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ON THE REVENUE MAXIMISING RATE OF INFLATION WHEN THERE IS UNANTICIPATED INFLATION*

D. CHAPPELL and D. A. PEEL

INTRODUCTION

In a recent paper Cathcart [3] derived the formula for the inflation rate which will maximise the present value of government revenue from inflation as a fraction of income for economies in which there is exogenous growth of real output and unanticipated inflation along the transition path. Cathcart's analysis would seem to represent an important extension of Bailey's original static analysis [1] in which inflationary expectations are always realized, which in a deterministic setting is equivalent to rational expectations.

Cathcart's formula was obtained in the following manner. The government revenue from inflation as a fraction of income (g) is given by

$$g=(\pi+\lambda-a\dot{\pi}^{*})\cdot k \qquad (1)^{1}$$

where π is the actual rate of inflation, $\dot{\pi}^{*}$ is the change in the expected rate of inflation, λ is the exogenously given growth rate of real income, k equals the ratio of real balances to real income and is assumed equal to

$$k=k_0 e^{-a\pi^{*}} \qquad (2)$$

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[1] The simplest intuitive method of understanding why (1) yields the government's revenue from inflation is to suppose that the government finances nominal expenditure pG where p is the price level, G is real government expenditure by monetary expansion. Consequently

$$pG=\dot{M} \qquad (a)$$

where $\dot{M}$ is the change in monetary expansion.

It follows that

$$\frac{G}{y}=\frac{Mm}{py} \qquad (b)$$

where y is the level of real income and $m\equiv\dot{M}/M$.

Consequently

$$\frac{G}{y}=(\pi+\lambda-a\dot{\pi}^{*})k \qquad (c)$$

where the symbols are defined in the text and the semi-logarithmic demand function for money is assumed.

More generally real government receipts are equal to real balances held by the public times the rate of monetary expansion (the tax).

a, k_0 are constants, e is the exponential.
Inflationary expectations are formed adaptively so that

$$\dot{\pi}^* = b(\pi - \pi^*); \qquad b > 0 \tag{3}$$

where b is the constant coefficient of adaptive expectations; substitution of (3) and (2) into (1) gives

$$g = \left[\pi^* + \dot{\pi}^*\left(\frac{1}{b} - a\right) + \lambda\right] k_0 e^{-a\pi^*} \tag{4}$$

Assuming a rate of discount ϕ, Cathcart employs calculus of variations to maximise the present value of g. Thus, he maximises:—

$$Z = \int_0^\infty \left[\pi^* + \dot{\pi}^*\left(\frac{1}{b} - a\right) + \lambda\right] k_0 e^{-a\pi^*} \cdot e^{-\phi t} dt \tag{5}$$

and obtains the maximising condition.

$$\pi^*(t) = \frac{1}{a} - \lambda - \phi\left(1 - \frac{1}{ab}\right) \tag{6}$$

(6) is a constant and implies a π in equilibrium equal to π^*.

Unfortunately, (6) does not, in general, yield a maximum.[2] This may be demonstrated quite simply by employing the more modern technique of the Maximum Principle of optimal control theory.

We commence by noting that since

$$\pi + \lambda - a\dot{\pi}^* = m \tag{7}$$

(where m is the proportional rate of change of the money stock) we may eliminate π from the equations and recast the problem as:—

$$\max_{(m)} \int_0^\infty m k_0 e^{-(a\pi^* + \phi t)} dt \tag{8}$$

[2] We might note that the same type of results are obtained if output is not fixed, but responds to inflation. There are two competing hypotheses. First the deviation of the level of output from trend is a function of the *level* of unanticipated prices. (See e.g. Lucas [7]). Second, deviations of the level of output from trend respond to *unanticipated* inflation, *via* an augmented Phillips curve (see e.g. Frenkel and Rodriguez [5]). For the former case we note that Burmeister and Turnovsky [2] show that the assumption of adaptive expectations concerning the rate of inflation is only derivable in a consistent manner if it is assumed that the level of prices is equal to that expected and consequently we can interpret Cathcart's analysis as being consistent with a Lucas supply response with the *level* of prices equal to that expected so that output is changing at the trend rate. In the latter case, with adaptive expectations,

$$\lambda = \lambda^* + j\dot{\pi}^*$$

where λ^* is the trend rate of change of output and j is a constant. Consequently the points made in the text are still valid here.

s.t. $$\dot{\pi}^* = \frac{b}{1-ab}(m - \pi^* - \lambda); \qquad \pi^*(0) = \pi_0^* \tag{9}$$

Clearly, for this expectations formation mechanism to make economic sense, we require $ab \leqq 1$. In what follows we shall assume that the strict inequality holds.

