Title	A NOTE ON THE DISINTEGRATION OF MEASURES A CONVERGENCE THEOREM
Sub Title	
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Publisher	Keio Economic Society, Keio University
Publication year	1979
Jtitle	Keio economic studies Vol.16, No.1/2 (1979.) ,p.57- 64
JaLC DOI	
Abstract	
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19790001-0057

A NOTE ON THE DISINTEGRATION OF MEASURES
A CONVERGENCE THEOREM

TORU MARUYAMA

1. INTRODUCTION

Aumann and Perles [2] rigorously examined a kind of variational problem arising in mathematical economics. Berliocchi and Lasry [3] as well as Artstein [1] generalized the formulation of this problem and gave interesting existence proofs of optimal solutions. Maruyama [8] presented a further generalization as follows.

Let X be a compact metric space and $\bar{\mu}$ be a non-negative, non-atomic Borel measure on X such that $\bar{\mu}(X)=C<+\infty$. We denote by $\mathfrak{M}_{\bar{\mu}}$ the set of all non-negative Borel measures μ on X which satisfy the following two conditions:

(i) $\mu \ll \bar{\mu}$,

(ii) $\mu(X) \leqq C$.

Let Y be a locally compact Polish space and consider the functions:

$$u: \quad X \times Y \to \boldsymbol{R},$$

$$g_i: \quad X \times Y \to [0, \ +\infty]; \qquad i=1, 2, \cdots, n.$$

The problem is to

$$\textit{Maximize} \qquad \int_X u(x, f(x))\, d\mu$$

subject to

$$\mu \in \mathfrak{M}_{\bar{\mu}}$$

$$f: \quad X \to Y \quad \text{is Borel-measurable}$$

$$\int_X g_i(x, f(x))\, d\mu \leqq \alpha_i; \qquad i=1, 2, \cdots, n$$

where

$$(\alpha_1, \alpha_2, \cdots, \alpha_n) \text{ is a fixed vector.}$$

Under what conditions on u and g_i's, does an optimal solution (μ^*, f^*) exist? An

The author is grateful to Professor Akio Koyama for his invaluable comments on both of this paper and [8].

answer to this question is given in Maruyama [8] and the present paper supplies a necessary stepping-stone to the solution of the problem.

Let

$$\gamma_n = \int_X \delta_x \otimes \nu_n[x] d\mu_n ; \qquad n=1, 2, \cdots$$

be a sequence of disintegrations of measures, the definition of which will be given at the begining of §3. We are going to investigate a sufficient condition of the weak*-convergence of $\{\gamma_n\}$ in relation to the weak*-convergences of $\{\nu_n[x]\}$ and $\{\mu_n\}$.

2. THE SPACE $\mathfrak{M}$

Let (X, ρ) be a metric space and $\mathfrak{M}$ be the set of all non-negative Borel measures on X which satisfy the condition:

$$\mu(X) \leqq C < +\infty .$$

The topology of $\mathfrak{M}$ is defined by the weak*-convergence: i.e. a net $\{\mu_\alpha\}$ in $\mathfrak{M}$ converges to $\mu_\infty \in \mathfrak{M}$ (symbolically w^*-lim $\mu_\alpha = \mu_\infty$) iff

$$\lim_\alpha \int_X f d\mu_\alpha = \int_X f d\mu_\infty$$

for any bounded, continuous real-valued function f defined on X. The Banach space of those functions endowed with the sup-norm is denoted by $\mathscr{C}^b(X)$.

The topological and analytical properties of the space of all the probability measures on X have been systematically elucidated recently by Billingsley [4], Maruyama [7] or Parthasarathy [9]. Most of the corresponding statements in the more general space $\mathfrak{M}$ can be proved analogously, out of which the following two theorems are useful in this paper.

THEOREM A. *$\mathfrak{M}$ is separably metrizable iff X is a separable metric space.*

THEOREM B. *Let X be a separable metric space, $\{\mu_n\}$ a sequence in $\mathfrak{M}$ and $\mu_\infty \in \mathfrak{M}$. Then the following two statements are equivalent.*

(i) w^*-lim $\mu_n = \mu_\infty$.

(ii) *For any equi-continuous, uniformly bounded subfamily $\mathscr{A} \subset \mathscr{C}^b(X)$,*

$$\lim_{n\to\infty} \sup_{f \in \mathscr{A}} \left| \int_X f d\mu_n - \int_X f d\mu \right| = 0 .$$

The following basic result seems to have been obtained independently by several authors (for example, Grandmont [6]). A self-contained proof of this theorem (for the space of probability measures) can be found in Maruyama [7]. For the sake of the readers' convenience, I will repeat it again in this paper for the space $\mathfrak{M}$. We must prepare a lemma.

