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TASTE CHANGE AND THE STABILITY OF COMPETITIVE EQUILIBRIUM

Ryuzo SATO and Victor CHOLEWICKI

I

The paucity of concrete results in the area of taste change is in no small part due to the very nature of the topic. It is clear to the casual observer that tastes do in fact change from one period to another. The difficulty lies in isolating the effects of taste change from the income and substitution effects, which are themselves likely to be altered by a change in tastes. The issue is further complicated by the changing characteristics of what is considered the same good over time, and the emergence of new goods.

At the level of the individual consumer it is known that a change taste will cause the consumer to change his utility maximizing choice within his attainable set. However, since there is no a priori method of determining the direction of the taste change, there is no a priori method of determining the direction of change in the consumer's consumption pattern. The fact that in the general case the effects of taste change in the absence of advertising are not predictable is a major reason for the relative dearth of literature on the subject.

Throughout this paper tastes will be treated as being exogenously determined. For our system to accord with the traditional competitive model we must assume the absence of advertising. This is a major omission only if the primary effect of advertising is to induce people to buy a type of product they would not otherwise buy. However, if advertising simply induces consumers to buy a particular brand of a type of product they had already decided to purchase then our system must be regarded as being a highly relevant paradigm. Clearly, tastes can and do change in the absence of and even despite advertising. In that case shifts of demand between product categories are at least partially attributable to exogenous phenomena, the remainder being attributable to standard income and substitution effects; shifts of demand within product categories may result primarily from advertising. However, since our system deals with general product categories, the phenomena we intend to investigate are adequately represented in our model.

In equilibrium the consumer maximizes utility by attaining a commodity bundle and supplying services such that the marginal rate of substitution between every pair of goods and services is equal to the ratios of their prices. If the consumer's utility function contains parameters which are exogenous functions of time and which cause some or all of the marginal rates of substitution to also depend exogenously on time then we would expect the consumer's equilibrium of the initial time period to be non-optimal in a subsequent time period. His adjustment to a new equilibrium in the second time period would result in a new bundle of goods demanded and services supplied. However, if the production relationships have not also shifted, the result is market disequilibrium at the old set of prices. Hence, taste change, if it is empirically observable, would be expected to imply an equilibrium price vector which itself is a function of time.

Empirically meaningful taste change will be understood to denote changes in the consumer's utility which cause a change in the shape of the set C_R where

$$C_R = \{x : xRx_0\} \tag{1.1}$$

i.e., the "at least as good as" set. This occurs if at least some of the marginal rates of substitution change. The equilibrium, however, will not be displaced unless at least some marginal rates of substitution change when evaluated at x_0, the initial equilibrium. Any utility functions which contains taste change parameters in a separable form, i.e.,

$$U(X, t) = a(t) + b(t)U(x)$$

clearly will not result in displacement of equilibrium. A sufficient condition for taste change to result in equilibrium displacement is

$$\frac{\partial}{\partial t}\left[\frac{Ux_i}{Ux_j}\right] \neq 0 \tag{1.2}$$

for some $i \neq j$. Condition (1.2) is also sufficient for the slope of the individual consumer's demand function in each direction to be functions of time. If the taste change parameters are continuously differentiable functions of time and, as is assumed, the partial derivatives of the utility function with respect to the taste change parameters exist, then the slopes of the individual's demand function will be continuously differentiable. Thus barring the possibility that each consumer's change in taste is exactly offset by changes in tastes of the other consumers, empirically significant taste change, as defined, will result in excess demand functions with continuous partial derivatives with respect to time.

The system of excess demand functions implied by the existence of exogeneously changing tastes over time is considerably more complicated than its counterpart in traditional analysis. A general excess demand function is particularly difficult to deal with since the null solution is not the critical point; transformation of the system to consideration of the null solution is not feasible in the general case since it results in a system which has no intuitive economic meaning. If the null solution cannot be considered for stability purposes, most of the techniques of Liapunov, indispensable for this kind of analysis, cannot be applied. Finally, linearization is not an acceptable alternative due to the presence of the exogeneously time dependent parameters which prevents arbitrarily close approximations for the entire time interval under consideration. Thus, we find the linear competitive model with time dependent coefficients to be the most efficient model for our investigation: it incorporates the basic properties of taste change, the exogenous time dependence of the excess demand system, while remaining amenable to the available mathematical techniques.

