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A VARIATIONAL PROBLEM RELATING TO THE THEORY OF RESOURCE ALLOCATIONS

Toru MARUYAMA*

1. INTRODUCTION

Aumann-Perles [5] rigorously examined the following variational problem and established a sufficient condition for the existence of optimal solutions.

Let $T=[0, 1]$ with Lebesgue measure $\mathrm{d}t$ and $X=\mathbf{R}^l_+$ (non-negative orthant of $\mathbf{R}^l$). Furthermore let $u: T\times X\to\mathbf{R}$ be measurable on $T\times X$. The problem is to

$$\underset{x:T\to X}{\textit{Maximize}} \quad \int_T u(t, x(t))\mathrm{d}t$$

subject to

$$\int_T x(t)\,\mathrm{d}t=(1, 1, \cdots, 1).$$

The motivation of this type of the variational problem comes from mathematical economics. (*cf.* Aumann-Shapley [6], Kawamata [17], and Yaari [23].) For example, we can interprete $u(t, x)$ as the utility of the agent t when his consumption vector is x. The total quantity of each consumption good is assumed to be equal to 1. In this setting, $x: T\to X$ can be interpreted as an allocation of consumption goods among agents. And our problem is to find out an allocation which maximizes the sum (or integral) of the utilities of all agents corresponding to this allocation.

Some generalizations of the Aumann-Perles' problem were given by Artstein [3], [4] and Berliocchi-Lasry [8]. (See also Arkin [1], Arkin-Levin [2], Balder [7], and Ekeland-Temam [16], pp. 361–373.)

The present paper is a revised and enlarged version of Maruyama [18], and aims at a further sophistification of the above problem, which is required for a certain kind of economic analysis (*cf.* Kawamata [17]).

Sketch of the problem: Let T be a compact metric space, and $\bar{\mu}$ be a non-atomic positive Radon measure on T which satisfies

(i) $\bar{\mu}(T)=C<+\infty$,

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(ii) $\bar{\mu}(\partial E)=0$ for every measurable set E in T.

We designate by $\mathfrak{M}_\chi$ the set of all non-negative Radon measures μ on T such that μ is absolutely continuous with respect to $\bar{\mu}$ (denoted by $\mu \ll \bar{\mu}$) and

(*) the Radon-Nikodým derivative of μ is a characteristic function of some measurable set of T.

Let X be a locally compact Polish space, and let

$$u: T\times X \to \mathbf{R}$$

$$g_i: T\times X \to \bar{\mathbf{R}}_+ ; \qquad i=1, 2, \cdots, l.$$

Then our problem is:

(I)
$$\underset{\mu,\, x}{Maximize} \int_T u(t, x(t))\,\mathrm{d}\mu$$

subject to

a) $\int_T g_i(t, x(t))\,\mathrm{d}\mu \leqq \omega_i; \qquad i=1, 2, \cdots, l$

b) $\mu \in \mathfrak{M}_\chi$

c) $x: T\to X$ is measurable

where $(\omega_1, \omega_2, \cdots, \omega_l)$ is a fixed vector.

Our purpose is to establish a set of sufficient conditions which assures the existence of optimal solutions for the problem (I).

Let $\mu \in \mathfrak{M}_\chi$ and let h be its Radon-Nikodým derivative. Then h and $x: T\to X$ jointly determine a Radon measure on $T\times X$ of the form:

$$(\dagger) \qquad \gamma = \int_T \delta_t \otimes \delta_{x(t)} h(t)\,\mathrm{d}\bar{\mu}.$$

Hence our problem is equivalent to the problem:

(II)
$$\underset{\gamma}{Maximize} \int_{T\times X} u(t, x)\,\mathrm{d}\gamma$$

subject to

a) $\int_{T\times X} g_i(t, x)\,\mathrm{d}\gamma \leqq \omega_i; \qquad i=1, 2, \cdots, l$

b) γ is of the form $(\dagger)$.

I am indebted to Berliocchi-Lasry [8] for such a transformation of the original problem (I) into the form (II) and a full use of disintegration theory in this problem. In contrast with Berliocchi-Lasry [8], we introduce the new control variable h as well as x.

II. SOME RESULTS ON DISTINTEGRATION OF MEASURES

Let γ be a Radon measure on $T\times X$ which can be expressed as

$$\gamma=\int_T \delta_t\otimes\nu[t]\,\mathrm{d}\mu(t)\,,$$

where δ_t is the Dirac measure at t, μ is a Radon measure on T, and $\nu: t\mapsto \nu[t]$ is a weak*-measurable mapping on T into the set of all Radon probability measures on X. If such a expression is possible, γ is said to have a *μ-disintegration.*

It may be convenient to collect here a few results on distintegration of measures which are useful in later discussions.

T and X are assumed to be compact throughout this section.

PROPOSITION 1 (Castaing [12]). *Let $\Gamma: T\twoheadrightarrow X$ be a measurable multi-valued mapping such that $\Gamma(t)\subset X$ is compact for all $t\in T$. Then a Radon measure γ on $T\times X$ has a disintegration of the form:*

$$\gamma=\int_T \delta_t\otimes\nu[t]\,\mathrm{d}\mu$$

$$\operatorname{supp}\ \nu[t]\subset\Gamma(t)\ \text{a.e.}\ (t)$$

if and only if

$$\int_{T\times X} f(t,x)\,\mathrm{d}\gamma\leqq\int_T \sup_{x\in\Gamma(t)} f(t,x)\,\mathrm{d}\mu$$

for all $f\in C(T\times X)$, the set of all continuous real-valued functions on $T\times X$.

