Title	A GAME THEORETICAL INTERPRETATION OF THE STACKELBERG DISEQUILIBRIUM
Sub Title	
Author	ITO, Yozo KANEKO, Mamoru
Publisher	Keio Economic Society, Keio University
Publication year	1980
Jtitle	Keio economic studies Vol.17, No.1 (1980.) ,p.29- 40
JaLC DOI	
Abstract	We regard a leader in the Stackelberg disequilibrium as a duopolist who decides his output level before- his rival's decision and then informs him of it, and regard a follower as one who waits to be informed of it and then decides his output level under the information. We formulate this situation as a game in extensive form and provide the subgame perfect equilibrium of the game to give an interpretation to the Stackelberg disequilibrium. Further we discuss a variation of the game.
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19800001-0029

A GAME THEORETICAL INTERPRETATION OF THE STACKELBERG DISEQUILIBRIUM

Yozo Ito and Mamoru Kaneko

Abstract. We regard a leader in the Stackelberg disequilibrium as a duopolist who decides his output level before his rival's decision and then informs him of it, and regard a follower as one who waits to be informed of it and then decides his output level under the information. We formulate this situation as a game in extensive form and provide the subgame perfect equilibrium of the game to give an interpretation to the Stackelberg disequilibrium. Further we discuss a variation of the game.

1. INTRODUCTION

In many imperfect competition theories, the Stackelberg disequilibrium plays an important role to show that an imperfect competition—even the simplest duopoly—is very complicated and subtle, and that we must treat it with the greatest care. The significance of the Stackelberg disequilibrium, however, has not been made fully clear until now. The Stackelberg disequilibrium is usually interpreted as providing a criticism against the assumption of each duopolist's behavior in Nash (Cournot) equilibrium in which each adjusts his strategy assuming that his rival sticks to a strategy. However it may be also possible to interpret it as a situation where a "leader" decides an output level before a "follower" does and then informs him of the decision and each duopolist can choose to behave as a leader or a follower. We make often this interpretation, which is, however, not discussed explicitly nor formally. So, we will seek to make this interpretation clear in this paper.

In this paper, we employ the above interpretation of "leader" and "follower." More precisely, we regard a "leader" in the Stackelberg disequilibrium as a duopolist who decides his output level before his rival's decision and then informs him of it, and regard a follower as one who waits to be informed of it and then decides his output level under the information. Then we formulate the full argument of the Stackelberg disequilibrium as a game in extensive form and apply Selten's [5, 6] subgame perfect equilibrium to this game. We obtain the result that in the equilibrium, a kind of prisoner's dilemma appears but not the Stackelberg disequilibrium. Further we discuss a variation of the game to clarify and specify our interpretation.

2 THE STACKELBERG DISEQUILIBRIUM

Before discussing the Stackelberg disequilibrium in the context of game theory, we must review briefly the standard discussion of the Stackelberg disequilibrium. In this paper we employ a simple quantity duopoly model as follows. Duopolists 1 and 2 sell the same homogeneous commodity, and have the linear cost functions $C_1(q)$ and $C_2(q)$, respectively:

$$C_1(q)=c_1q \quad \text{and} \quad C_2(q)=c_2q \qquad \text{for all} \quad q\geqq 0\,, \tag{1}$$

where c_1 and c_2 are positive real numbers. The duopolists confront the objective market demand function $P(q)$:

$$P(q)=\begin{cases}-aq+b & \text{if} \quad 0\leqq q\leqq b/a \\ 0 & \text{if} \quad q\geqq b/a\,,\end{cases} \tag{2}$$

where a and b are positve real numbers. Hence their profit functions are given as

$$\begin{aligned} &f_1(q_1, q_2)=q_1P(q_1+q_2)-c_1q_1 \quad \text{and} \\ &f_2(q_1, q_2)=q_2P(q_1+q_2)-c_2q_2 \qquad \text{for all} \quad (q_1, q_2)\in E^2_+\,, \end{aligned} \tag{3}$$

where E^2_+ is the nonnegative orthant of the 2-dimensional Euclidean space.

To simplify the following discussion, we assume:

$$b\geqq 3\max(c_1, c_2)\,. \tag{4}$$

In the story of the Stackelberg disequilibrium, a *follower* and a *leader* play important roles. A "follower" means a duopolist who adjusts his strategy (output level) to maximize his profit, assuming that his rival sticks to a strategy. When both behaves as followers, a resulting equilibrium is a Nash equilibrium (Cournot equilibrium). Namely, a strategy pair (q_1^F, q_2^F) is called a *Nash equilibrium* if

$$\begin{aligned} &f_1(q_1^F, q_2^F)\geqq f_1(q_1, q_2^F) \qquad \text{for all} \quad q_1\geqq 0 \\ &f_2(q_1^F, q_2^F)\geqq f_2(q_1^F, q_2) \qquad \text{for all} \quad q_2\geqq 0\,. \end{aligned} \tag{5}$$