Introducing a co-state variable, $\mu(t)$, we define the (current valued) Hamiltonian;

$$H(m, \pi^*, \mu, t) \equiv e^{-\phi t}\left[mk_0 e^{-a\pi^*} + \frac{\mu b(m - \pi^* - \lambda)}{1-ab}\right] \tag{10}$$

and let:—

$$\psi(\pi^*, \mu, t) = \sup_{(m)} H(m, \pi^*, \mu, t) \tag{11}$$

We remark at once that since the Hamiltonian is linear in m (the control variable) the problem as it stands is not well posed since no restriction is imposed on the control set, i.e. Maximising (10) with respect to m implies that we set:—

$$m = \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} \qquad \text{as } k_0 e^{-a\pi^*} + b\mu/(1-ab) \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \tag{12}$$

It is appropriate in such cases to restrict the control variable to belong to some closed set and adopt the relevant impulse (bang–bang) control. What Cathcart has done in employing calculus of variations is analyse the singular arc which may be derived by setting $\partial H/\partial m = 0$; clearly, this in general will not constitute a maximum at every point in time over the infinite horizon. For a discussion of the optimality of singular arcs the reader is referred to Lewis [6].

The following implications are suggested by these results. In a world in which price expectations are formed as a weighted sum of previous actual inflation rates and if the government's objective function is to maximise revenue from inflation then the optimal policy is to instantaneously adjust the money supply so as to effect an immediate jump in the rate of inflation to some particular level (infinite in Cathcart's framework). This offers one potential explanation of the stylized facts of hyper-inflations. In particular the observed rates of inflation are substantially in excess of those required to maximise government revenue *in the steady state*. (See, for example, the tables in Cathcart [3]). The analysis suggests that on the *transitional path* the government can exploit the lag of the expected on the actual rate of inflation, implicit in the adaptive expectations mechanism, to generate substantially higher revenues. One problem with this interpretation is that jumps in inflation rates, particularly to infinite rates, are not observed in practice.

Within the spirit of the type of model under consideration, an alternative suggestion is that the government maximises revenue net of a cost of adjusting its instrument, i.e. the money supply.

Consequently our new objective becomes:

$$\max \int_0^\infty (mk_0 e^{-a\pi^*} - \gamma \dot{m}^2/2) e^{-\phi t} dt \tag{13}$$ [3]

[3] It should be noted that similar results are obtained if the authorities maximise $g - \gamma\dot{\pi}2/2$ where changes in the inflation rate are a surrogate for "social strife."

where γ is a positive constant, subject to:

$$\dot{\pi}^* = b(m - \pi^* - \lambda)/(1 - ab); \qquad \pi^*(0) = \pi_0^* \tag{9}$$

The notion that costs are incurred in changing the rate of monetary expansion can be rationalized on the basis of 'insititutional factors' and is a feature of the optimal stabilization literature. (See for example, Turnovsky [8], chapter 14.)

As it stands this problem is not amenable to solution by the Maximum Principle. First we must define a new variable, x, such that:

$$\dot{m} = x; \qquad m(0) = m_0 \tag{14}$$

Now the problem is to:

$$\max_{(x)} \int_0^\infty (mk_0 e^{-a\pi^*} - \gamma x^2/2) e^{-\phi t} dt \tag{15}$$

subject to (9) and (14).