PROPOSITION 2 (Maruyama [19]). *Consider*

$$\gamma_n=\int_T \delta_t\otimes\nu_n[t]\,\mathrm{d}\mu_n;\qquad n=1,2,\cdots$$

$$\gamma=\int_T \delta_t\otimes\nu[t]\,\mathrm{d}\mu\,.$$

(i) *If*

a) $w^*\text{-}\lim \mu_n=\mu$
b) $t_p\to t$ *implies* $w^*\text{-}\lim \nu_n[t_p]=\nu_n[t]$ *for all* n
⟨*continuity*⟩
c) $t_n\to t$ *implies* $w^*\text{-}\lim \nu_n[t_n]=\nu[t]$ *for all* $t\in T$,
⟨*continuous convergence*⟩

then $w^*\text{-}\lim \gamma_n = \gamma$.

(ii) $w^*\text{-}\lim \gamma_n = \gamma$ *implies* a). *But* b) *and* c) *are not necessarily true.*

Let $\mathfrak{M}_S$ be the set of all those Radon measures on T which are absolutely continuous with respect to $\bar{\mu}$ and the Radon-Nikodým derivatives belong to

$$S \equiv \{h \in L^1(\bar{\mu}) \mid 0 \leqq h(t) \leqq 1 \quad \text{a.e.}\}.$$

Before we proceed to the next lemma, we should remind of the following fact:

> If T is a compact metric space, then the set of all Radon measures μ on T such that $0 \leqq \mu(T) \leqq C$ endowed with the weak*-topology is metrizable. (*cf.* Maruyama [21], Theorem 5.23.)

LEMMA 1. $\mathfrak{M}_S$ *is convex and weak*-compact.*

Proof. Since the convexity is almost obvious, it is enough to prove the weak*-compactness. Since the set $\mathfrak{M}$ of all non-negative Radon measures μ on T such that $\mu(T) \leqq C$ is weak*-compact, we have only to prove that $\mathfrak{M}_S$ is a closed subset of $\mathfrak{M}$. Let

$$d\mu_n = h_n d\bar{\mu}; \qquad h_n \in S$$

be a sequence in $\mathfrak{M}_S$ which converges, in the weak*-topology, to some Radon measure μ on T; that is

$$\int_T f\,\mathrm{d}\mu_n = \int_T f h_n\,\mathrm{d}\bar{\mu} \to \int_T f\,\mathrm{d}\mu \qquad \text{for every} \quad f \in C(T), \tag{1}$$

where $C(T)$ is the set of all the continuous functions on T into $\mathbf{R}$. Our aim is to show that $\mu \in \mathfrak{M}_S$.

S is weakly relatively compact in $L^1(\bar{\mu})$ because it is L^1-bounded and uniformly integrable. (*cf.* Maruyama [21], Theorem 5.18.) Hence, by Eberlein-Šmulian's Theorem, $\{h_n\}$ has a weakly convergent subsequence $\{h_{n_m}\}$ to some $h \in L^1(\bar{\mu})$; that is

$$\int_T g h_{n_m}\,\mathrm{d}\bar{\mu} \to \int_T g h\,\mathrm{d}\bar{\mu} \qquad \text{for every} \quad g \in L^\infty(\bar{\mu}). \tag{2}$$

Since $C(T) \subset L^\infty(\bar{\mu})$ in this case, (1) and (2) jointly imply that

$$\int_T f h_{n_m}\,\mathrm{d}\bar{\mu} \to \int_T f h\,\mathrm{d}\bar{\mu} = \int_T f\,\mathrm{d}\mu \qquad \text{for every} \quad f \in C(T). \tag{3}$$

Therefore we must have $\mathrm{d}\mu = h\mathrm{d}\bar{\mu}$. Since S is strongly closed, it is also weakly closed. Consequently h is an element of S as it is a weak limit of a sequence in S. Thus we have completed the proof of the desired result: $\mu \in \mathfrak{M}_S$. Q.E.D.

REMARK. In Lemma 1, the uniform boundedness of the Radon-Nikodým derivatives of the measures in $\mathfrak{M}_S$ is crucial. We have to keep in mind the fact that the set of all those Radon measures μ such that

$$\text{(i)} \quad \mu \ll \bar{\mu}\,, \qquad \text{(ii)} \quad 0 \leqq \mu(T) \leqq C$$

(without any specification for the Radon-Nikodým derivatives)

is *not* weak*-compact. For example, let $T=[0, 1]$, $\bar{\mu}$ be the Lebesgue measure on T. Then we can find an approximating sequence $\{\mu_n\}$ in this set, which converges to the Dirac measure δ_0 at 0. But δ_0 is not absolutely continuous with respect to $\bar{\mu}$. The condition of the uniform boundedness of the Radon-Nikodým derivatives excludes such possibilities.