In our model, there exists a unique Nash equilibrium, which is given as

$$q_1^F=(b-2c_1+c_2)/3a \quad \text{and} \quad q_2^F=(b+c_1-2c_2)/3a\,. \tag{6}$$

A "leader" means a duopolist who maximizes his profit, assuming that his rival behaves as a follower. Let us consider the case where duopolists 1 and 2 behave as a leader and a follower, respectively. Then duopolist 1 can expect duopolist 2's reaction $q_2(q_1)$ to his strategy q_1, where $q_2(q_1)$ is defined by

$$f_2(q_1, q_2(q_1))\geqq f_2(q_1, q_2) \qquad \text{for all} \quad q_2\geqq 0\,. \tag{7}$$

This is a natural conclusion from the assumption. In this case duopolist 1 can use the reaction function $q_2(q_1)$ to maximize his profit, and so he chooses his strategy q_1^L as follows:

(8) $$f_1(q_1^L, q_2(q_1^L)) \geqq f_1(q_1, q_2(q_1)) \quad \text{for all} \quad q_1 \geqq 0 .$$

In our model, (7) and (8) determine a unique equilibrium (q_1^L, q_2^F), which is given as

(9) $$q_1^L = (b-2c_1+c_2)/2a \quad \text{and} \quad q_2^F = q_2(q_1^L) = (b+2c_1-3c_2)/4a .$$

(q_1^L, q_2^F) is called a *Stackelberg equilibrium* in the case where 1 is a leader and 2 a follower. In the case where duopolists 1 and 2 behave as a follower and a leader respectively, our model has also a unique Stackelberg equilibrium (q_1^F, q_2^L), which is given as

(10) $$q_1^F = q_1(q_2^L) = (b-3c_1+2c_2)/4a \quad \text{and} \quad q_2^L = (b+c_1-2c_2)/2a .$$

It is easily seen from the definition of leader that it is impossible to assume that both behave as leaders.

From the above discussion we get the following payoff matrix;

1 \ 2	F	L
F	$\left(\frac{(b-2c_1+c_2)^2}{9a}, \frac{(b+c_1-2c_2)^2}{9a}\right)$	$\left(\frac{(b-3c_1+2c_2)^2}{16a}, \frac{(b+c_1-2c_2)^2}{8a}\right)$
L	$\left(\frac{(b-2c_1+c_2)^2}{8a}, \frac{(b+2c_1-3c_2)^2}{16a}\right)$	

It is recognized from this matrix that there exists an incentive for each duopolist to play the role of a leader. Henderson and Quandt [2, page 230] say as follows[1]: "If both desire to be leaders, each assumes that the other's behavior is governed by this reaction function, but, in fact, neither of the reaction function is obeyed, and a Stackelberg disequilibrium is encountered. Stackelberg believed that this disequilibrium is the most frequent outcome. The final result of a Stackelberg disequilibrium can not be predicted a priori. If Stackelberg was correct, this situation will result in economic warfare, and equilibrium will not be achieved until one has succumbed to the leadership of the other or a collusive agreement has been reached."

3. A GAME THEORETICAL INTERPRETATION OF THE DISEQUILIBRIUM

In this section, we provide another interpretation of "leader" and "follower" and formulate the argument of the Stackelberg disequilibrium as a game in extensive form.

From an informational point of view, we propose to interpret a "leader" and a "follower" as duopolists i and j, respectively, who behave as follows:

[1] For more details, see Fellner [1].

(1): Duopolist i decides his strategy q_i without any information about his rival's decision and then informs the rival of his decision q_i.

(2): Duopolist j waits to be informed of his rival's decision q_i and then decides his strategy.

Initially, let us consider the case where duopolists 1 and 2 behave as a leader and a follower mentioned above, respectively. This situation is formulated as a game in extensive form, Γ_{LF}. The game tree of Γ_{LF} is drawn as Fig. 1. The payoff functions are, of course, $f_1(q_1, q_2)$ and $f_2(q_1, q_2)$.

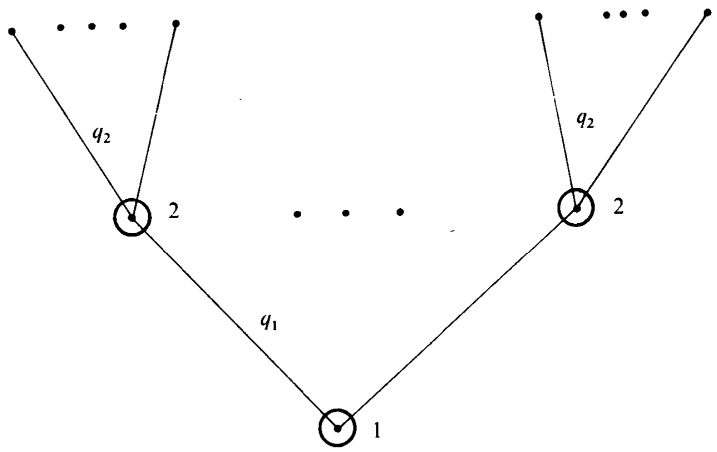

Fig. 1. Γ_{LF}.