We proceed by introducing a pair of co-state variables, $\mu_1(t)$ and $\mu_2(t)$, and defining the Hamiltonian:

$$H(m, x, \mu_1, \mu_2, t) \equiv e^{-\phi t}[mk_0 e^{-a\pi^*} - \gamma x^2/2 + \mu_1 b(m - \pi^* - \lambda)/(1 - ab) + \mu_2 x] \tag{16}$$

and,

$$\psi(m, \mu_1, \mu_2, t) = \sup_{(x)} H(m, x, \mu_1, \mu_2, t) \tag{17}$$

The optimality conditions are then as follows:

$$\left.\begin{aligned} H(m, x, \mu_1, \mu_2, t) &\equiv \psi(m, \mu_1, \mu_2, t) \\ \dot{\mu}_1 &= \phi\mu_1 - \frac{\partial H}{\partial \pi^*} \cdot e^{\phi t} \\ \dot{\mu}_2 &= \phi\mu_2 - \frac{\partial H}{\partial m} \cdot e^{\phi t} \\ \{\mu_1, \mu_2\} &\neq \forall t \end{aligned}\right\} \tag{18}$$

Maximisation of $H(\cdot\,\cdot)$ with respect to x yields $x \equiv \mu_2/\gamma$. Substituting this as appropriate we see that the following set of differential equations must be satisfied:

$$\left.\begin{aligned} \dot{\mu}_1 &= \left[\frac{\phi(1-ab)+b}{1-ab}\right]\mu_1 + amk_0 e^{-a\pi^*} \\ \dot{\mu}_2 &= \phi\mu_2 - k_0 e^{-a\pi^*} - \frac{b\mu_1}{1-ab} \\ \dot{m} &= \frac{\mu_2}{\gamma} \\ \dot{\pi}^* &= b(m - \pi^* - \lambda)/(1 - ab) \end{aligned}\right\} \tag{19}$$

The associated boundary and transversality conditions are as follows:

$$\left.\begin{aligned} &\pi^*(0)=\pi_0^* \\ &m(0)=m_0 \\ &\mathrm{Lim}_{t\to\infty} e^{-\phi t}\mu_1(t)\cdot\pi^*(t)=0 \\ &\mathrm{Lim}_{t\to\infty} e^{-\phi t}\mu_2(t)\cdot m(t)=0 \end{aligned}\right\} \quad (20)$$

Clearly; because of the non-linearities involved, it is likely that (19), (20) can only be solved numerically. Consequently our analysis is confined to an examination of the steady state and its stability. Setting the right hand side of equations (19) equal to zero the steady state is as follows:

$$\left.\begin{aligned} &m=\frac{\phi(1-ab)+b}{ab}; \qquad \pi^*=\frac{\phi(1-ab)+b-ab\lambda}{ab} \\ &\mu_1=\frac{-(1-ab)k_0\exp\left\{-\left(\dfrac{\phi(1-ab)+b-ab\lambda}{b}\right)\right\}}{b}; \qquad \mu_2=0 \end{aligned}\right\} \quad (21)$$

It is interesting to note that the steady state values of m and π^* are precisely those derived (erroneously) by Cathcart *in the absence of costs of adjustment.*

Let us now examine the stability characteristics of (21). Expanding (19) in a Taylor series around the stationary point and ignoring terms of higher than the first order we obtain.[4]

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\mu}_2 \\ \dot{m} \\ \dot{\pi}^* \end{bmatrix} \simeq \begin{bmatrix} \phi+A & 0 & B & -B(1+\phi/A) \\ -A & \phi & 0 & B \\ 0 & 1/\gamma & 0 & 0 \\ 0 & 0 & A & -A \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ m \\ \pi^* \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ 0 \\ -A\lambda \end{bmatrix} \quad (22)$$

where the bars over variables denote their steady state values and:

$$A\equiv b/(1-ab)<0$$

$$B\equiv ak_0\exp\left\{-\left(\frac{\phi(1-ab)+b-ab\lambda}{b}\right)\right\}>0$$

$$C_1\equiv-(\phi+A)\bar{\mu}_1-B\bar{m}+B(1+\phi/A)\bar{\pi}^*$$

$$C_2\equiv A\bar{\mu}_1-B\bar{\pi}^*$$

If Z is a characteristic root of the Jacobian in (22) then the characteristic equation is:

$$F(Z)=Z^4-2\phi Z^3-(A^2+A\phi-\phi^2)Z^2+A\phi(A+\phi)Z+A^2B/\gamma=0 \quad (23)$$

[4] Note that the $\dot{\pi}^*$ equation is written exactly since it is already linear.

Analysis of the function $F(Z)$ (in particular, its first and second derivatives) shows that the characteristic roots conform to one of the following patterns[5]:

(i) All roots are real with two negative and two positive.