We will consider an economy in which the demand functions are approximated by the form

$$x_i^D = \gamma_i(t) + \sum_{j=i}^{n} \alpha_{ij}(t) P_j \qquad i = 1, \cdots, n \tag{1.3}$$

and the supply functions by the form

$$x_i^s = \delta_i + \sum_{j=i}^{n} \beta_{ij} P_j \qquad i = 1, \cdots, n \tag{1.4}$$

yielding the system of excess demand functions

$$E = A(t)P + b(t) \tag{1.5}$$

where $A(t)$ is an $n \times n$ matrix whose elements are functions of time and $b(t)$ is an $n \times 1$ vector, also time dependent. Price is assumed to change in direct proportion to excess demand; speeds of adjustment are assumed to be unity to simplify exposition. The market price vector, $P^*(t)$, can be computed by solving the system when excess demand is equal to zero. We assume that $A(t)$ is nonsingular. Thus

$$P^*(t) = -A^{-1}(t)b(t)\,. \tag{1.6}$$

We would not expect $\dot{P}$ to tend to zero in our system; equilibrium in our model is defined to be zero deviation from the market-clearing price vector $P^*(t)$. Thus, we are interested in the deviation from equilibrium over time and whether the deviation tends toward zero. Changes in the actual price vector, $P(t)$ over time are governed by excess demand and can be solved for explicitly.

Assume for simplicity unit speed of adjustment in all markets. The standard assumption that market price changes as a function of excess demand yields

$$\dot{P} = A(t)P(t) + b(t) \tag{1.7}$$

$$= A(t)[P(t) - P^*(t)] \tag{1.8}$$

where (1.7) and (1.8) follow from (1.5) and (1.6) respectively. Since we are interested in the dynamic behavior of *deviations* from equilibrium price we subtract $\dot{P}^*(t)$ from both sides of (1.8) which yields

$$\dot{z} = A(t)z(t) + h(t) \tag{1.9}$$

where

$$z(t) \equiv P(t) - P^*(t) \tag{1.10}$$

and

$$h(t) = \dot{P}^*(t) \tag{1.11}$$

$$= -\dot{A}^{-1}b(t) - A^{-1}\dot{b}(t) \tag{1.12}$$

The term $\dot{A}^{-1}$ is the derivative of the inverse of $A(t)$, *not* the inverse of the derivative of $A(t)$. Also, it can be shown that the same derivation of the basic equation of motion holds for any set of strictly positive speeds of adjustment.

Our model takes the form of a non-autonomous, non-homogeneous system of first order, ordinary differential equations, which includes the traditional linear model as a special case. When A and b are constant we are left with the far simpler autonomous case. In our analysis we will find it necessary to use definitions of stability commonly found in the literature on the stability of non-autonomous systems. The equation

$$\dot{y} = A(t)y \tag{1.13}$$

is said to be

(i) UNIFORMLY STABLE if and only if there exists a positive constant K such that

$$|Y(t)Y^{-1}(s)| \le K \qquad \text{for} \quad t_0 \le s \le t < \infty \tag{1.14}$$

(ii) UNIFORMLY ASYMPTOTICALLY STABLE if and only if there exist positive constants K and α such that

$$|Y(t)Y^{-1}(s)| \le Ke^{-\alpha(t-s)} \qquad \text{for} \quad t_0 \le s \le t < \infty \tag{1.15}$$

In contrast, the more familiar definitions of stability are of the form: STABILITY if and only if $|Y(t)| < K$ and ASYMPTOTIC STABILITY if and only if $|Y(t)| \to 0$ as $t \to \infty$. Throughout, $Y(t)$ denotes the fundamental matrix of (1.13). In subsequent analysis we will find that the stability of our system will depend on the bounds we can compute on the characteristic roots of $A(t)$, the rate of change of $A(t)$ and $b(t)$, the boundedness of $\dot{A}^{-1}(t)$, $A(t)$ and $b(t)$, and the magnitude of the changes of $A(t)$.

A final note before proceeding: Our model can be regarded either as an approximation of an equilibrium system, in which case we seek a tendency to return to equilibrium; or disequilibrium system in which the short side of the market prevails and in which goods are not carried from one period to the next.