If duopolist 2 knows the rival's strategy when he decides his strategy, then it is the best for duopolist 2 to maximize his profit using the information. Namely, duopolist 2 maximizes $f_2(q_1, q_2)$ by fixing q_1 and adjusting q_2. This is the behavior of a follower in the sense of Section 2. Since duopolist 1 reasons well and can know this fact, it is the best way for duopolist 1 to behave as a leader in the sense of Section 2. This reasoning corresponds to a *subgame perfect equilibrium* of Selten [5, 6] in Γ_{LF}.[2] A behavior strategy of duopolist 1 in Γ_{LF} is a quantity b_1^{LF} in E_+ and one of duopolist 2 is a function b_2^{LF} from E_+ to E_+. It is easily verified that Γ_{LF} has a unique subgame perfect equilibrium (b_1^{LF}, b_2^{LF}), which is given as

$$b_1^{LF}=q_1^L \quad \text{and} \quad b_2^{LF}(q)=q_2(q) \qquad \text{for all} \quad q \geqq 0 .$$

Here q_1^L is given as (9) and $q_2(q)$ is 2's reaction function defined by (7). The final result of this game is the pair $(b_1^{LF}, b_2^{LF}(b_1^{LF}))=(q_1^L, q_2^F)$, which is the Stackelberg equilibrium in the case where 1 is a leader and 2 a follower. Thus we have constructed a game in extensive form Γ_{LF}, in which a Stackelberg equilibrium is derived as a subgame perfect equilibrium.

[2] Let G be a game in extensive form. A player's *behavior strategy* is a function which assigns a local strategy to each information set of his. A subtree of the game tree is called a *subgame* if it forms a game in extensive form. A pair of behavior strategies is called a *subgame perfect equilibrium* if it is an equilibrium in any subgame. For exact definition, see Selten [5, 6].

In the same way we can construct a game in extensive form Γ_{FL}, which corresponds to the case where duopolists 1 and 2 behave as a follower and a leader, respectively. The game tree of Γ_{FL} is drawn as Fig. 2. The game Γ_{FL} has also the unique subgame perfect equilibrium (b_1^{FL}, b_2^{FL}) which is given as

$$b_1^{FL}(q)=q_1(q) \qquad \text{for all} \quad q \geqq 0 \quad \text{and} \quad b_2^{FL}=q_2^L,$$

where $q_1(q)$ is 1's reaction function and q_2^L is given as (10).

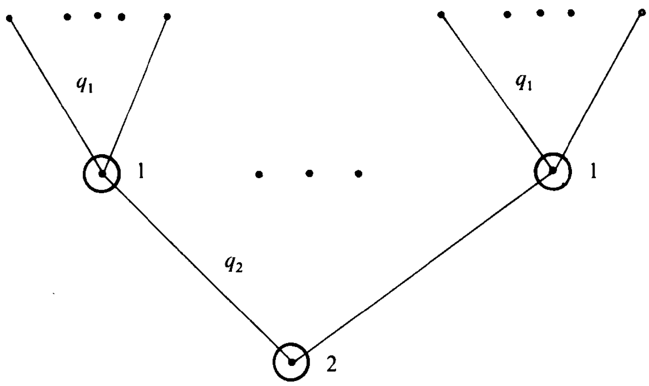

Fig. 2. Γ_{FL}.

Let us consider the case where both duopolists behave as followers. In our interpretation, a follower waits to be informed of the other's strategy and then decides his strategy. But if both behave in such a way, then they must decide independently their strategies without any information. This situation can be formulated as a game in extensive form Γ_{FF}, which has the game tree drawn as Fig. 3. This game is also a normal form game. This Γ_{FF} has the unique subgame perfect equilibrium $(b_1^{FF}, b_2^{FF})=(q_1^F, q_2^F)$, which is given as (6).

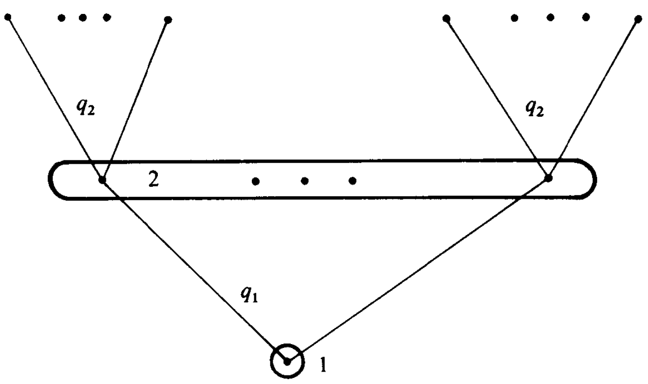

Fig. 3. Γ_{FF} and Γ_{LL}.