We designate by $\Delta(\mu)$ the set of all Radon measures on $T \times X$ which have μ-disintegrations, and put

$$\Delta(\mathfrak{M}_S) = \bigcup_{\mu \in \mathfrak{M}_S} \Delta(\mu)\,.$$

LEMMA 2. (Castaing [12]). *Let μ be a non-negative Radon measure on T. Then $\Delta(\mu)$ is convex and weak*-compact.*

LEMMA 3. *The multi-valued mapping*

$$\Delta : \mu \longmapsto\!\!\!\!\rightarrow \Delta(\mu)$$

is compact-valued and upper hemi-continuous (u.h.c.) on $\mathfrak{M}_S$.

Proof. The compact-valuedness of Δ is an immediate consequence of Lemma 2. Hence it is sufficient to show the u.h.c.

Let $\{\mu_n\}$ be a sequence in $\mathfrak{M}_S$ which converges to $\mu_0 \in \mathfrak{M}_S$. Pick up any element

$$\gamma_n = \int_T \delta_t \otimes \nu_n[t]\, \mathrm{d}\mu_n\,; \qquad n=1, 2, \cdots$$

of $\Delta(\mu_n)$. As is well-known, Δ is u.h.c. at μ_0 if and only if there exists a convergent subsequence $\{\gamma_{n_m}\}$ of $\{\gamma_n\}$ whose limit belongs to $\Delta(\mu_0)$. (*cf.* Maruyama [21], Theorem 2.28.)

Since $T \times X$ is compact, the set $\mathfrak{M}$ of all non-negative Radon measures on $T \times X$ where total variations are uniformly bounded by C is weak*-compact. Hence $\{\gamma_n\}$ has a convergent subsequence $\{\gamma_{n_m}\}$. Let

$$w^*\text{-}\lim_{m \to \infty} \gamma_{n_m} = \gamma_0\,.$$

We have to show that $\gamma_0 \in \Delta(\mu_0)$. Since each γ_n $(n=1, 2, \cdots)$ has μ_n-disintegration,

$$\int_{T \times X} f(t, x)\, \mathrm{d}\gamma_n \leqq \int_T \left[\sup_{x \in X} f(t, x)\right] \mathrm{d}\mu_n \tag{4}$$

for every $f \in C(T \times X)$ by Proposition 1. It can easily be seen, by Berge's Maximum Theorem (Maruyama [21], Thorem 2.32), the function

$$t \longmapsto \sup_{x \in X} f(t, x)$$

is continuous on T. Therefore $w^*\text{-}\lim_{n\to\infty} \mu_n = \mu_0$ implies that

$$\lim_{n\to\infty} \int_T [\sup_{x\in X} f(t, x)]\, \mathrm{d}\mu_n = \int_T [\sup_{x\in X} f(t, x)]\, \mathrm{d}\mu_0 . \tag{5}$$

Furthermore it follows from (4) and (5) that

$$\int_{T\times X} f(t, x)\, \mathrm{d}\gamma_0 \leqq \int_T [\sup_{x\in X} f(t, x)]\, \mathrm{d}\mu_0 .$$

Hence, again by Proposition 1, $\gamma_0 \in \Delta(\mu_0)$. This proves the u.h.c. of Δ. Q.E.D.

PROPOSITION 3. *$\Delta(\mathfrak{M}_S)$ is convex and weak*-compact.*

Proof. By Lemma 1, $\mathfrak{M}_S$ is weak*-compact. Hence $\Delta(\mathfrak{M}_S)$ is weak*-compact as an image of a compact set by a compact-valued u.h.c. multi-valued mapping Δ (Maruyama [21], Theorem 2.27). Convexity is almost obvious. Q.E.D.

III. CARATHÉODORY FUNCTIONS AND NORMAL INTEGRANDS

In this section, we are going to examine the continuity property of the mapping of the form:

$$\psi : \gamma \mapsto \int_{T\times X} f(t, x)\, \mathrm{d}\gamma , \qquad \gamma \in \Delta(\mathfrak{M}_S) .$$

If the function $f: T\times X \to \mathbf{R}$ is continuous and $\operatorname{supp} f$ (support of f) is compact, then ψ is obviously continuous. However the continuity or semi-continuity of ψ is assured even for larger classes of functions.

DEFINITION. Let $(T, \mathscr{E}, \mu)$ be a measure space and X be a topological space. A function $f: T\times X \to \mathbf{R}$ is called a *Carathéodory function* if it satisfies the following two conditions:

(i) $t \mapsto f(t, x)$ is μ-measurable for every $x \in X$,

(ii) $x \mapsto f(t, x)$ is continuous for almost every $t \in T$,

The following lemma is well-known as Scorza-Dragoni's Theorem. (*cf.* Berliocchi-Lasry [8], pp. 132–133; Maruyama [21], Theorem 6.34.)

LEMMA 4. *Assume that $(T, \mathscr{E}, \mu)$ is an exterior regular, Borel finite measure space, X is a second countable topological space, and $f: T\times X \to \mathbf{R}$ is a Carathéodory function. Then for any $\varepsilon > 0$, there is a closed set $F \subset T$ such that*

$$\mu(T \backslash F) < \varepsilon$$

$$f\,|_{F\times X} \quad \textit{is continuous}$$

(where $f\,|_{F\times X}$ means the restriction of f to $F\times X$).