LEMMA 1. *Let $\{f_n\}$ be a sequence of real-valued functions defined on a metric space X such that*

(i) *f_n is continuous at $x \in X$ for all n;*

(ii) *$\{f_n\}$ is continuously convergent to f at x.* (i.e. *$x_n \to x$ implies $f_n(x_n) \to f(x)$.*) *Then $\{f_n\}$ is equi-continuous at x.*

Proof. Assume that $\{f_n\}$ is *not* equi-continuous at x. Then for some $\varepsilon > 0$, we can find a sub-sequence $\{f_{n_p}\}$ of $\{f_n\}$ such that

$$\begin{array}{ll} |f_{n_1}(x_1) - f_{n_1}(x)| \geqq \varepsilon & \text{for some } x_1 \in B_1(x) \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ |f_{n_p}(x_p) - f_{n_p}(x)| \geqq \varepsilon & \text{for some } x_p \in B_{1/p}(x) \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array} \tag{1}$$

where

$$B_{1/p}(x) = \left\{ y \in X \mid \rho(x, y) < \frac{1}{p} \right\}.$$

Then, by construction,

$$x_p \to x \quad \text{as } p \to \infty.$$

Hence, by (ii), we must have

$$\lim_{p \to \infty} f_{n_p}(x_p) = f(x). \tag{2}$$

However

$$|f_{n_p}(x_p) - f(x)| \geqq |f_{n_p}(x_p) - f_{n_p}(x)| - |f_{n_p}(x) - f(x)|.$$

Since

$$|f_{n_p}(x) - f(x)| \to 0 \quad \text{as } p \to \infty,$$

$$\lim_{p \to \infty} |f_{n_p}(x_p) - f(x)| \geqq \varepsilon$$

which contradicts to (2). Q.E.D.

THEOREM 1. *Let X be a separable metric space and $\{\mu_n\}$ be a sequence in $\mathfrak{M}$ such that*

(i) $w^*\text{-lim}\ \mu_n = \mu_\infty \in \mathfrak{M}$.

Furthermore assume that a sequence $\{f_n\}$ in $\mathscr{C}^b(X)$ satisfy:

(ii) *$\{f_n\}$ is uniformly bounded;*

(iii) *$\{f_n\}$ is continuously convergent to f at any $x \in X$.*

Then we have

$$\lim_{n\to\infty} \int_X f_n\, d\mu_n = \int_X f\, d\mu_\infty .$$

Proof. By Lemma 1, $\{f_n\}$ is equi-continuous at any $x \in X$. Hence by Theorem B,

$$\lim_{n\to\infty} \sup_p \left| \int_X f_p\, d\mu_n - \int_X f_p\, d\mu_\infty \right| = 0 . \tag{3}$$

Next consider

$$\left| \int_X f d\mu_\infty - \int_X f_n d\mu_n \right| \leqq \left| \int_X f d\mu_\infty - \int_X f_n d\mu_\infty \right| + \left| \int_X f_n d\mu_\infty - \int_X f_n d\mu_n \right| \tag{4}$$

The first term on the right hand side of (4) tends to 0 by (ii), (iii) and the Bounded Convergence Theorem. And the second term tends to 0 by (3).

Thus we get the desired result. Q.E.D.

3. WEAK*-CONVERGENCE OF DISINTEGRATIONS

Let X and Y be compact metric spaces and $\mu \in \mathfrak{M}$ throughout this section.

Then we can define the crucial concept of μ-disintegration of a measure as follows.

DEFINITION. A Borel positive measure γ on $X \times Y$ is said to have a μ-*disintegration* if there is a weak*-measurable mapping

$$x \to \nu[x]$$

of X into the space of Borel probability measures on Y such that

$$\gamma = \int_X \delta_x \otimes \nu[x] d\mu , \tag{5}$$

where δ_x is the Dirac measure concentrating at x. (cf. Bourbaki [5] pp. 39–42 for detailed discussions.)

We denote by $\mathfrak{D}(\mu)$ the set of all Borel measures on $X \times Y$ which have μ-disintegrations and put

$$\mathfrak{D}(\mathfrak{M}) \equiv \cup \{\mathfrak{D}(\mu) \mid \mu \in \mathfrak{M}\}.$$

Since X and Y are compact metric space, so is $X \times Y$. Hence $\mathfrak{D}(\mathfrak{M})$ is separably metrizable by Theorem A.

THEOREM 2. *Let $\{\gamma_n\}$ be a sequence in $\mathfrak{D}(\mathfrak{M})$ such that*

$$\gamma_n = \int_X \delta_x \otimes \nu_n[x] d\mu_n; \qquad n = 1, 2, \cdots.$$

Assume that

(i) $w^*\text{-}\lim_{n \to \infty} \mu_n = \mu_\infty \in \mathfrak{M}$;

(ii) *if* $x_n \to x$, then

$$w^*\text{-}\lim_{n \to \infty} \nu_n[x_n] = \nu_\infty[x] \qquad \langle\text{continuous convergence}\rangle$$

$$w^*\text{-}\lim_{k \to \infty} \nu_n[x_k] = \nu_n[x]; \qquad n = 1, 2, \cdots \quad \langle\text{continuity}\rangle$$

for any $x \in X$.