II

As our first result we state

THEOREM 1. (a) *If, for the multi-market system,*

$$\dot{z} = A(t)z + h(t) \tag{2.1}$$

(i) *The real parts of the characteristic roots of* $A(t)$ *are all less than or equal to* $-\varepsilon$, *where* $\varepsilon > 0$, *for all* $t \ge t_0$,

(ii) *taste change is bounded and results in the boundedness of* $|A(t)|$ *and* $|b(t)|$ *for all* $t \geq t_0$, *and*

(iii) *tastes change occurs at a sufficiently small rate where "sufficiently small" is defined as* $|\dot{A}(t)|$ *satisfying the bound derived in the proof below and also satisfying*

$$\int_{t_0}^{\infty} |\dot{A}(s)|\, ds \leq M_1 < \infty \tag{2.2}$$

$$\int_{t_0}^{\infty} |\dot{b}(s)|\, ds \leq M_2 < \infty \tag{2.3}$$

then the multi-market system is stable.

(b) *If, in addition, tastes converge to an ultimate taste pattern at a sufficiently fast rate, where "sufficiently fast" is defined to be a rate which satisfies*

$$|\dot{A}(t)| \leq K_1 e^{-\sigma t} \qquad \sigma > 0 \qquad for \quad t \geq t_0 \tag{2.4}$$

$$|\dot{b}(t)| \leq K_2 e^{-\sigma t} \qquad \sigma > 0 \qquad for \quad t \geq t_0 \tag{2.5}$$

then the system is globally asymptotically stable and the absolute value of $z(t)$, *the deviation of* $P(t)$ *from* $P^*(t)$, *satisfies an exponential bound*

$$|P(t) - P^*(t)| \leq K e^{-\alpha t} \quad \alpha > 0 \qquad for \quad t \geq t_0 \tag{2.6}$$

Proof. (a) Consider the Liapunov function

$$V(x, t) = X'Q(t)X \tag{2.7}$$

where $Q(t)$ is a bounded, differentiable symmetric, positive definite matrix which satisfies

$$QA + A'Q = -I\,. \tag{2.8}$$

By a theorem of Liapunov we know that a symmetric positive definite matrix satisfying (2.8) exists for a given matrix A whose roots all have negative real parts less than $-\varepsilon$. Therefore, a matrix $Q(t)$ can be found for each $A(t)$. Since $A(t)$ is differentiable, $Q(t)$ must also be differentiable. The boundedness and non-singularity of $A(t)$ assures the boundedness of $Q(t)$. Taking the derivative of (2.8) with respect to time we have

$$Q\dot{A} + Q\dot{A} + \dot{A}'Q + A'\dot{Q} = 0\,. \tag{2.9}$$

The smaller the bound we impose on $\|\dot{A}\|$, the smaller will be the bound on $\|\dot{Q}\|$ since $A(t)$ and $Q(t)$ are bounded.

The time derivative of our Liapunov function is

$$\dot{V} = x'[A'Q + QA + \dot{Q}]x \tag{2.10}$$

$$= x'Bx \tag{2.11}$$

where

$$B=[-I+\dot{Q}]\,. \tag{2.12}$$

Clearly, if B, a symmetirc matrix, is negative definite over the entire interval, the null solution of

$$\dot{x}=A(t)x \tag{2.13}$$

is exponentially stable. This is guaranteed if the elements of $\dot{Q}$ are sufficiently small. The fact that $Q(t)$ is bounded means that $V(x, t)$ possesses an infinitely small upper bound, ensuring the uniform asymptotic stability of (2.13).

We now return to our system

$$\dot{z}=A(t)z-\dot{p}^* \tag{2.14}$$

which has the solution

$$z(t)=X(t)X^{-1}(t_0)\eta+\int_{t_0}^{t} X(t)X^{-1}(s)h(s)ds\,. \tag{2.15}$$

The boundedness of $A(t)$ totether with the strict negativity of its characteristic roots ensures that $A^{-1}(t)$ exists and is bounded for all $t\geq t_0$. Continuous differentiability of both $A(t)$ and $b(t)$ ensures that $\dot{A}(t)$ and $\dot{b}(t)$ are also bounded for $t\geq t_0$. Since

$$\dot{A}^{-1}=A^{-1}\dot{A}A^{-1} \tag{2.16}$$

and

$$|\dot{A}^{-1}|=|A^{-1}\dot{A}A^{-1}|\leq|A^{-1}|^2|\dot{A}|\,. \tag{2.17}$$

This allows us to deduce that

$$\int_{t_0}^{\infty}|\dot{A}^{-1}(s)|ds\leq M_3<\infty \qquad t\leq\infty \tag{2.18}$$