PROPOSITION 4. *Assume that T and X are compact metric spaces. Then for any*

Carathéodory's function f, the mapping $\psi:\Delta(\mathfrak{M}_S)\to\mathbf{R}$ *defined by*

$$\psi:\gamma \;\mapsto\; \int_{T\times X} f(t,x)\,\mathrm{d}\gamma$$

is continuous.

Proof. Let $\{\gamma_n\}$ be a sequence in $\Delta(\mathfrak{M}_S)$ which converges to γ_0, where

$$\begin{aligned}\gamma_n&=\int_T \delta_t\otimes\nu_n[t]\,\mathrm{d}\mu_n; \qquad n=1,2,\cdots\\ \gamma_0&=\int_T \delta_t\otimes\nu_0[t]\,\mathrm{d}\mu_0.\end{aligned} \tag{1}$$

For any $\varepsilon>0$, choose $0<\delta<\varepsilon/8\|f\|$. Then, by Lemma 4, there is a compact set $K\subset T$ such that

$$\begin{aligned}&\mu_0(T\backslash K)<\delta\\ &f\big|_{K\times X}\quad\text{is continuous.}\end{aligned} \tag{2}$$

Since $w^*\text{-}\lim_{n\to\infty}=\gamma_0$ implies $w^*\text{-}\lim_{n\to\infty}\mu_n=\mu_0$ by Proposition 2, there exists an $n'\in\mathbf{N}$ such that

$$|\mu_n(T)-\mu_0(T)|<\delta \qquad \text{for all}\quad n\geqq n'. \tag{3}$$

(*cf.* Maruyama [21], Theorem 5.20 or Parthasarathy [22], pp. 40–42.) Since $\mu_0(\partial K)=0$, there exists an $n''\in\mathbf{N}$ such that

$$|\mu_n(K)-\mu_0(K)|<\delta \qquad \text{for all}\quad n\geqq n''. \tag{4}$$

Therefore

$$\begin{aligned}\mu_n(T\backslash K)&=\mu_n(T)-\mu_n(K)\\ &\leqq(\mu_0(T)+\delta)-(\mu_0(K)-\delta)\qquad\text{(by (3))}\\ &\leqq\mu_0(T)-\mu_0(K)+2\delta\\ &=\mu_0(T\backslash K)+2\delta<3\delta\qquad\text{(by (2))}\\ &\qquad\qquad\text{for all}\quad n\geqq\text{Max}\,(n',n'').\end{aligned} \tag{5}$$

Next, remark that f is continuous on the closed set $K\times X$. Hence, by Tietze Extension Theorem, $f\big|_{K\times X}$ has a continuous extension $\tilde{f}$ to the whole space $T\times X$ such that $\|\tilde{f}\|=\|f\|$ (sup-norm). Since $w^*\text{-}\lim\gamma_n=\gamma_0$, there is an $n'''\in\mathbf{N}$ such that

$$\left|\int_{T\times X}\tilde{f}\,\mathrm{d}\gamma_n-\int_{T\times X}\tilde{f}\,\mathrm{d}\gamma_0\right|<\varepsilon \qquad \text{for all}\quad n\geqq n'''. \tag{6}$$

Hence

$$\left|\int_{T\times X} f\,\mathrm{d}\gamma_n - \int_{T\times X} f\,\mathrm{d}\gamma_0\right|$$

$$\leqq \left|\int_{T\times X} f\,\mathrm{d}\gamma_n - \int_{T\times X} \tilde{f}\,\mathrm{d}\gamma_n\right|$$

$$+\left|\int_{T\times X} \tilde{f}\,\mathrm{d}\gamma_n - \int_{T\times X} \tilde{f}\,\mathrm{d}\gamma_0\right| + \left|\int_{T\times X} \tilde{f}\,\mathrm{d}\gamma_0 - \int_{T\times X} f\,\mathrm{d}\gamma_0\right|$$

$$\leqq 2\|f\|\,|\gamma_n(K^c\times X)+\gamma_0(K^c\times X)|$$

$$+\left|\int_{T\times X} \tilde{f}\,\mathrm{d}\gamma_n - \int_{T\times X} \tilde{f}\,\mathrm{d}\gamma_0\right|$$

$$\leqq 8\delta\|f\|+\varepsilon \qquad \text{(by (2), (4), (5), and (6))}$$

$$<2\varepsilon \qquad \text{for all} \quad n\geqq \mathrm{Max}\,(n', n'', n''').$$

Hence

$$\lim_{n\to\infty}\int_{T\times X} f\,\mathrm{d}\gamma_n = \int_{T\times X} f\,\mathrm{d}\gamma_0\,.$$

Q.E.D.

REMARK. Berliocchi-Lasry [8] proved a theorem corresponding to our Proposition 4 for the case

$$\psi:\Delta(\mu)\to\mathbf{R}$$

where μ is fixed and T and X are not necessarily compact. But when we regard μ as a variable, the problem becomes somewhat harder as we have seen.

The following definition specifies a more general kind of functions.