Finally let us consider the case where both duopolists behave as leaders. In our interpretation, a leader decides his strategy and then informs his rival of it. Of course, a leader must do this without any information about his rival's decision. Hence if both behave in such a way, then both decide their strategies before they inform each other of their decisions. So they must decide their strategies without any information. This situation has the same structure as that in the case of Γ_{FF}. Namely, the game Γ_{LL} of this situation is the same as Γ_{FF}. Of course, Γ_{LL} has the unique subgame perfect equilibrium $(b_1^{LL}, b_2^{LL})=(q_1^F, q_2^F)$.

We have interpreted the concepts of leader and follower from an informational point of view. In this context, a duopolist's choice of a leader or a follower means whether he informs his rival of his strategy or not. Including this choice, we can formulate the full argument of the Stackelberg disequilibrium as a game in extensive form Γ as follows. The game Γ has two stages.

The 1st Stage: Each duopolist i decides independently whether or not he informs his rival of the decision of an output level q_i at the second stage. "F" and "L" denote these decisions (to inform his rival of q_i or not, respectively). Then each duopolist informs each other of his decision of L or F.

The 2nd Stage: There are four cases (FF), (FL), (LF) and (LL). In each case, the duopolists play the corresponding game which has been discussed above.

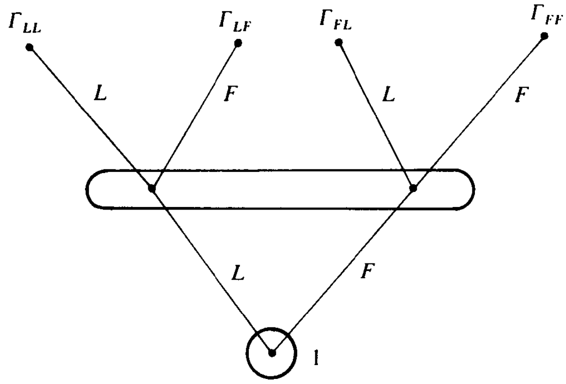

Fig. 4. Γ.

The game tree of Γ is drawn as Fig. 4. A behavior strategy B_i of duopolist i $(i=1, 2)$ is a quintuple $(b_i, b_i^{FF}, b_i^{FL}, b_i^{LF}, b_i^{LL})$ such that

$$\begin{array}{lllll} b_i \in \{L, F\},\ b_i^{LL},\ b_i^{FF} \in E_+ & \text{and} & & & \\ b_i^{LF} \in E_+ & \text{and} & b_i^{FL} & \text{is a function from } E_+ \text{ to } E_+ & \text{if } i=1 \\ b_i^{FL} \in E_+ & \text{and} & b_i^{LF} & \text{is a function from } E_+ \text{ to } E_+ & \text{if } i=2. \end{array}$$

A behavior strategy assigns a local strategy (say, L, F or q_i) to each information set. The payoff function F_i of duopolist i $(i=1, 2)$ is defined by

$$F_i(B_1, B_2)=\begin{cases} f_i(b_1^{FF}, b_2^{FF}) & \text{if } b_1=F \ \&\ b_2=F \\ f_i(b_1^{LL}, b_2^{LL}) & \text{if } b_1=L \ \&\ b_2=L \\ f_i(b_1^{LF}, b_2^{LF}(b_1^{LF})) & \text{if } b_1=L \ \&\ b_2=F \\ f_i(b_1^{FL}(b_2^{FL}), b_2^{FL}) & \text{if } b_1=F \ \&\ b_2=L \end{cases} \tag{11}$$

Thus we have completed to formulate the full argument as the game in extensive form from the informational point of view.

It is natural and consistent to consider this game Γ in terms of subgame perfect equilibrium. The following theorem is the main result of this section.

THEOREM 1. *There exists a unique subgame perfect equilibrium* $(B_1, B_2)=((b_1, b_1^{FF}, b_1^{FL}, b_1^{LF}, b_1^{LL}), (b_2, b_2^{FF}, b_2^{FL}, b_2^{LF}, b_2^{LL}))$ *in the game* Γ, *which is given as*

$$b_1=b_2=L\,, \tag{12}$$

$$\begin{aligned} &b_1^{FF}=b_1^{LL}=(b-2c_1+c_2)/3a,\ b_2^{FF}=b_2^{LL}=(b+c_1-2c_2)/3a,\\ &b_1^{LF}=(b-2c_1+c_2)/2a,\ b_2^{FL}=(b+c_1-2c_2)/2a,\\ &b_1^{FL}(q)=q_1(q) \quad \text{and} \quad b_1^{LF}(q)=q_2(q) \qquad \text{for all} \quad q\geqq 0\,. \end{aligned} \tag{13}$$

This theorem says that in the subgame perfect equilibrium, each duopolist desires to be a leader and decides his output level without any information about his rival's decision. Then the Nash (Cournot) equilibrium results. In Γ, even when both desire to be leaders, the Stackelberg disequilibrium is not encountered, but a kind of dilemma appears. This is the important difference between the standard argument of Stackelberg disequilibrium and our interpretation of it.