and, therefore, that

$$\int^{\infty}|h(s)|ds\leq M_4<\infty \qquad t\leq\infty \tag{2.19}$$

since each individual term in $h(s)$ is either bounded or has a convergent integral. We have

$$|z(t)|\leq|X(t)X^{-1}(t_0)\eta|+\int_{t_0}^{t}|X(t)X^{-1}(s)||h(s)|ds \tag{2.20}$$

$$\leq K_1e^{-\beta t}+K_2M_3 \tag{2.21}$$

due to the uniform asymptotic stability of the homogeneous system, and

$$|P(t)-P^*(t)|\leq K_2M_3+K_1e^{-\beta t} \qquad t\geq t_0 \tag{2.22}$$

(b) If $|\dot{A}(t)|$ converges at a sufficiently fast rate then $|\dot{A}^{-1}(t)|$ does also since

$$|\dot{A}^{-1}(t)| \leq |A^{-1}(t)|^2 |\dot{A}(t)| A^{-1}(t)|^2 K_1 e^{-\sigma t} \qquad t \geq t_0 \tag{2.23}$$

Thus, $|h(t)|$, consisting of bounded terms multiplied by exponentially declining terms must also satisfy an exponential bound

$$|h(t)| \leq K_3 e^{-\sigma t} \qquad t \geq t_0 \tag{2.24}$$

Using (2.21) we have

$$z(t) \leqq K_1 e^{-\beta t} + K_4 \int e^{-\beta(t-s)} e^{-\sigma s} ds \tag{2.25}$$

$$\leq K_5 e^{-\gamma t} \qquad \gamma > 0 \tag{2.26}$$

where γ is the maximum of $(\beta - \sigma, \sigma - \beta, \beta)$. Q.E.D.

The theorem allows us to deduce stability of the linear competitive system on the basis of the negativity of the real parts of all characteristic roots of $A(t)$ over the entire interval, the boundedness of our excess demand functions, and a "sufficiently" small rate of change of the coefficients of the excess demand matrix, $A(t)$. It should be stressed that the boundedness of $\|\dot{A}(t)\|$ is not in itself sufficient when combined with the conditions on the real parts of the characteristic roots of $A(t)$. We need $\|\dot{A}(t)\|$ and therefore, $\|Q(t)\|$ to satisfy an a priori bound. Since $Q(t)$ is unique for each $t \in [0, \infty)$, and $\|\dot{Q}(t)\|$ is directly related to $\|\dot{A}(t)\|$ it is clear that for sufficiently small $\|A(t)\|$ our Liapunov function has a negative time derivative along all solutions of the homogeneous portion of our system.

The restrictions on $\dot{A}(t)$ can be viewed as representing bounds on the rate of change of price elasticities of demand which, in principle, are empirically observable phenomena. For the purpose of the theorem, the rate of change of the intercept terms need only be bounded, thereby ensuring the boundedness of $\dot{P}^*(t)$. The standard case, in which no exogenous changes occur, can be seen as being a special case of the above system since $\dot{P}^*(t)$ is identically zero. However, it should be noted that as long as exogenous change occurs, i.e., if either A or b is time dependent, the standard analysis is inappropriate since $\dot{P}^*(t)$ will not be zero.

The behavior of the actual price, $P(t)$, over time is evident from the estimates (2.25) and (2.26) above. Given an initial set of prices $P(0)$, all deviations from $P^*(t)$ will be bounded by a function which decreases over time. If $\|\dot{P}^*(t)\|$ is simply bounded the function approaches a positive constant asymptotically; thus, the set has the form as shown inside the shaded areas in Fig. 1. All solutions will remain within the outer boundaries of the shaded region. The boundaries of the region tend to a horizontal shape as $t \to \infty$. In the case of exponential convergence of the coefficients of our excess demand function the boundaries of our attraction set approach zero as $t \to \infty$. (Shaded area is region of attraction. Area between dotted line is stability region.)