(ii) The roots are two complex conjugate pairs, one with negative real parts and one with positive real parts.

Thus we always have two stable roots which are real if the following inequality holds and complex otherwise:

$$\frac{(A+\phi)^2}{4} \geqq \frac{B}{\gamma}$$

For example, if the ratio of real balances to real income is high and/or the 'cost' of adjusting the money supply is low the roots are likely to be complex, giving rise to cyclical behaviour along the optimal path.

We may ensure convergence to the stationary point given in (21) by restricting our attention to the plane spanned by the characteristic vectors associated with the two stable roots.[6] In this way we synthesize a locally optimal feedback rule for the control variable $x = x(m, \pi^*)$. The two relevant differential equations describing motion in this plane are:

$$\begin{aligned} \dot{m} &= x(m, \pi^*) \\ \dot{\pi}^* &= b(m - \pi^* - \lambda)/(1 - ab) \end{aligned} \tag{24}$$

[5] This may be seen from the following arguments. (See Mathematical Appendix for details).

(i) $\text{Lim}_{z \to \infty} F(Z) = \text{Lim}_{z \to -\infty} F(Z) = \infty$

(ii) $F(Z)$ had a *local* maximum at $z = \phi/2$ and $F(\phi/2) > 0$

(iii) $F(Z)$ has equal *global* minima at $Z = \frac{\phi}{2} \pm \frac{1}{2}(\phi^2 + 2A(A+\phi))^{1/2}$ and for each of these values:

$F(Z) = A^2 \left\{ \frac{B}{\gamma} - \frac{(A+\phi)^2}{4} \right\}$. Clearly,

if $\frac{B}{\gamma} \geqq \frac{(A+\phi)^2}{4}$ all roots are real; otherwise, all roots are complex.

(iv) $F(0) > 0$ and $F'(0) > 0$. Thus if all roots are real then two are positive and two are negative.

(v) If all roots are complex then from the fact that the sum of the roots is positive:—

$$\left(\sum_{i=1}^{4} Zi = 2\phi > 0 \right),$$

at least two must have positive real parts. Thus there are three possible cases to consider but it is relatively easy to show that the only pattern of roots compatible with the signs of the coefficients of $F(Z)$ (particularly the coefficient on Z) is two with positive real parts and two with negative real parts.

Thus there are *always* two stable roots as asserted in the text.

[6] This is a standard procedure in the stability analysis of non-linear ordinary differential equations. The interested reader is referred to Coddington and Levinson [4]. For another application of the technique in economic analysis see Ryder and Heal [9].

Taking a linear approximation around the stationary point gives:

$$\begin{bmatrix} \dot{m} \\ \dot{\pi}^* \end{bmatrix} \simeq \begin{bmatrix} x_m & x_{\pi^*} \\ A & -A \end{bmatrix} \begin{bmatrix} m \\ \pi^* \end{bmatrix} - \begin{bmatrix} C_3 \\ A\lambda \end{bmatrix} \tag{25}$$

where $C_3 \equiv x_m \bar{m} + x_{\pi^*} \bar{\pi}^*$

The characteristic equation associated with (25) is:

$$Z^2 - (x_m - A)Z - A(x_m + x_{\pi^*}) = 0 \tag{26}$$

However, since we 'know' the relevant characteristic roots, Z_1 and Z_2 say, we may solve for the partial derivatives of the feedback rule, x_m and x_{π^*}

$$\begin{aligned} x_m &= Z_1 + Z_2 + A < 0 \\ x_{\pi^*} &= \frac{-(Z_1 + A)(Z_2 + A)}{A} < 0 \end{aligned} \tag{27}$$

The signs of these partial derivatives accord with a priori logic and are obviously self-stabilizing.[7]