Proof of Theorem 1. It has been already shown that the subgames Γ_{FF}, Γ_{LF}, Γ_{FL} and Γ_{LL} have the equilibria which are given as (13). Hence it is sufficient to consider the game Γ assuming that both stick to their strategies given as (13) in each subgame. Then the game is reduced to a game with one stage, which is a game in normal form with the following payoff matrix:

1 \ 2	F	L
F	$\left(\dfrac{(b-2c_1+c_2)^2}{9a}, \dfrac{(b+c_1-2c_2)^2}{9a}\right)$	$\left(\dfrac{(b-3c_1+2c_2)^2}{16a}, \dfrac{(b+c_1-2c_2)^2}{8a}\right)$
L	$\left(\dfrac{(b-2c_1+c_2)^2}{8a}, \dfrac{(b+2c_1-3c_2)^2}{16a}\right)$	$\left(\dfrac{(b-2c_1+c_2)^2}{9a}, \dfrac{(b+c_1-2c_2)^2}{9a}\right)$

Since $(b-2c_1+c_2)^2/8a>(b-2c_1+c_2)^2/9a$, strategy pair (F, F) can not be an equilibrium in this game. It is not difficult to verify that

$$\frac{(b-3c_1+2c_2)^2}{16a}<\frac{(b-2c_1+c_2)^2}{9a} \qquad \text{iff} \quad b+c_1-2c_2>0\,,$$

$$\frac{(b+2c_1-3c_2)^2}{16a}<\frac{(b+c_1-2c_2)^2}{9a} \qquad \text{iff} \quad b-2c_1+c_2>0\,.$$

It holds by Assumption (4) that $b+c_1-2c_2>0$ and $b-2c_1+c_2>0$. It follows from these inequalities that strategy pair (L, L) is an equilibrium but not (L, F) nor (F, L). We have shown that this normal form game has the unique equilibrium point, which is given as (12).

Q.E.D.

4. A VARIATION OF Γ

We provided the game in extensive form Γ by which we formulated an argument of the Stackelberg disequilibrium from the informational point of view. In this section, we would like to clarify and specify further the significance of our interpretation, and for this purpose, we will provide another game Γ^*.

The game Γ^* has the economic characteristics—the cost functions and the demand function—given in Section 2, and is played in the time interval [0, 1] as follows. Each duopolist i $(i=1, 2)$ can decide his output level $q_i \geqq 0$ at any time $t \in [0, 1]$. Let i decide q_i at time t_i. If the rival j has not decided his output level earlier than $t(\geqq t_i)$, then j has been informed of q_i (this case is denoted by (A)) with probability $g_j(t-t_i)$ but not (Case (B)) with probability $1-g_j(t-t_i)$ by t. Of course, when neither of the duopolists has decided his output level, each knows nothing about his rival's decision. Here $g_j(s)$ $(j=1, 2, s \in [0, 1])$ is an increasing function with $0 \leqq g_j(s) \leqq 1$ and $g_j(0)=0$.

This situation can be also formulated as a game in extensive form. The game has two stages and a random mechanism as follows.

The 1st Stage: Each duopolist i decides independently the time when he will decide his output level, i.e., he chooses independently t_i from [0, 1].

The Random Mechanism: Let $t_i \leqq t_j$. Then the random mechanism indicates the Case (A) happens with probability $g_j(t_j-t_i)$ and Case (B) does with $1-g_j(t_j-t_i)$ at the second stage.

The 2nd Stage: Duopolist i decides his output level q_i at time t_i without any information. Duopolist j decides his output level q_j (Case (A)) under the information of q_i or (Case (B)) without it at time t_j.

The game tree of Γ^* is drawn as Fig. 5.

In fact, if duopolist j is informed of q_i at time t with $t_i<t<t_j$, there exists a possibility such that duopolist j changes time t_j after the information. But such a change does not have any effect on his utility (profit) at all, and so we assume that in this case, duopolist j decides his output level at time t_j.

Here we assume that each duopolist has the risk-neutral utility function with respect to profit.