DEFINITION. Let $(T, \mathscr{E}, \mu)$ be a measure space and X be a topological space. A function $f:T\times X\to\bar{\mathbf{R}}$ is called a *normal integrand* if the multi-valued mapping

$$t \longmapsto\!\!\!\to \mathrm{Epi}\ f(t)\equiv\{(x,\alpha)\in X\times\bar{\mathbf{R}}\,|\,f(t,x)\leqq\alpha\}$$

is measurable and closed-valued.

The following lemma gives a characterization of positive normal integrands. (For the proof, see Berliocchi-Lasry [8], pp. 138–139 and Maruyama [21], Theorem 6.36, Theorem 6.38.)

LEMMA 5. *Let T and X be a locally compact Polish space, and μ be a finite Radon measure on T. Then the following three statements are equivalent for a function $f:T\times X\to\bar{\mathbf{R}}_+$.*

(i) *f is a positive normal integrand.*

(ii) *There exists a Borel function $g:T\times X\to\bar{\mathbf{R}}_+$ such that*

1. $x \mapsto g(t, x)$ *is* l.s.c.,
2. $f(t, x) = g(t, x)$

for almost every $t \in T$.

(iii) *There exists a sequence of Carathéodory functions* $\{f_n : T \times X \to \mathbf{R}_+\}$ *such that*

$$f(t, x) = \sup_n f_n(t, x) \qquad \textit{for almost every} \quad t \in T .$$

Then we can get the following fact as an immediate corollary of Proposition 4.

COROLLARY 1. *Assume that T and X are compact metric spaces. Then for any positive normal integrand f, the mapping*

$$\psi : \gamma \mapsto \int_{T \times X} f \, \mathrm{d}\gamma$$

is lower semi-continuous on $\Delta(\mathfrak{M}_S)$.

IV. ADMISSIBLE MEASURES

Let $g_1, g_2, \cdots, g_l : X \times Y \to [0, +\infty]$ be a positive normal integrands. In this section, we are interested in a sufficient condition which assures the compactness of $\Delta(\mathfrak{M}_S : g_1, g_2, \cdots, g_l)$ defined as follows:

$$\Delta(\mathfrak{M}_S;\, g_1, g_2, \cdots, g_l) \equiv \left\{ \gamma \in \Delta(\mathfrak{M}_S) \,\middle|\, \int_{T \times X} g_i \, d\gamma \leqq \omega_i \quad \text{for all} \quad i = 1, 2, \cdots, l \right\}$$

The following lemma is an easy consequence of Proposition 3 and Corollary 1.

LEMMA 6. *If T and X are compact metric spaces, then $\Delta(\mathfrak{M}_S;\, g_1, g_2, \cdots, g_l)$ is convex and weak*-compact.*

Once the above results is established, we can extend it, under some restriction, to the case where X is not necessarily compact. The idea of the proof is due to Berliocchi-Lasry [8], p. 150.

Let X be locally compact and $X^* = X \cup \{x_\infty\}$ be the one-point compactification of X. Then we say that a function $h : X \to \bar{\mathbf{R}}$ *diverges to* $+\infty$ *at* x_∞ (symbolically $h(x) \to +\infty$ as $x \to x_\infty$) if there is, for each $B > 0$, a compact set $K_B \subset X$ such that

$$h(x) \geqq B \qquad \text{for any} \quad x \in X \setminus K_B .$$

PROPOSITION 5. *Let T be a compact metric space and X be a locally compact Polish space. If*

$$\sum_{i=1}^{l} g_i(t, x) \to +\infty \quad \text{a.e.} \qquad \text{as} \quad x \to x_\infty ,$$

then $\Delta(\mathfrak{M}_S;\, g_1, g_2, \cdots, g_l)$ is convex and weak-compact.*

Proof. Since the convexity is obvious, we have only to show the compactness. If we define the normal integrand

$$g_{l+1}=\sum_{i=1}^{l} g_i\,,$$

then

$$g_{l+1}(t, x)\to +\infty \quad \text{a.e.} \qquad \text{as} \quad x\to x_\infty$$

and

$$\Delta(\mathfrak{M}_S\,;\, g_1, g_2, \cdots, g_l)=\Delta(\mathfrak{M}_S\,;\, g_1, g_2, \cdots, g_l, g_{l+1})\,.$$

where the right hand side is the set of all those elements γ of $\Delta(\mathfrak{M}_S\,;\, g_1, g_2, \cdots, g_l)$ which also satisfy the inequality

$$\int_{T\times X} g_{l+1}\,\mathrm{d}\gamma \leqq \sum_{i=1}^{l} \omega_i\,.$$

Hence we can assume, without loss of generality, that

$$g_1(t, x)\to +\infty \quad \text{a.e.} \qquad \text{as} \quad x\to x_\infty\,.$$

If we extend g_1 to $T\times X^*$ by

$$\tilde g_1(t, x)=\begin{cases} g_1(t, x) & \text{for} \quad x\neq x_\infty \\ +\infty & \text{for} \quad x=x_\infty\,, \end{cases}$$

then $\tilde g_1$ is also a normal integrand on $T\times X^*$. On the other hand, $g_2, g_3, \cdots, g_l$ can be extended to normal integrands on $T\times X^*$ by

$$\tilde g_i(t, x)=\begin{cases} g_i(t, x) & \text{for} \quad x\neq x_\infty \\ 0 & \text{for} \quad x=x_\infty\,. \end{cases}$$