Then

$$w^*\text{-}\lim_{n \to \infty} \gamma_n = \int_X \delta_x \otimes \nu_\infty[x]\, d\mu_\infty.$$

Proof. Let $f \in \mathscr{C}^b(X \times Y)$. Then by definition of γ_n,

$$\int_{X \times Y} f(x, y) d\gamma_n = \int_X d\mu_n \int_Y f(x, y) d\nu_n[x].$$

If we define $\psi_n : X \to \boldsymbol{R}$ by

$$\psi_n(x) = \int_Y f(x, y) d\nu_n[x]; \qquad n = 1, 2, \cdots, \tag{6}$$

then we can claim:

Claim (I). $\psi_n(x)$ is continuous.

For the sake of simplicity, we omit the index n. What we have to show is: $x_k \to x$ implies

$$\int_Y f(x_k, y) d\nu[x_k] \longrightarrow \int_Y f(x, y) d\nu[x]. \tag{7}$$

If we denote $f_k(y) = f(x_k, y)$, then $\{f_k\}$ is uniformly bounded and continuously convergent to $f(x, y)$ at any $y \in Y$. And by assumption (ii),

$$w^*\text{-}\lim_{k \to \infty} \nu[x_k] = \nu[x]$$

Hence by Theorem 1, we get the desired result.

Claim (II).

$$\int_X \psi_n(x) d\mu_n \longrightarrow \int_X \psi_\infty(x) d\mu_\infty.$$

We have already checked the continuity of $\psi_n(x)$ ($n = 1, 2, \cdots$) in *Claim (I)*. And the uniform boundedness of $\{\psi_n\}$ is clear because

$$\|\psi_n\| \leqq \|f\| < +\infty; \qquad n = 1, 2, \cdots. \tag{8}$$

Next we have to show that ψ_n is continuously convergent to ψ_∞ at any $x \in X$.

Assume $x_n \to x$ (as $n \to \infty$). If we denote $f_n(y) = f(x_n, y)$ as in *Claim (I)*, then $\{f_n\}$ is uniformly bounded and continuously convergent to $f(x, y)$ at any $y \in Y$. Furthermore, by assumption (ii),

$$w^*\text{-}\lim_{n \to \infty} \nu_n[x_n] = \nu_\infty[x].$$

Hence, again by Theorem 1,

$$\int_Y f_n(y) d\nu_n[x_n] \longrightarrow \int_Y f(x, y) d\nu_\infty[x] \qquad \text{as } n \to \infty.$$

$$\text{i.e.} \quad \psi_n(x_n) \longrightarrow \psi_\infty(x) \qquad \text{as } n \to \infty. \tag{9}$$

(8), (9), (i) and Theorem 1 imply *Claim (II)*.

By the above discussions, we have seen that

$$\int_{X \times Y} f(x, y) d\gamma_n \longrightarrow \int_X d\mu_\infty \int_Y f(x, y) d\nu_\infty[x]$$

for every $f \in \mathscr{C}^b(X \times Y)$; that is

$$w^*\text{-}\lim_{n\to\infty} \gamma_n = \gamma_\infty .$$

Q.E.D.

REMARK. Define γ_n, $\gamma_\infty \in \mathfrak{D}(\mathfrak{M})$ by

$$\gamma_n = \int_X \delta_x \otimes \nu_n[x] d\mu_n ; \qquad n = 1, 2, \cdots$$

$$\gamma_\infty = \int_X \delta_x \otimes \nu_\infty[x] d\mu_\infty$$

and assume $w^*\text{-}\lim_{n\to\infty} \gamma_n = \gamma_\infty$.

Then we can easily verify that

$$w^*\text{-}\lim_{n\to\infty} \mu_n = \mu_\infty .$$

In fact, if $g \in \mathscr{C}^b(X)$, then

$$\int_X g(x) d\mu_n = \int_{X\times Y} g(x) d\gamma_n \longrightarrow \int_{X\times Y} g(x) d\gamma_\infty = \int_X g(x) d\mu_\infty \qquad \text{as } n\to\infty .$$

$$\therefore \quad \int_X g(x) d\mu_n \longrightarrow \int_X g(x) d\mu_\infty \qquad \text{as } n\to\infty .$$

However $\gamma_n \to \gamma_\infty$ does *not* necessarily imply the condition (ii) in Theorem 2.

COUNTER EXAMPLE. Let us define

$$\gamma_n = \int_X \delta_x \otimes \nu_n[x] d\mu_n$$

as follows:

$$\mu_\infty = \mu_n = \delta_{x_0} \qquad \text{for all } \; n$$

$$\nu_\infty[x_0] = \nu_n[x_0] \qquad \text{for all } \; n \qquad (\textit{no other specification})$$

where x_0 is any fixed point of X. Then for any $f \in \mathscr{C}^b(X \times Y)$,

$$\int_{X\times Y} f(x, y) d\gamma_n = \int_Y f(x_0, y) d\nu_n[x_0]$$

$$= \int_Y f(x_0, y) d\nu_\infty[x_0]$$

$$= \int_{X \times Y} f(x, y) d\gamma_{\infty} .$$

$$\therefore \quad w^*\text{-}\lim_{n \to \infty} \gamma_n = \gamma_{\infty} .$$

But the condition (ii) in the above theorem is not necessarily satisfied.

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