A behavior strategy of duopolist i $(i=1, 2)$ in Γ^* is a triple $B_i=(t_i, q_i, b_i)$ such

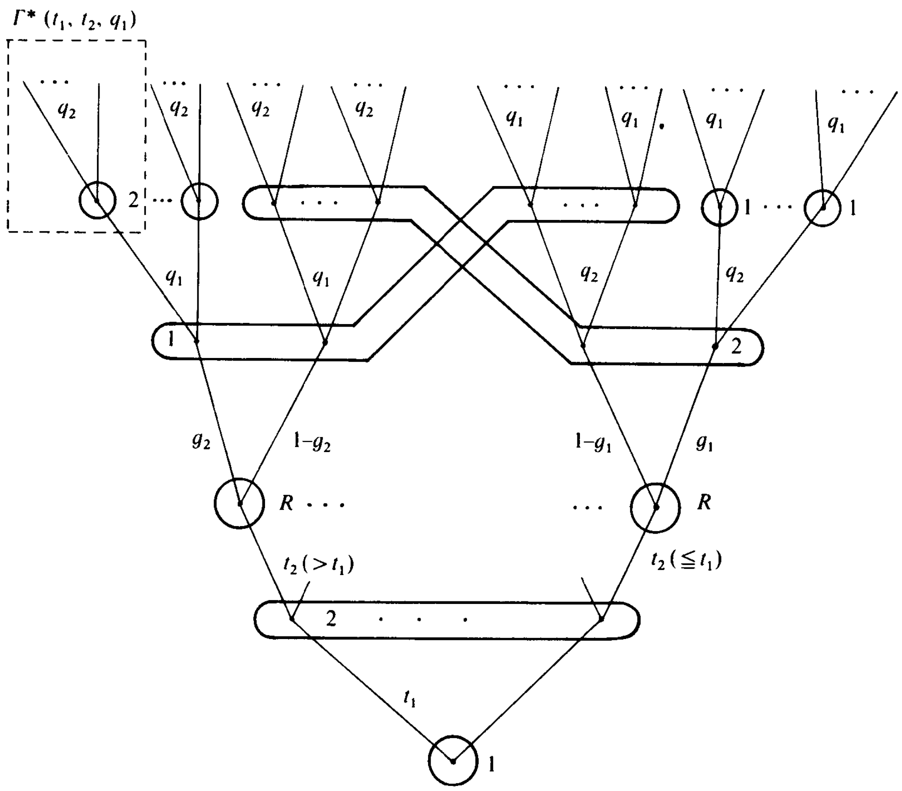

Fig. 5. Γ^*.

that

(14) $$t_i \in [0, 1] \quad \text{and} \quad q_i \in E_+ \,,$$

(15) $$b_i \quad \text{is a function from} \quad E_+ \quad \text{to} \quad E_+ \,.$$

A behavior strategy B_i assigns a local strategy to each information set. t_i is the time when i intends to decide an output level. q_i is duopolist i's output level in the case where i is not informed of his rival's decision and $b_i(q_j)$ is his output level in the case where he is informed of his rival's decision q_j.

The game Γ^* has a kind of proper subgames. They are ones in which a duopolist i has been informed of his rival's decision q_j and then decides his output level q_i. They are one person games, a representative game of which is denoted by $\Gamma^*(t_1, t_2, q_j)$. In $\Gamma^*(t_1, t_2, q_j)$, only duopolist i adjusts his output level q_i to maximize his profit under the information q_j. Hence the following lemma is clear.

LEMMA 1. *Let* (B_1, B_2) *be a subgame perfect equilibrium in* Γ^*. *Then* $b_i(q) = q_i(q)$ *for all* $q \geqq 0$ *and* $i = 1, 2$, *where* $q_i(q)$ *is* i*'s reaction function defined by* (7).

THEOREM 2. *There exists a unique subgame perfect equilibrium* $(B_1^*, B_2^*) = ((t_1^*, q_1^*, b_1^*), (t_2^*, q_2^*, b_2^*))$ *in the game* Γ^*, *which is given as*

(16) $$t_1^* = t_2^* = 0\,,$$

(17) $$q_1^* = (b - 2c_1 + c_2)/3a \quad \textit{and} \quad q_2^* = (b + c_1 - 2c_2)/3a\,,$$

(18) $$b_1^*(q) = q_1(q) \quad \textit{and} \quad b_2^*(q) = q_2(q) \qquad \textit{for all} \quad q \in E_+\,.$$

Proof. By Lemma 1 we can assume that b_i^* ($i=1, 2$) is always given as (18). Hence this game Γ^* is already a game in normal form. The payoff function $P_i((t_1, q_1), (t_2, q_2))$ ($i=1, 2$) is defined by

$$P_i((t_1, q_1), (t_2, q_2)) = \begin{cases} g_2(t_2 - t_1) f_i(q_1, q_2(q_1)) + (1 - g_2(t_2 - t_1)) f_i(q_1, q_2) & \text{if} \quad t_1 \leqq t_2\,. \\ g_1(t_1 - t_2) f_i(q_1(q_2), q_2) + (1 - g_1(t_1 - t_2)) f_i(q_1, q_2) & \text{if} \quad t_1 \geqq t_2\,. \end{cases}$$