We designate by $\Delta^*(\mathfrak{M}_S\,;\, \tilde g_1, \tilde g_2, \cdots, \tilde g_l)$ the set of all non-negative Radon measures γ^* on $T\times X^*$ such that

(i) $\gamma^*\in\Delta^*(\mathfrak{M}_S)$ where $\Delta^*(\mathfrak{M}_S)$ is defined in the same manner as $\Delta(\mathfrak{M}_S)$;

(ii) $\displaystyle\int_{T\times X} \tilde g_i(t, x)\,\mathrm{d}\gamma^* \leqq \omega_i\,; \qquad i=1, 2, \cdots, l\,.$

Then by Lemma 6, $\Delta^*(\mathfrak{M}_S;\, \tilde g_1, \tilde g_2, \cdots, \tilde g_l)$ is compact and convex.

Let $\theta(\gamma^*)$ be the restriction of $\gamma^*\in\Delta^*(\mathfrak{M}_S;\, \tilde g_1, \tilde g_2, \cdots, \tilde g_l)$ to the set of all continuous functions with compact supports in $T\times X$. Then clearly the mapping

$$\theta:\gamma^*\mapsto\theta(\gamma^*)$$

is a continuous bijection of $\Delta^*\,(\mathfrak{M}_S;\, \tilde g_1, \tilde g_2, \cdots, \tilde g_l)$ onto $\Delta(\mathfrak{M}_S;\, g_1, g_2, \cdots, g_l)$. Therefore we get the desired result. Q.E.D.

V. OPTIMAL SOLUTIONS

In this final section, we are going to formulate the problem exactly and find out a sufficient condition which assures the existence of optimal solutions.

Before we proceed to our main theorem, we need one more new concept.

DEFINITION. Let f and g be real-valued functions on $T\times X$. We write $f\prec g$ if for any $\varepsilon>0$, there exists a $\xi_\varepsilon\in L^1(\bar{\mu})$ such that

$$f(t, x)\geqq \xi_\varepsilon(t)\Rightarrow f(t, x)\leqq \varepsilon g(t, x) \quad \text{a.e.}$$

PROPOSITION 6. *Assume the following three conditions for $u: T\times X\rightarrow \mathbf{R}$.*

(i) *u is Borel measurable,*

(ii) *$u(t, x)$ is upper semi-continuous in x for almost every t,*

(iii) *$u^+\prec g$;* i.e. *for any $\varepsilon>0$, there exists a $\xi_\varepsilon\in L^1(\bar{\mu})$ such that*

$$u^+(t, x)\geqq \xi_\varepsilon(t)\Rightarrow u^+(t, x)\leqq \varepsilon g(t, x)$$

where

$$u^+(t, x)=\text{Max }\{u(t, x), 0\}\,.$$

Then the mapping

$$\gamma \mapsto \int_{T\times X} u(t, x)\,d\gamma$$

is upper semi-continuous on $\Delta(\mathfrak{M}_S;\, g_1, g_2, \cdots, g_l)$.

Proof. Let $X^*=X\cup\{x_\infty\}$ be the one-point compactification of X. And we can assume without loss of generality, that $g_1(t, x)\rightarrow +\infty$ as $x\rightarrow x_\infty$. Define $\tilde{g}_1$ and $\tilde{g}_i$ $(i=2, \cdots, l)$ exactly as in Proposition 5. Then $\Delta^*(\mathfrak{M}_S;\, \tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_l)$ is homeomorphic to $\Delta(\mathfrak{M}_S;\, g_1, g_2, \cdots, g_l)$ under the restriction mapping θ (*cf.* Proposition 5).

Furthermore if we define

$$\tilde{u}(t, x)=\begin{cases} u(t, x) & \text{for } x\neq x_\infty \\ +\infty & \text{for } x=x_\infty\,, \end{cases}$$

then $\tilde{u}$ is u.s.c. on $T\times X^*$. If we put

$$\tilde{u}^+(t, x)=\text{Max }\{\tilde{u}(t, x), 0\}$$

$$\tilde{u}^-(t, x)=\tilde{u}^+(t, x)-\tilde{u}(t, x)\,,$$

then $\tilde{u}^-$ is a positive normal integrand. Then by Corollary 1, the mapping

$$\gamma^* \mapsto \int_{T\times X^*} \tilde{u}^-(t, x)\,d\gamma^*$$

is l.s.c. on $\Delta^*(\mathfrak{M}_S;\, \tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_l)$. Hence we can conclude that the mapping

$$\gamma \mapsto \int_{T\times X} u^{-}\mathrm{d}\gamma$$

is l.s.c. on $\Delta(\mathfrak{M}_S; g_1, g_2, \cdots, g_l)$. (*cf.* Fig. 1).

$$\begin{tikzcd}
\Delta^*(\mathfrak{M}_S; \tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_l) \ni \gamma^* \arrow[r, "\text{l.s.c.}"] & \displaystyle\int_{T\times X^*} \tilde{u}^{-}(t, x)\,\mathrm{d}\gamma^* \arrow[d, equal] \\
\Delta(\mathfrak{M}_S; g_1, g_2, \cdots, g_l) \ni \gamma \arrow[u, "\theta^{-1}", "\text{cont.}"'] \arrow[r, "\psi"'] & \displaystyle\int_{T\times X} u^{-}(t, x)\,\mathrm{d}\gamma
\end{tikzcd}$$

Fig. 1.