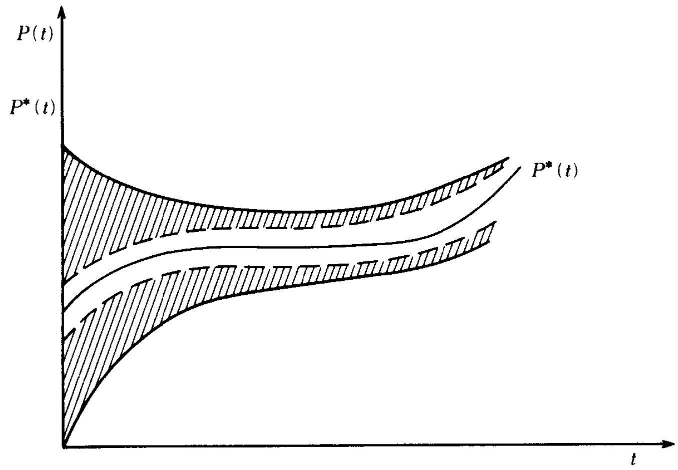

Fig. 1.

We now seek to establish stability conditions on the linear competitive system when the rate of change of the coefficients of our excess demand functions, and therefore, the rate of taste change, are bounded but not required to satisfy an *a priori* bound. However, we must first develop some preliminary concepts. The measure of a matrix A, $\mu(A)$ is defined to be

$$\mu(A) = \lim_{h \to +0} \frac{|I+hA|-1}{h} \tag{2.27}$$

where $|I+hA|$ is the norm of $I+hA$. The above formulation can be used with any norm which translates an array, such as a matrix or a vector, into a nonnegative distance on the real line. An important property of a matrix is

$$\mu(A) \leq |A| \tag{2.28}$$

which can be deduced from the definition. A second important property of the measure of a matrix is

$$\mu(A) - \mu(B) \leq |A-B|\,. \tag{2.29}$$

This follows from

$$\lim_{h \to +0} \left| \frac{|I+hA|-1}{h} - \frac{|I+hB|-1}{h} \right| \leq \lim_{h \to +0} \left| \frac{1}{h} + \frac{|hA|}{h} - \frac{1}{h} - \frac{1}{h} - \frac{|hB|}{h} + \frac{1}{h} \right|. \tag{2.30}$$

The importance of (2.29) is that if $A = A(t)$ and $B = A(t+h)$ we can deduce that $\mu[A(t)]$ possesses a right-hand derivative.[1]

[1] Much of the material regarding the continuous differentiability of the measure of $A(t)$ is drawn from Coppel [10], Chapters 1 and 2.

THEOREM 2. (a) *If*

(i) *taste change is bounded and results in bounded demand elasticities and intercepts; and*

(ii) *tastes change smoothly and continuously yielding excess demand functions whose coefficients and intercepts have bounded time derivatives and satisfy* (2.2) *and* (2.3), *and*

(iii) *the own price effect is negative and greater in absolute value than the sum of the absolute values of the cross price effects for each market over the entire time interval,* $0 \leq t \leq \infty$ *then the linear competitive system is stable.*

(b) *If the conditions of part* (a) *hold and if tastes converge to an ultimate taste regime exponentially, resulting in the exponential convergence of* $b(t)$ *and* $A(t)$, *the linear competitive system is uniformly asymptotically stable.*

Proof. (a) Let $r(t) = |Y(t)|$ where $Y(t)$ is a fundamental solution of

$$\dot{y} = A(t)y \tag{2.31}$$

and $r_+'(t)$ denote the right-hand derivative of $|Y(t)|$.

$$r_+' = \lim_{h \to +0} \frac{|Y(t) + h\dot{Y}(t)| - |Y(t)|}{h} \tag{2.32}$$

$$= \lim_{h \to +0} \frac{|Y(t) + hA(t)Y(t)| - |Y(t)|}{h} \tag{2.33}$$

$$r_+' \leq \lim_{h \to +0} \left| \frac{|I + hA(t)| - 1}{h} \right| |Y(t)| \,. \tag{2.34}$$

Since (2.22) satisfies the Kamke condition for differential inequalities we can state

$$r_+'(t) \leq \mu[A(t)] r(t) \,. \tag{2.35}$$

Integrating (2.23) we have

$$|Y(t)| \leq |Y(t_0)| \exp \left| \int_{t_0}^{t} \mu[A(s)] ds \right| . \tag{2.36}$$

Provided that $\mu[A(t)] < -\gamma < 0$ over the entire interval (t_0, ∞) we have uniform asymptotic stability.[2] The strict negativity of the measure of $A(t)$ throughout precludes convergence of the integral on the right-hand side, i.e.,

$$|Y(t)| < |Y(t_0)| \exp[-\gamma t] \,. \tag{2.37}$$

Choosing as a norm

$$|Y| = \sup_{1 \leq i \leq n} |Y_i| \,. \tag{2.38}$$

[2] The mathematics which forms the basis of the theorem is due to Lozinskii [22].