Initially we show that (B_1^*, B_2^*) is a subgame perfect equilibrium. It is sufficient to show that $((t_1^*, q_1^*), (t_2^*, q_2^*))$ is an equilibrium point of the above normal form game. It is clear that

$$P_1((0, q_1^*), (0, q_2^*)) \geqq P_1((0, q_1), (0, q_2^*)) \qquad \text{for all} \quad q_1 \geqq 0$$
$$P_2((0, q_1^*), (0, q_2^*)) \geqq P_2((0, q_1^*), (0, q_2)) \qquad \text{for all} \quad q_2 \geqq 0\,.$$

Let $t_1 > 0$. Since $q_1^* = q_1(q_2^*)$ and $f_1(q_1^*, q_2^*) \geqq f_1(q_1, q_2^*)$ for all $q_1 \geqq 0$, we have

$$\begin{aligned} P_1((0, q_1^*), (0, q_2^*)) = f_1(q_1^*, q_2^*) &\geqq g_1(t_1) f_1(q_1(q_2^*), q_2^*) + (1 - g_1(t_1) f_1(q_1, q_1^*) \\ &= P_1((t_1, q_1), (0, q_2^*)) \qquad \text{for all } q_1 \geqq 0\,. \end{aligned}$$

Similarly we can show that

$$P_2((0, q_1^*), (0, q_2^*)) \geqq P_2((0, q_1^*), (t_2, q_2)) \qquad \text{for all} \quad (t_2, q_2)\,.$$

Conversely we show that if $(\tilde{B}_1, \tilde{B}_2) = ((\tilde{t}_1, \tilde{q}_1, b_1^*), (\tilde{t}_2, \tilde{q}_2, b_2^*))$ is a subgame perfect equilibrium, then it satisfies (16) and (17). Let $(\tilde{t}_1, \tilde{t}_2)$ be fixed and suppose $\tilde{t}_1 \leqq \tilde{t}_2$ without loss of generality. Then $(\tilde{q}_1, \tilde{q}_2)$ must satisfy

(19) $$\begin{aligned} &P_1((\tilde{t}_1, \tilde{q}_1), (\tilde{t}_2, \tilde{q}_2)) \geqq P_1((\tilde{t}_1, q_1), (\tilde{t}_2, \tilde{q}_2)) \qquad \text{for all} \quad q_1 \geqq 0 \\ &P_2((\tilde{t}_1, \tilde{q}_1), (\tilde{t}_2, \tilde{q}_2)) \geqq P_2((\tilde{t}_1, \tilde{q}_1), (\tilde{t}_2, q_2)) \qquad \text{for all} \quad q_2 \geqq 0\,, \end{aligned}$$

which is equivalent to

(20) $$\begin{aligned} &\tilde{g}_2 f_1(\tilde{q}_1, q_2(\tilde{q}_1)) + (1 - \tilde{g}_2) f_1(\tilde{q}_1, \tilde{q}_2) \geqq \tilde{g}_2 f_1(q_1, q_2(q_1)) \\ &\qquad\qquad + (1 - \tilde{g}_2) f_1(q_1, \tilde{q}_2) \qquad \text{for all} \quad q_1 \geqq 0\,, \\ &f_2(\tilde{q}_1, \tilde{q}_2) \geqq f_2(\tilde{q}_1, q_2) \qquad \text{for all} \quad q_2 \geqq 0\,, \end{aligned}$$

where $\tilde{g}_2 = g_2(\tilde{t}_2 - \tilde{t}_1)$. This implies $\tilde{q}_2 = q_2(\tilde{q}_1)$. So it is easy to verify that

$$\tilde{q}_1 = \frac{b - 2c_1 + c_2}{(3 - g_2(\tilde{t}_2 - \tilde{t}_1))a}$$

$$\tilde{q}_2 = \frac{(2 - g_2(\tilde{t}_2 - \tilde{t}_1))b + 2c_1 - (4 - g_2(\tilde{t}_2 - \tilde{t}_1))c_2}{2(3 - g_2(\tilde{t}_2 - \tilde{t}_1))a}.$$

Hence if $\tilde{t}_1 = \tilde{t}_2 = 0$, then $(\tilde{q}_1, \tilde{q}_2)$ coincides with (q_1^*, q_2^*) given as (17). So it is sufficient to show that if $(\tilde{t}_1, \tilde{t}_2) \neq (0, 0)$, then $((\tilde{t}_1, \tilde{q}_1), (\tilde{t}_2, \tilde{q}_2))$ is not an equilibrium of the above normal form game. Suppose $0 \leqq \tilde{t}_1 < \tilde{t}_2$. We put $\bar{g}_2 = g_2(\tilde{t}_2 - \bar{t}_1)$, where $\tilde{t}_1 < \bar{t}_1 < \tilde{t}_2$. Of course, $\tilde{g}_2 > \bar{g}_2$. It can be easily verified that $f_1(q_1, q_2(q_1))$ and $f_1(q_1, \tilde{q}_2)$ are differentiable with respect to q_1 in a neiborhood of $\tilde{q}_1$. By (20), $(\tilde{q}_1, \tilde{q}_2)$ satisfies

$$\tilde{g}_2\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_1}{\partial q_2} \cdot \frac{dq_2(q_1)}{dq_1}\right)\Bigg|_{q_1 = \tilde{q}_1} = -(1 - \tilde{g}_2)\frac{\partial f_1}{\partial q_1}\Bigg|_{(q_1, q_2) = (\tilde{q}_1, \tilde{q}_2)}.$$

