\}$. Let $p_S=1/(n-1)$ for every $S\in T'$ and $p_S=0$ for every $S\in T-T'$. Then $\sum_{i\in S\in T}p_S=1$ for every $i\in N$. Hence T is balanced.

Let $u\in\bigcap\{V(S)$: $S\in T\}$. Then for every $S\in T$ there is an $x_S(S)\in X^S$ such that if $S=N$ or $x_{)S(}\in F_{)S(}(x_S(S))$ then $U_i(x_S(S), x_{)S(})\geqq u_i$ for every $i\in S$. If $N\in T$ then clearly $u\in V(N)$. Hence suppose $N\notin T$. For every $i\in N$, let $x_i=\sum_{i\in S\in T}p_S x_i(S)$. Then

$$x_i=\sum_{i\in S\in T'}p_S x_i(S)=\frac{1}{n-1}\sum_{j\in N-\{i\}}x_i(N-\{j\})\,.$$

Let $x=(x_1, x_2, \cdots, x_n)$. Then

$$\begin{aligned}x&=\frac{1}{n-1}\left(\sum_{j\in N-\{1\}}x_1(N-\{j\}),\ \sum_{j\in N-\{2\}}x_2(N-\{j\}),\ \cdots,\ \sum_{j\in N-\{n\}}x_n(N-\{j\})\right)\\&=\frac{1}{n-1}((x_{N-\{n\}}(N-\{n\}), x_n(N-\{1\}))+(x_{N-\{n-1\}}(N-\{N-1\}), x_{n-1}(N-\{1\}))\\&\qquad+\cdots+(x_{N-\{2\}}(N-\{2\}), x_2(N-\{1\}))\\&=\sum_{i=2}^{n}\frac{1}{n-1}(x_{N-\{i\}}(N-\{i\}), x_i(N-\{1\}))\,.\end{aligned}$$

Since $x_{N-\{i\}}(N-\{i\}) \in X^{N-\{i\}}$ for every $i=2, 3, \cdots, n$ and $x_i(N-\{1\}) \in X_i$ for every $i=2, 3, \cdots, n$, it follows that $(x_{N-\{i\}}(N-\{i\}), x_i(N-\{1\})) \in X$ for every $i=2, 3, \cdots, n$. Hence $x \in X$ by convexity. Furthermore $U_1(x_{N-\{i\}}(N-\{i\}), x_i(N-\{1\})) \geqq u_1$ for every $i=2, 3, \cdots, n$, so that $U_1(x) \geqq u_1$ by the quasi-concavity of U_1. Similarly $U_i(x) \geqq u_i$ for every $i=2, 3, \cdots, n$. Thus $u \in V(N)$.

PROPOSITION 11. *If there is an $i \in N$ such that $i \in S$ for every $S \in T$, then either $N \in T$ or T is not balanced.*

Proof. Suppose that $N \notin T$ and T is balanced. If $i \in S$ for every $i \in N$ and every $S \in T$ then $T=\{N\}$. But $N \notin T$ by assumption, so that $T \neq \{N\}$. Hence $i \notin S$ for some $i \in N$ and $S \in T$. Let $T(i)=\{S \in T\colon i \notin S\}$ and $T'(i)=\{S \in T\colon i \in S\}$. Then $T(i) \neq \varnothing$ for some $i \in N$. Clearly $\bigcup_{i=1}^n T(i) \subseteq T$. Let $S \in T$. Since $N \notin T$, $S \neq N$. Hence $i \notin S$ for some $i \in N$, so that $S \in T(i)$. Therefore $S \in \bigcup_{i=1}^n T(i)$. Thus $T \subseteq \bigcup_{i=1}^n T(i)$, implying that $T = \bigcup_{i=1}^n T(i)$. Since T is balanced, there is a $p=(p_S\colon S \in T)$ such that $p_S \geqq 0$ for every $S \in T$ and $\sum_{S \in T'(i)} p_S = 1$ for every $i \in N$. But $T=T'(i^*)$ for some $i^* \in N$ by hypothesis, so that $\sum_{S \in T} p_S = 1$. Let S' be any coalition in T. Then $S' \neq T$, so that $i' \notin S'$ for some $i' \in N$, that is, $S' \in T(i')$ for some $i' \in N$. Since $T=T(i') \cup T'(i')$ and $T(i') \cap T'(i') = \varnothing$, it follows that

$$\sum_{S \in T(i')} p_S + \sum_{S \in T'(i')} p_S = \sum_{S \in T} p_S = 1 = \sum_{S \in T'(i')} p_S$$

and therefore

$$\sum_{S \in T(i')} p_S = 0 .$$

Thus $p_{S'}=0$. Since S' is arbitrary in T, $p_S=0$ for every $S \in T$, which contradicts that T is balanced. Hence $N \in T$ or T is not balanced.