It remains to examine the (local) nature of convergence to the stationary point. Clearly, from (25),

$$\dot{m} \gtreqless 0 \quad \text{as} \quad m \gtreqless \bar{m} - \frac{x_{\pi^*}\pi^*}{x_m} + \frac{x_{\pi^*}\bar{\pi}^*}{x_m} \tag{28}$$

and

$$\dot{\pi}^* \gtreqless 0 \quad \text{as} \quad m \gtreqless \lambda + \pi^*$$

From (21) we know that $\bar{m} > 0$ and that $\bar{\pi}^* = \bar{m} - \lambda$. In what follows we will assume, without loss of generality, that λ is 'small' so that $\bar{\pi}^*$ is positive. Under this assumption behaviour in the (π^*, m) plane is illustrated in the following phase diagrams (Figs. I and II below). These illustrate several interesting features of the model. Firstly, if Z_1 and Z_2 are complex conjugates then any deviation from equilibrium will result in damped cyclical behaviour (Fig. II). Secondly, as Fig. I shows, even if Z_1 and Z_2 are real, 'overshooting' of ($\bar{m}$, $\bar{\pi}^*$) is a strong possibility. However, we should note that both cases are, of course, asymptotically stable.

CONCLUSIONS

The purpose of Cathcart's analysis was to derive the path for the expected inflation rate which will maximise the government revenue from inflation when price expectations are formed adaptively. Cathcart's suggested optimal solution is

[7] The signs may be ascertained by applying the formula for the roots of a quartic equation and showing that A lies within a certain range.

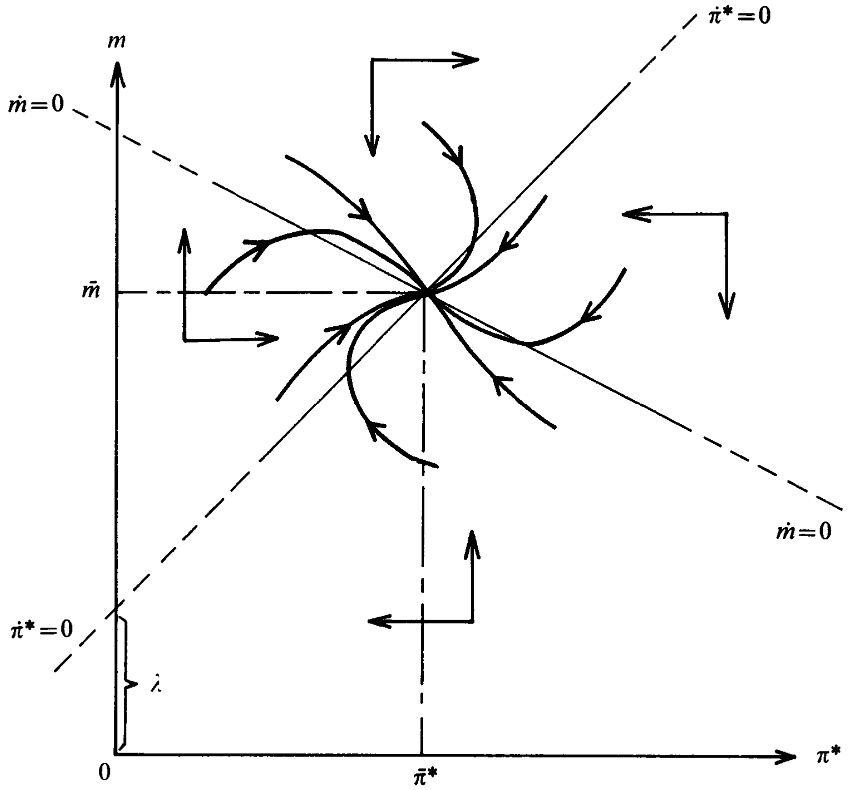

Fig. I. Z_1, Z_2 real and negative (stable improper node).