It is also not difficult to verify that

$$\frac{\partial f_1}{\partial q_1}\Bigg|_{(q_1, q_2) = (\tilde{q}_1, \tilde{q}_2)} = \frac{-\tilde{g}_2(b - 2c_1 + c_2)}{2(3 - \tilde{g}_2)} < 0.$$

Hence we have

$$\bar{g}_2\left(\frac{\partial f_1}{\partial q_1} + \frac{\partial f_1}{\partial q_2} \cdot \frac{dq_2(q_1)}{dq_1}\right)\Bigg|_{q_1 = \tilde{q}_1} < -(1 - \bar{g}_2)\frac{\partial f_1}{\partial q_1}\Bigg|_{(q_1, q_2) = (\tilde{q}_1, \tilde{q}_2)},$$

i.e.,

$$\frac{\partial P_1}{\partial q_1}\Bigg|_{\substack{(t_1, q_1) = (\bar{t}_1, \tilde{q}_1) \\ (t_2, q_2) = (\tilde{t}_2, \tilde{q}_2)}} < 0.$$

This implies that there is a $\bar{q}_1$ in a neiborhood of $\tilde{q}_1$ such that

$$P_1((\bar{t}_1, \tilde{q}_1), (\tilde{t}_2, \tilde{q}_2)) < P_1((\bar{t}_1, \bar{q}_1), (\tilde{t}_2, \tilde{q}_2)).$$

Since $q_2(\tilde{q}_1) = \tilde{q}_2$ as shown above, $P_1((\bar{t}_1, \tilde{q}_1), (\tilde{t}_2, \tilde{q}_2)) = P_1((\tilde{t}_1, \tilde{q}_1), (\tilde{t}_2, \tilde{q}_2))$. Then we have

$$P_1((\tilde{t}_1, \tilde{q}_1), (\tilde{t}_2, \tilde{q}_2)) < P_1((\bar{t}_1, \bar{q}_1), (\tilde{t}_2, \tilde{q}_2)).$$

This means that $((\tilde{t}_1, \tilde{q}_1), (\tilde{t}_2, \tilde{q}_2))$ is not an equilibrium in the above normal form game. In the case where $0 < \tilde{t}_1 = \tilde{t}_2$, we can similarly prove that $((\tilde{t}_1, \tilde{q}_1), (\tilde{t}_2, \tilde{q}_2))$ can not be an equilibrium of the above normal form game.

Q.E.D.

In the game Γ^*, it is the role of a leader or a follower for a duopolist to decide an output level before or after his rival's decision, respectively. It is more profitable to become a leader, i.e., to decide an output level before his rival's decision, and the probability of becoming a leader increases if he decides it sooner. So, both try to decide their output levels as soon as possible. In consequence, both decide output levels at time 0 and without any information. This situation is the same as the Nash's, and so the Nash equilibrium appears as the final outcome. Thus our result is not a disequilibrium but a dilemma, a result quite similar to that of Section 3.

We can construct several variations of Γ and Γ^* by changing their rules. For example, let us consider the variation of Γ^* such that when a duopolist decides an output level, his rival certainly and simultaneously knows this time but not necessarily the decision itself (The decision intself is known with probability $g_j(t-t_i)$ in the same way as in the game Γ^*.) This situation can be also formulated as a game in extensive form, which is more complicated because the game gives a follower the information on the time when a leader decides an output level. But the almost same result as Theorem 2 is true in this game. Thus the results of several variations of Γ and Γ^* would be hardly different from those of Γ and Γ^*.

Senshu University *University of Tsukuba*

REFERENCES

[1] W. Fellner, *Competition among the Few-Oligopoly and Similar Market Structures*, 2nd. 1965, Augustus M. Kelley Pub.

[2] J. M. Henderson and R. E. Quandt, *Microeconomic Theory=A Mathematical Approach*, 2nd ed. 1971, McGraw-Hill.

[3] J. P. Mayberry, J. F. Nash and M. Shubik, "A Comparison of Treatments of a Duopoly Situation," *Econometrica*, **21**, 141–154, 1953.

[4] J. F. Nash, "Non-Cooperative Games," *Annals of Mathematics*, **54**, 286–295, 1951.

[5] R. Selten, "A Simple Model of Imperfect Competition, where 4 are Few and 6 are Many," *International Journal of Game Theory*, **2**, 141–201, 1973.

[6] ———, "Reexamination of the Perfectness Concept for Equilibrium Point in Extensive Games," *International Journal of Game Theory*, **4**, 25–55, 1975.