6. TWO OR THREE-PERSON GAMES

THEOREM 1. *Suppose that* (i) *X is compact,* (ii) *X^i is nonempty for every $i \in N$, and* (iii) *U_i is a continuous real-valued function on X for every $i \in N$. If $n=2$ then there exists a cooperative solution.*

Proof. There can be at most two balanced collections of coalitions: $T_1 = \{\{1\}, \{2\}\}$ and $T_2 = \{\{1, 2\}\}$. Both are structural. Hence the theorem follows from Propositions 2, 5, 6, 7, and 9 and Scarf's Theorem.

We note that the above theorem for two-person games requires no convexity assumptions. On the other hand, we note also that cooperative behaviors play no crucial roles in two-person games.

Let us now turn to three-person games.

LEMMA 1. *If $n=3$ and T is balanced but not structural then $T=\{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$.*

Proof. Clearly, $\{1, 2, 3\} \notin T$. Suppose $\{1\} \in T$. Since T is not structural, $\{2, 3\} \notin T$ and either $\{2\} \notin T$ or $\{3\} \notin T$. But, since T is balanced, it contains at least one coalition owning 3, so that $\{3, 1\} \in T$. Then, letting $T' = \{\{2\}, \{3, 1\}\}$, we see that T is structural, a contradiction. Hence $\{2\} \notin T$. Similarly, $\{3\} \notin T$. Since T is balanced, for every $i \in N$, T contains at least one coalition owning i, so that $T = \{\{1\}, \{1, 2\}, \{3, 1\}\}$. But this is not balanced by Proposition 11. Therefore $\{1\} \notin T$. Similarly, $\{2\} \notin T$ and $\{3\} \notin T$. Thus $T \subseteq \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$. Clearly, $T \neq \{\{1, 2\}\}$, $T \neq \{\{2, 3\}\}$, and $T \neq \{\{3, 1\}\}$. Furthermore, by Proposition 11, $T \neq \{\{1, 2\}, \{2, 3\}\}$, $T \neq \{\{2, 3\}, \{3, 1\}\}$, and $T \neq \{\{3, 1\}, \{1, 2\}\}$. Hence it follows that $T = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$.

THEOREM 2. *Suppose that* (i) *X is compact and convex,* (ii) *X^i is nonempty for every $i \in N$, and* (iii) *for every $i \in N$, U_i is a continuous real-valued function on X such that $U_i((1-t)x + tx') \geqq U_i(x')$ whenever $U_i(x) \geqq U_i(x')$ and $0 \leqq t \leqq 1$. If $n = 3$ then there exists a cooperative solution.*

Proof. In view of Propositions 2, 5, 6, and 7 and Scarf's Theorem, it suffices to show that the game (V, N) is balanced. Let T be any balanced collection of coalitions. We wish to show that $\bigcap\{V(S)\colon S \in T\} \subseteq V(N)$. By Proposition 9, we have only to consider the case for which T is not structural. By Lemma 1, we may then let $T = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$. But then $\bigcap\{V(S)\colon S \in T\} \subseteq V(N)$ by Proposition 10, as desired.

7. FOUR-PERSON GAMES

In this section, we prove the following theorem in a series of lemmas.

THEOREM 3. *Suppose that* (i) *X is compact and convex,* (ii) *X^i is nonempty for every $i \in N$, and* (iii) *for every $i \in N$, U_i is a continuous real-valued function on X such that $U_i((1-t)x + tx') \geqq U_i(x')$ whenever $U_i(x) \geqq U_i(x')$ and $0 \leqq t \leqq 1$. If $n = 4$ then there exists a cooperative solution.*

This is exactly the same as Theorem 2 for three-person games. But its proof is rather lengthy. In the following proof, we assume that all the assumptions of this theorem are always satisfied. It is again sufficient to show that the game (V, N) is balanced. Let T be any balanced collection of coalitions. By Proposition 9, we have only to show that $\bigcap\{V(S)\colon S \in T\} \subseteq V(N)$ whenever T is not structural. Hence from here on we assume that T is not structural. Moreover, by Proposition 10, we may assume that T contains at most three 3-person coalitions.