Since $u^+ \prec g$, it is clear that $\tilde{u}^+ \prec \tilde{g} = \sum_{i=1}^{l} \tilde{g}_i$; i.e. for any $\varepsilon > 0$, there exists a function $\xi_\varepsilon \in L^1(\bar{\mu})$ (we can, and do, assume that $\xi_\varepsilon \geqq 0$ in this case) such that

$$\tilde{u}^+(t, x) \leqq \sup\{\xi_\varepsilon(t), \varepsilon\tilde{g}(t, x)\}\,.$$

Therefore $\tilde{u}^+ \in L^1(\gamma^*)$ for all $\gamma^* \in \Delta^*\{\mathfrak{M}_S; \tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_l\}$. If we put

$$h(t, x) = \inf\{\tilde{u}^+(t, x), \xi_\varepsilon(t)\}\,,$$

then

$$h(t, x) \leqq \tilde{u}^+(t, x) \leqq h(t, x) + \varepsilon\tilde{g}(t, x)\,. \tag{1}$$

Hence

$$0 \leqq \int_{T\times X^*} h\,\mathrm{d}\gamma^* \leqq \int_{T\times X^*} h\,\mathrm{d}\gamma^* + \varepsilon \sum_{i=1}^{l} \omega_i\,.$$

Since $\xi_\varepsilon(t) - h(t, x)$ is a positive normal integrand on $T\times X^*$, the mapping

$$\gamma^* \mapsto \int_{T\times X^*} (\xi_\varepsilon - h)\,\mathrm{d}\gamma^* = \int_{T\times X^*} \xi_\varepsilon(t)\chi_{X^*}(x)\,\mathrm{d}\gamma^* - \int_{T\times X^*} h(t, x)\,\mathrm{d}\gamma^*$$

is l.s.c. on $\Delta^*(\mathfrak{M}_S; \tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_l)$.

Here the first integrand $\xi_\varepsilon(t)\chi_{X^*}(x)$ is a Carathéodory function. So, by Proposition 4, the mapping

$$\gamma^* \mapsto \int_{T\times X^*} \xi_\varepsilon(t)\chi_{X^*}(x)\,\mathrm{d}\gamma^*$$

is continuous. Therefore the mapping

$$\gamma^* \mapsto -\int_{T\times X^*} h(t, x)\,\mathrm{d}\gamma^*$$

is l.s.c., and hence, by (1)

$$\gamma^* \mapsto \int_{T\times X^*} \tilde{u}^+(t, x)\,\mathrm{d}\gamma^*$$

is u.s.c. on $\Delta^*(\mathfrak{M}_S; \tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_l)$. By the same method as in the case u^-, we can conclude that

$$\gamma \mapsto \int_{T\times X} u^+(t, x)\,\mathrm{d}\gamma$$

is u.s.c. on $\Delta(\mathfrak{M}_S; g_1, g_2, \cdots, g_l)$.

Thus we get the desired result. Q.E.D.

By Propositions 5 and 6, the following problem (A) has a solution.

$$\text{(A)} \qquad \begin{aligned} &\mathop{Maximize}_{\gamma} \int_{T\times X} u(t, x)\,\mathrm{d}\gamma \\ &\qquad\qquad on \quad \Delta(\mathfrak{M}_S; g_1, g_2, \cdots, g_l) \end{aligned}$$

Let

$$\gamma^* = \int_T \delta_t \otimes \nu^*[t]\,\mathrm{d}\mu^*$$

be a solution of (A). Then γ^* is obviously a solution of the problem:

$$\text{(B)} \qquad \begin{aligned} &\mathop{Maximize}_{\gamma} \int_T u(t, x)\,\mathrm{d}\gamma \\ &\qquad\qquad on \quad \Delta(\mu^*; g_1, g_2, \cdots, g_l) \end{aligned}$$

(Note that $\Delta(\mu^*; g_1, g_2, \cdots, g_l)$ is also weak*-compact and convex.)

In order to approach our final goal, we have to prepare a couple of results from convex analysis. Proposition 7 comes from Carathéodory's theorem, and Proposition 8 is an easy corollary of Ljapunov's convexity theorem.

PROPOSITION 7. *Let $\mathfrak{X}$ be a locally convex topological linear space and K be a compact convex subset of $\mathfrak{X}$. Let $\psi_i: \mathfrak{X} \to \mathbf{R}$ $(i=1, 2, \cdots, l)$ be affine functions and define*

$$H = \{x \in K \mid \psi_i(x) \leqq 0;\ i=1, 2, \cdots, l\}.$$

Then any extreme point of H can be expressed as a convex combination of at most $(l+1)$ extreme points of K.