We find that the measure of $A(t)$, for this choice of norm for $A(t)$ real, is defined to be

$$\mu[A(t)] = \sup_{1 \leqq i \leqq n} \left[a_{ii}(t) + \sum_{j \neq i}^{n} |a_{ij}(t)| \right]. \tag{2.39}$$

The strict negativity of (2.27) is the condition on the slopes of the excess demand functions imposed by the theorem.

As in the proof of the previous theorem we consider the explicit solution of the linear competitive system

$$z(t) = X(t)X^{-1}(t_0)\eta + \int_{t_0}^{t} X(t)X^{-1}(s)h(s)ds. \tag{2.40}$$

The boundedness of the elements of $A^{-1}(t)$, $b(t)$, $\dot{A}^{-1}(t)$ and $b(t)$ again ensures the boundedness of the elements of $|\dot{P}^*(t)|$ over the entire interval, $[0, \infty)$. Using the same argument as in Theorem 1 we can deduce stability.

(b) Again, the argument is the same as in part b of Theorem 1. Q.E.D.

The results of Theorem 2 depend on our choice of norm. By choosing the norm

$$|Y| = \sum_{i=1}^{n} |Y_i| \tag{2.41}$$

we obtain another sufficiency condition since for this norm with $A(t)$ real, yields, as a computation for the norm of $A(t)$

$$\mu[A(t) = \sup_{1 \leqq j \leqq n} \left[a_{jj}(t) + \sum_{j \neq i} |a_{ji}(t)| \right]. \tag{2.42}$$

We can now state:

THOREM 3. (a) *If conditions* (i) *and* (ii) *of Thorem* 2 *hold and if the effect of a change in the* i^{th} *price has a greater effect on excess demand in the* i^{th} *market than the sum of the absolute values of its effects on all other markets, for all i, and over the entire interval* $[t_0, \infty)$, *i.e., if*

$$a_{ii}(t) + \sum_{i \neq j}^{n} |a_{ij}(t)| \leq -\gamma < 0 \qquad j = 1, \cdots, n \tag{2.43}$$

for all $t \geq t_0$, *then the linear competitive system is uniformly stable.*

(b) *If, in addition, part* (b) *of Theorem* 2 *holds then the system is uniformly asymptotically stable.*

Theorems 2 and 3 are *the non-autonomous counterparts of the dominant diagonal theorems of the traditional analysis. It is interesting to note that since the signs of the off-diagonal elements need not be specified, and since these elements are functions of time, the transition of one good for being a substitute for, say the* k^{th} *good to being a complement for the same good in a subsequent time period need not destabilize the* *system, providing that the conditions of the theorems are met.* Also, it should be noted that the sufficiency conditions derived from the use of the measure of $A(t)$ are slightly stronger than the requirement of the strict negativity of the real parts of all the characteristic roots of $A(t)$ over the entire interval, $[t_0, \infty)$. It can be shown that $\mu[A(t)]$ is an upper bound for the $R[\lambda_i(t)]$ for any $t \in [t_0, \infty)$.

Finally, it should be clear that none of the theorems above is a special case of any of the others. In Theorem 1 it was necessary to impose restrictions on the norm of $\dot{A}(t)$ to derive sufficient conditions for the convergence of the homogeneous system. Thus, not just any bound on $\|\dot{A}(t)\|$ would suffice. However, *it is evident that systems which do not possess the dominant diagonal properties of Theorems* 2 *or* 3 *could satisfy the sufficiency criteria of Theorem* 1.

We now consider the question of whether the linear competitive system can be shown to be stable if the magnitudes of the taste change effects in the excess demand matrix are "sufficiently small." Intuition suggests that infinitesimally small changes in the price effect terms would not significantly alter the stability properties of the system. To determine just how small these changes must be we consider a special case of our system where

$$A(t) = [A + B(t)] \tag{2.44}$$

where A is a matrix of constants and whose characteristic roots all have strictly negative real parts. $B(t)$ is a matrix whose elements are continuous functions of time. We can now state

THEOREM 4. (a) *If taste change is of a sufficiently small magnitude, resulting in sufficiently small deviations of the excess demand function matrix $A(t)$ from a constant matrix A, all of whose characteristic roots have real parts strictly negative, i.e., if $|B(t)|$ is "sufficiently small," which is defined to be*

$$|B(t)| < \frac{\sigma}{K} \tag{2.45}$$

where $-\sigma$ is greater than the largest characteristic root of A, and K is a positive constant whose magnitude depends on the characteristic roots of A, and conditions (i) *and* (ii) *of Theorem* 2 *hold.*

$$\left(\textit{in particular} \int^{\infty} |\dot{b}(s)|\, ds \le m_1 < +\infty, \int^{\infty} |\dot{B}(s)|\, ds \le m_2 < +\infty \right)$$

over the entire interval, $t \in [t_0, \infty]$, then the linear competitive system is stable.