LEMMA 2. *If T contains three 3-person coalitions then $\bigcap\{V(S)\colon S \in T\} \subseteq V(N)$.*

Proof. Without loss of generality, we may assume that $\{\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\} \subset T$ and $\{1, 2, 3\} \notin T$. Since T is not structural, $\{1\} \notin T$, $\{2\} \notin T$, and $\{3\} \notin T$. If $\{1, 2\} \notin T$, $\{2, 3\} \notin T$, and $\{3, 1\} \notin T$ then $4 \in S$ for every $S \in T$. Since $\{1, 2, 3, 4\} \notin T$, it follows from Proposition 11 that T is not balanced, a contradiction. Hence $\{1,$

$2\} \in T$ or $\{2, 3\} \in T$ or $\{3, 1\} \in T$. It suffices to consider the case for which $\{1, 2\} \in T$. Now let $u \in \bigcap\{V(S)\colon S \in T\}$. Then for every $S \in T$ there is an $x_S(S) \in X^S$ such that $U_i(x_S(S), x_{)S(}) \geqq u_i$ for every $i \in S$ and every $x_{)S(} \in X_{)S(}$. Let $x = \frac{1}{2}(x_{134}(134), x_2(234)) + \frac{1}{2}(x_{12}(12), x_{34}(234))$, where we simplify notation by writing, say, $(x_{12}(12), x_{34}(234))$ for $(x_1(\{1, 2\}), x_2(\{1, 2\}), x_3(\{2, 3, 4\}), x_4(\{2, 3, 4\}))$ and $(x_{134}(134), x_2(234))$ for $(x_1(134), x_2(234), x_{34}(134))$. Clearly, $(x_{134}(134), x_2(234)) \in X$ and $(x_{12}(12), x_{34}(234)) \in X$, so that $x \in X$ by convexity. Furthermore $U_1(x_{134}(134), x_2(234)) \geqq u_1$ and $U_1(x_{12}(12), x_{34}(234)) \geqq u_1$, so that $U_1(x) \geqq u_1$ by the quasi-concavity of U_1. Noticing that $x = \frac{1}{2}(x_1(134), x_{234}(234)) + \frac{1}{2}(x_{12}(12), x_{34}(134))$, we see that $U_2(x) \geqq u_2$. Similarly $U_3(x) \geqq u_3$ and $U_4(x) \geqq u_4$. Thus $u \in V(N)$, which completes the proof.

LEMMA 3. *If T contains two 3-person coalitions then* $\bigcap\{V(S)\colon S \in T\} \subseteq V(N)$.

Proof. Without loss of generality, we may assume that $\{1,2,3\} \notin T$, $\{1,2,4\} \notin T$, and $\{\{1, 3, 4\}, \{2, 3, 4\}\} \subseteq T$. Since T is not structural, $\{1\} \notin T$ and $\{2\} \notin T$. If $\{1, 2\} \in T$ then the same argument applies as that in the proof of Lemma 2. We assume that $\{1, 2\} \notin T$ in what follows. Suppose further that $\{1, 3\} \in T$ and $\{1, 4\} \in T$. Then $\{2, 4\} \notin T$ and $\{2, 3\} \notin T$. Consider the linear equation

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

It is easy to see that this system has no nonnegative solution. Hence T is not balanced, a contradiction. Therefore $\{1, 3\} \notin T$ or $\{1, 4\} \notin T$. Suppose that $\{1, 3\} \in T$ or $\{1, 4\} \in T$. It suffices to consider the case for which $\{1, 3\} \in T$. Then $\{1, 4\} \notin T$ and $\{2, 4\} \notin T$. The system of linear equations

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

has no nonnegative solution, so that T is not balanced. Hence $\{1, 3\} \notin T$. Similarly $\{1, 4\} \notin T$. But the linear system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

has no nonnegative solution. Thus it is impossible that $\{1, 2\} \notin T$. This completes the proof of the lemma.

LEMMA 4. *If T contains one 3-person coalition then* $\bigcap\{V(S)\colon S \in T\} \subseteq V(N)$.

Proof. Without loss of generality, we may assume that $\{1, 2, 3\} \notin T$, $\{1, 2, 4\} \notin T$, $\{1, 3, 4\} \notin T$, and $\{2, 3, 4\} \in T$. Then $\{1\} \notin T$. Let $u \in \bigcap\{V(S)\colon S \in T\}$. Then for every $S \in T$ there is an $x_S(S) \in X^S$ such that $U_i(x_S(S), x_{)S(}) \geqq u_i$ for every $i \in S$ and every $x_{)S(} \in X_{)S(}$. Two cases will be considered separately.