PROPOSITION 8. *Ler μ be a finite non-atomic measure of T and consider the* *formulas*:

$$\sum_{j=1}^{p} \lambda_j \int_T f_{ij}(t)\, d\mu; \qquad i=1, 2, \cdots, n$$

$$\lambda_j \geqq 0, \quad \sum_{j=1}^{p} \lambda_j = 1 .$$

Then there exists a decomposition $T_1, T_2, \cdots, T_p$ *of* T *such that*

$$\sum_{j=1}^{p} \lambda_j \int_T f_{ij}(t)\, d\mu = \sum_{j=1}^{p} \int_{Tj} f_{ij}(t)\, d\mu \qquad (i=1, 2, \cdots, n) .$$

(See Berliocchi-Lasry [8] or Maruyama [21], Theorem 6.6 for Proposition 7, and Maruyama [21], Corollary 6.3 for Proposition 8.)

Since the mapping

$$\gamma \mapsto \int_{T \times X} u(t, x)\, d\gamma$$

is linear and $\Delta(\mu^*; g_1, g_2, \cdots, g_l)$ is convex, γ^* can be assumed to be an extreme point of $\Delta(\mu^*; g_1, g_2, \cdots, g_l)$ without loss of generality.

According to Proposition 7, the extreme points of $\Delta(\mathfrak{M}_S; g_1, g_2, \cdots, g_l)$ can be expressed as a convex combination of those of $\Delta(\mathfrak{M}_S)$. Hence we are motivated to find out the concrete forms of the extreme points of $\Delta(\mathfrak{M}_S)$. Let

$$\gamma = \int_T \delta_t \otimes v[t] h(t)\, d\bar{\mu}$$

be an extreme point of $\Delta(\mathfrak{M}_S)$. Then we can claim that

h must be a characteristic function of some measurable set.

Proof. Assume that h is not a characteristic function of any measurable subset of T. Then there exists a non-zero integrable function $g: T \to [0, 1]$ such that

$$0 \leqq h(t) \pm g(t) \leqq 1 .$$

(*cf.* Castaing-Valadier [13], pp. 108–109 or Maruyama [21], Theorem 6.3.) If we define

$$\gamma_+ = \int_T \delta_t \otimes v[t](h+g)\, d\bar{\mu}$$

$$\gamma_- = \int_T \delta_t \otimes v[t](h-g)\, d\bar{\mu} ,$$

then γ_+ and γ_- are distinct elements of $\Delta(\mathfrak{M}_S)$ and clearly

$$\gamma = \frac{1}{2}(\gamma_+ + \gamma_-) .$$

This contradicts to our assumption that γ is an extreme point of $\Delta(\mathfrak{M}_S)$.

Q.E.D.

Consequently, any extreme point of $\Delta(\mathfrak{M}_S)$ must be of the form

$$\gamma=\int_T \delta_t \otimes \delta_{x(t)} \chi_E(t)\,\mathrm{d}\bar{\mu}$$

where χ_E is the characteristic function of E.

Hence by Proposition 8, there exists measurable mappings $x_j: T \to X$ ($j=1, 2, \cdots, l+1$) and measurable sets E_j ($j=1, 2, \cdots, l+1$) such that

$$\gamma^*=\sum_{j=1}^{l+1} \lambda_j \int_T \delta_t \otimes \delta_{x_j(t)} \chi_{E_j}(t)\,\mathrm{d}\bar{\mu}$$

$$\lambda_j \geqq 0, \quad \sum_{j=1}^{l+1} \lambda_j=1 .$$

By Proposition 8, there exists a decomposition $T_1, T_2, \cdots, T_{l+1}$ of T such that

$$\begin{aligned}\int_{T\times X} u(t, x)\,\mathrm{d}\gamma^* &= \sum_{j=1}^{l+1} \int_{T_j} u(t, x_j(t)) \chi_{E_j}(t)\,\mathrm{d}\bar{\mu} \\ &= \sum_{j=1}^{l+1} \int_{T_j \cap E_j} u(t, x_j(t))\,\mathrm{d}\bar{\mu}\end{aligned}$$

$$\begin{aligned}\int_{T\times X} g_i(t, x)\,\mathrm{d}\gamma^* &= \sum_{j=1}^{l+1} \int_{T_j} g_i(t, x_j(t)) \chi_{E_j}(t)\,\mathrm{d}\bar{\mu} \\ &= \sum_{j=1}^{l+1} \int_{T_j \cap E_j} g_i(t, x_j(t))\,\mathrm{d}\bar{\mu}\end{aligned}$$

$$(i=1, 2, \cdots, l).$$

If we define

$$x^*(t)=\sum_{j=1}^{l+1} \chi_{T_j \cap E_j}(t) x_j(t)$$

$$\mu^*(E)=\int_E \chi_A\,\mathrm{d}\bar{\mu}$$

with

$$A=\bigcup_{j=1}^{l+1} (T_j \cap E_j),$$

then (μ^*, x^*) is a solution of our original problem (I).

Summing up, we have the following final result.

THEOREM. *Assume the followings*:

a) $u: T\times X\to \mathbf{R}$ *satisfies the conditions* (i), (ii) *and* (iii) *in Proposition* 6;

b) $g_i: T\times X\to \bar{\mathbf{R}}_+$ $(i=1, 2, \cdots, l)$ *is a positive normal integrand such that*

$$g(t, x)=\sum_{i=1}^{l} g_i(t, x)\to +\infty \quad (\text{a.e.}\bar{\mu}) \quad \text{as} \quad x\to x_\infty .$$

Then our problem (I) *has a solution.*

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