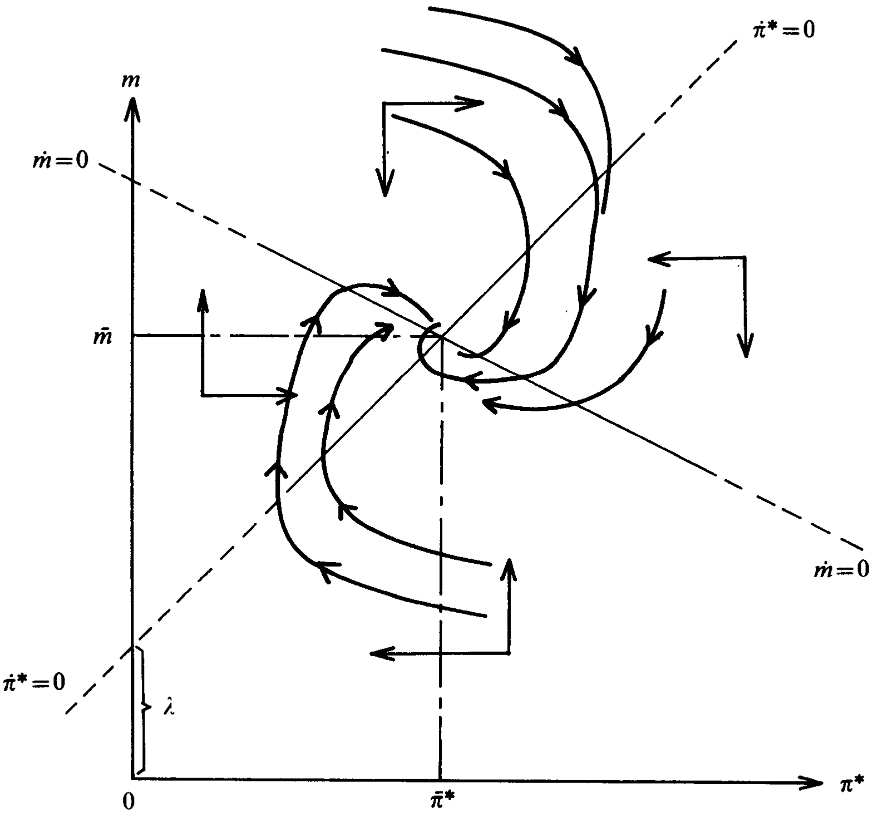

Fig. II. Z_1, Z_2 complex conjugates with negative real parts (stable focus).

that the expected and hence actual inflation rates are constant at a certain value. Unfortunately, Cathcart's solution is in general not optimal. We showed that when expectations are adaptive the optimal policy to maximise government revenue calls for an infinite jump in the rate of inflation in order to exploit the lagged response of the expected to actual inflation rate. Such a result seems to offer one possible explanation of why the actual rates of inflation in hyper-inflations are greatly in excess of those required to maximise the government revenue from inflation in the *steady* state. However, in practice we do not observe massive jumps in the inflation rate. Given the assumption that the authorities are attempting to maximise revenues this suggests two possibilities. Either, first expectations are not formed adaptively or second, there are costs in adjusting either the instrument, namely changes in the rate of monetary expansion or the rate of inflation, (a possible surrogate for 'social strife'). Both of these latter possibilities seem to preclude jumps in the inflation rate as policies to maximise the government revenue from inflation and can lead to higher inflation rates along the transition path than the actual steady state maximising rate of inflation.

University of Sheffield *University of Liverpool*

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MATHEMATICAL APPENDIX

PROPOSITION.

The roots of the quartic equation:

$$F(Z) \equiv Z^4 - 2\phi Z^3 - (A^2 + A\phi - \phi^2)Z^2 + A\phi(A+\phi)Z + A^2B/\gamma$$

(*where* ϕ, A, B *and* γ *are positive constants*) *conform to one of the following patterns*:

(i) *all roots are real with two negative and two positive, or*

(ii) *the roots are two complex conjugate pairs, one with negative real parts and one with positive real parts.*

Proof of the assertion may be established by the following steps:

(i) $\mathrm{Lim}_{z\to\infty} F(Z) = \mathrm{Lim}_{z\to-\infty} F(Z) = \infty$

(ii) $F'(Z) = 4Z^3 - 6\phi Z^2 - 2(A^2 + A\phi - \phi^2)Z + A\phi(A+\phi)$ and $F'(Z) = 0$ when:

(a) $Z = \hat{Z}_1 = \phi/2 > 0$

(b) $Z = \hat{Z}_2 = \phi/2 + 1/2(\phi^2 + 2A^2 + 2A\phi)^{1/2} > 0$

(c) $Z = \hat{Z}_3 = \phi/2 - 1/2(\phi^2 + 2A^2 + 2A\phi)^{1/2} < 0$

(iii) $F''(Z) = 12Z^2 - 12\phi Z - 2(A^2 + A\phi - \phi^2)$

$= -\phi^2 - 2A(A+\phi) < 0$ when $Z = \hat{Z}_1$

$= 4A(A+\phi) + 2\phi^2 > 0$ when $Z = \hat{Z}_2$ or $\hat{Z}_3$

Therefore $F(\hat{Z}_1)$ is a *local* maximum and $F(\hat{Z}_2)$ and $F(\hat{Z}_3)$ are (equal) *global* minima.