Case 1. $(\{1, 2\} \in T,\ \{1, 3\} \in T,\ \{4\} \in T)$ or $(\{1, 2\} \in T,\ \{1, 4\} \in T,\ \{3\} \in T)$ or $(\{1, 3\} \in T,\ \{1, 4\} \in T,\ \{2\} \in T)$.

It is sufficient to consider the subcase for which $\{1, 2\} \in T$, $\{1, 3\} \in T$, and $\{4\} \in T$. Then $\{3, 4\} \notin T$ and $\{2, 4\} \notin T$. Let $x = \frac{1}{2}(x_{12}(12), x_{34}(234)) + \frac{1}{2}(x_{13}(13), x_2(234), x_4(4))$. Then $x \in X$ and $U_1(x) \geqq u_1$. Since $x = \frac{1}{2}(x_1(13), x_{234}(234)) + \frac{1}{2}(x_{12}(12), x_3(13), x_4(4))$, $U_2(x) \geqq u_2$. Similarly $U_3(x) \geqq u_3$ and $U_4(x) \geqq u_4$. Therefore $u \in V(N)$.

Case 2. $(\{1, 2\} \notin T$ or $\{1, 3\} \notin T$ or $\{4\} \notin T)$, $(\{1, 2\} \notin T$ or $\{1, 4\} \notin T$ or $\{3\} \notin T)$, and $(\{1, 3\} \notin T$ or $\{1, 4\} \notin T$ or $\{2\} \notin T)$.

This case will be divided into two subcases.

Case 2.1. $\{1, 2\} \in T$, $\{1, 3\} \in T$, and $\{1, 4\} \in T$. Let $x = \frac{1}{3}(x_{12}(12), x_{34}(234)) + \frac{1}{3}(x_{13}(13), x_{24}(234)) + \frac{1}{3}(x_{14}(14), x_{23}(234))$. Then $x \in X$ and $U_1(x) \geqq u_1$. Since $x = \frac{1}{3}(x_1(13), x_{234}(234)) + \frac{1}{3}(x_1(14), x_{234}(234)) + \frac{1}{3}(x_{12}(12), x_3(13), x_4(14))$, it follows that $U_2(x) \geqq u_2$. Similarly $U_3(x) \geqq u_3$ and $U_4(x) \geqq u_4$. Thus $u \in V(N)$.

Case 2.2. $\{1, 2\} \notin T$ or $\{1, 3\} \notin T$ or $\{1, 4\} \notin T$. Let us first suppose that $(\{1, 2\} \in T, \{1, 3\} \in T)$ or $(\{1, 2\} \in T, \{1, 4\} \in T)$ or $(\{1, 3\} \in T, \{1, 4\} \in T)$. We have only to consider the case for which $\{1, 2\} \in T$ and $\{1, 3\} \in T$. Then $\{3, 4\} \notin T$, $\{2, 4\} \notin T$, $\{1, 4\} \notin T$, and $\{4\} \notin T$. But the linear system

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

has no nonnegative solution, so that T is not balanced. Hence $(\{1, 2\} \notin T$ or

$\{1, 3\} \notin T$), ($\{1, 2\} \notin T$ or $\{1, 4\} \notin T$), and ($\{1, 3\} \notin T$ or $\{1, 4\} \notin T$). Now suppose that $\{1, 2\} \in T$ or $\{1, 3\} \in T$ or $\{1, 4\} \in T$. It suffices to consider the case for which $\{1, 2\} \in T$. Then $\{1, 3\} \notin T$, $\{1, 4\} \notin T$, $\{3, 4\} \notin T$, and either $\{3\} \notin T$ or $\{4\} \notin T$. Suppose further that $\{3\} \in T$ or $\{4\} \in T$. If $\{3\} \in T$ then the linear system

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

has no nonnegative solution, and therefore T is not balanced. Hence $\{3\} \notin T$. Similarly $\{4\} \notin T$. But then $2 \in S$ for every $S \in T$, so that, by Proposition 11, T is not balanced. It is therefore impossible that $\{1, 2\} \in T$ or $\{1, 3\} \in T$ or $\{1, 4\} \in T$. Hence $\{1, 2\} \notin T$, $\{1, 3\} \notin T$, and $\{1, 4\} \notin T$. But then no coalition in T contains 1, so that T is not balanced. Thus Case 2.2 is impossible. This completes the proof of the lemma.