(b) *If all the conditions of part* (a) *hold, and if $|B(t)|$ and $|\dot{b}(t)|$ decline at an exponential rate then the competitive equilibrium system is uniformly asymptotically stable.*

Proof. Let $X(t)$ represent the solution of the autonomous system

$$\dot{x} = Ax \tag{2.46}$$

and $Q(t)$ can be expressed as

$$Q(t)=X(t)X^{-1}(t_0)Q(t_0)+X(t)\int_{t_0}^{t} X^{-1}(s)B(s)Q(s)ds \tag{2.47}$$

Using the fact that (2.46) is uniformly asymptotically stable we have

$$|Q(t)| \leq K \exp[-\sigma(t-t_0)]\,|Q(t_0)| + k\int_{t_0}^{t} \exp[-r(t-s)\,|B(s)|\,|Q(s)|\,ds\,. \tag{2.48}$$

Multiplying both sides by exp (σt), and denoting

$$w(t)=\exp(\sigma t)\,|Q(t)| \tag{2.49}$$

we have

$$w(t)\leq K \exp(\sigma t_0)|\,Q(t_0)\,| + K\int_{t_0}^{t} w(s)\,|B(s)|\,ds\,. \tag{2.50}$$

Application of Gronwall's inequality yields

$$w(t)\leq k \exp(\sigma t_0)\,|\,Q(t_0)\,|\exp\left[K\int_{t_0}^{t} |\,B(s)\,|\,ds\right]. \tag{2.51}$$

If $|B(t)|<\sigma/k$ for all $t\in[t_0, \infty]$ then our solution of $(x=(A+B(t))$ is uniformly asymptotically stable.

We again turn to the full system. As before we conclude that the boundedness of $\dot{P}^*$ together with the uniform asymptotic stability of the homogeneous segment yields the desired result. The proof of part (b) is also straightforward and is omitted. Q.E.D.

III

In conclusion, we outline some of the factors that account for the divergence of our analysis from the traditional results.

The fact that $A(t)$ is a function of t together with $b>0$ or $A(t)$ const and $b(t)$ a function of time makes the system non-autonomous since it insures that $P^*=P^*(t)$, a function of time. This is what prevents easy translation to origin, i.e., it always leaves one extra term.

The stability of $\dot{x}=A(t)x$ requires more restrictions than the constant A case to ensure stability. It can be shown that some $A(t)$ with rapidly increasing off-diagonal elements are unstable. Also, crossings of characteristic roots are a problem for exponential stability. This is similar to the standard constant A case in that it can cause divergence from equilibrium for a period of time. However, unlike the constant A case, where it is known that exponential terms in solution will soon dominate the t^n terms, in the $A(t)$ case it is not known when and how often roots will cross, making the establishment of an exponentially decreasing upper bound on the absolute divergence from equilibrium difficult. Thus, not just any $A(t)$ with all roots having negative real parts will do.

The restrictions on $|b(t)|$ and $|\dot{A}(t)|$ are due to the fact that, in our system, $A(t)$ and $b(t)$ are "driving forces" which would otherwise continuously drive the system from equilibrium. The assumption that tastes converge over time to an ultimate taste pattern is consistent with the hypothesis that taste is a function of the level of education which itself converges as a function of time.

As a final note it should be made clear that the case of variable speeds of adjustment in the individual markets is a far simpler matter than what has been discussed above. The possibility that the adjustment speeds are functions of time does not yield an equilibrium price which varies as a function of time. The resulting system is non-autonomous but is still homogeneous after the translation, $z(t)=(P(t)-P^e)$. For this case the system is stable whenever $|X(t)X^{-1}(s)|\leq K$ and is uniformly asymptotically stable whenever $|X(t)X^{-1}(s)|\leq Ke^{-\sigma t}$, for K, $\sigma>0$, in the theorems above.

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