(iv) $$F(\hat{Z}_1) = \frac{\phi^4}{16} + \frac{A\phi^2(A+\phi)}{4} + \frac{A^2B}{\gamma} > 0; \qquad F(0) = \frac{A^2B}{\gamma} > 0$$

$$F(\hat{Z}_2) = F(\hat{Z}_3) = A^2\left\{\frac{B}{\gamma} - \frac{(A+\phi)^2}{4}\right\} \gtreqless 0 \quad \text{as} \quad \frac{B}{\gamma} \gtreqless \frac{(A+\phi)^2}{4}$$

The three possible graphs of $F(Z)$ are depicted in Figs. 1, 2 and 3 below.

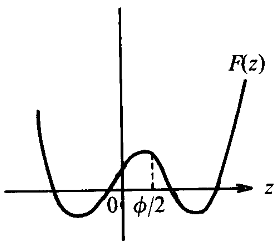

Fig. 1. $\frac{B}{\gamma} < \frac{(A+\phi)^2}{4}$

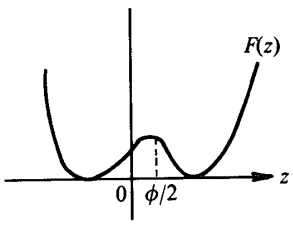

Fig. 2. $\frac{B}{\gamma} = \frac{(A+\phi)^2}{4}$

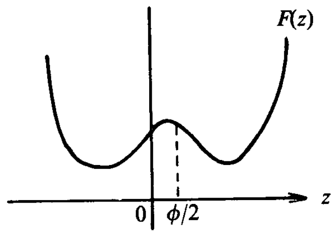

Fig. 3. $\frac{B}{\gamma} > \frac{(A+\phi)^2}{4}$

In Figs. 1 and 2 all roots are real and in Fig. 3 all roots are complex. Thus, if the roots are real there must be two negative (distinct in Fig. 1 and co-incident in Fig. 2) and two positive.

(v) Suppose, however, that all roots are complex (Fig. 3). Then from the fact that the sum of the roots is positive:—

$$\left(\sum_{i=1}^{4} Z_1 = 2\phi > 0\right)$$

at least one complex conjugate pair must have positive real parts. Thus, we must consider, for arbitrary roots, the following three possibilities:

(A) $Z_1,\ Z_2 = a \pm bi$
$Z_3,\ Z_4 = c \pm di$
(B) $Z_1,\ Z_2 = a \pm bi$ $\quad a,\ b,\ c,\ d > 0$ $\quad$ in all cases.
$Z_3,\ Z_4 = \ \pm di$
(C) $Z_1,\ Z_2 = a \pm bi$
$Z_3,\ Z_4 = -c \pm di$

The quartic equations associated with these three cases are as follows:

(A) $F_1(Z) = Z^4 - 2(a+c)Z^3 + (a^2+b^2+c^2+d^2+4ac)Z^2$
$\qquad -2[c(a^2+b^2)+a(c^2+d^2)]Z + (a^2+b^2)(c^2+d^2)$

(B) $F_2(Z) = Z^4 - 2aZ^3 + (a^2+b^2+d^2)Z^2 - 2ad^2Z + d^2(a^2+b^2)$

(C) $F_3(Z) = Z^4 - 2(a-c)Z^3 + (a^2+b^2+c^2+d^2-4ac)Z^2$
$\qquad +2[c(a^2+b^2)-a(c^2+d^2)]Z + (a^2+b^2)(c^2+d^2)$

Comparison with the equation under consideration, $F(Z)$, shows that only in case (*C*) are the signs of the co-efficients correct; in particular, the sign of the co-efficient on Z must be positive and this is violated in cases (A) and (B) above. Thus, if all roots are complex there must be one pair with negative real parts and one with positive real parts.

Thus we always have two stable roots which may be real or complex, as asserted in the text.