LEMMA 5. *If T contains at least one 2-person coalition then* $\bigcap\{V(S): S \in T\} \subseteq V(N)$.

Proof. The case for which T contains at least one 3-person coalition has been treated in Lemmas 2 to 4. So we assume in what follows that T contains no 3-person coalition. By hypothesis, T contains at least one 2-person coalition. We may, without loss of generality, that $\{1, 2\} \in T$. Then $\{3, 4\} \notin T$. We first show that $\{1, 3\} \notin T$ or $\{1, 4\} \notin T$. To this end, suppose on the contrary that $\{1, 3\} \in T$ and $\{1, 4\} \in T$. Then $\{2, 4\} \notin T$ and $\{2, 3\} \notin T$. Suppose further that $\{2\} \in T$ or $\{3\} \in T$ or $\{4\} \in T$. We consider the case for which $\{2\} \in T$. Then $\{3\} \notin T$ and $\{4\} \notin T$. Since the linear system

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

has no nonnegative solution, T is not balanced. Hence $\{2\} \notin T$. Similarly $\{3\} \notin T$ and $\{4\} \notin T$. But then $1 \in S$ for every $S \in T$ and hence T is not balanced by Proposition 11. Thus we cannot have $\{1, 3\} \in T$ and $\{1, 4\} \in T$ simultaneously. Therefore either $\{1, 3\} \notin T$ or $\{1, 4\} \notin T$. Similarly either $\{2, 3\} \notin T$ or $\{2, 4\} \notin T$. Two cases will be considered separately.

Case 1. ($\{1, 3\} \in T$, $\{2, 3\} \in T$, and $\{4\} \in T$) or ($\{1, 4\} \in T$, $\{2, 4\} \in T$, and $\{3\} \in T$).

It suffices to consider the case for which $\{1, 3\} \in T$, $\{2, 3\} \in T$, and $\{4\} \in T$. Let $u \in \bigcap \{V(S)\colon S \in T\}$. Then for every $S \in T$ there is an $x_S(S) \in X^S$ such that $U_i(x_S(S), x_{)S(}) \geqq u_i$ for every $i \in S$ and every $x_{)S(} \in X_{)S(}$. Let $x = \frac{1}{2}(x_{12}(12), x_3(23), x_4(4)) + \frac{1}{2}(x_{13}(13), x_2(23), x_4(4))$. Then $x \in X$, $U_1(x) \geqq u_1$, and $U_4(x) \geqq u_4$. Furthermore $U_2(x) \geqq u_2$ since $x = \frac{1}{2}(x_{12}(12), x_3(23), x_4(4)) + \frac{1}{2}(x_{13}(13), x_2(23), x_4(4))$. Similarly $U_3(x) \geqq u_3$. Thus $u \in V(N)$.

Case 2. $(\{1, 3\} \notin T$ or $\{2, 3\} \notin T$ or $\{4\} \notin T)$ and $(\{1, 4\} \notin T$ or $\{2, 4\} \notin T$ or $\{3\} \notin T)$.

Suppose $\{1, 3\} \in T$. Then $\{1, 4\} \notin T$, $\{2, 4\} \notin T$, and either $\{2, 3\} \notin T$ or $\{4\} \notin T$. If $\{4\} \notin T$ then no coalition in T contains 4, so that T is not balanced. Hence $\{4\} \in T$, $\{2, 3\} \notin T$, $\{2\} \notin T$, and $\{3\} \notin T$. But then the linear system

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

has no nonnegative solution, and therefore T is not balanced. Thus $\{1, 3\} \notin T$. Similarly $\{1, 4\} \notin T$. In view of the fact that the argument involved is symmetric with respect to 1 and 2, we may conclude that $\{2, 3\} \notin T$ and $\{2, 4\} \notin T$. Since $\{1, 2\} \in T$, either $\{3\} \notin T$ or $\{4\} \notin T$. If $\{3\} \notin T$ then no coalition in T contains 3, so that T is not balanced. Therefore $\{4\} \notin T$. But then a similar contradiction obtains. Thus Case 2 cannot occur. Hence the proof of Lemma 5 is complete.

All the cases for which T contains at least one 2 or 3-person coalition are covered by Lemmas 2 to 5. Only the case for which T contains only one-person coalitions is left with us. If T is balanced then $T = \{\{1\}, \{2\}, \{3\}, \{4\}\}$, which is structural. But we have ruled out the structural cases. This completes the proof of Theorem 3.

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