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Sub Title	
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Publisher	Keio Economic Society, Keio University
Publication year	1979
Jtitle	Keio economic studies Vol.16, No.1/2 (1979.) ,p.53- 56
JaLC DOI	
Abstract	
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19790001-0053

A NOTE ON THE EXISTENCE OF A CONTINUOUS UTILITY FUNCTION[†]

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1. INTRODUCTION

Let a consumer be given and his consumption set be X, which is a topological space. He has a weak preference relation $\succsim$ for pairs of elements in X satisfying the following:

(a) *Transitivity*:

$$x^1 \succsim x^2 \quad \text{and} \quad x^2 \succsim x^3 \quad \text{imply} \quad x^1 \succsim x^3;$$

(b) *Connexity*:

$$\text{For all } x^1 \text{ and } x^2, \text{ either } x^1 \succsim x^2 \quad \text{or} \quad x^2 \succsim x^1;$$

(c) *Continuity*:

For any given x^0, the sets $\{x \mid x \succsim x^0\}$ and $\{x \mid x^0 \succsim x\}$ are closed in X.

A real-valued function u, defined on X, is said to be a utility function on X if it has the property that $x^1 \succsim x^2$ if and only if $u(x^1) \geqq u(x^2)$. It is well known that Debreu [2], based on the work of Eilenberg, shows that if X is connected and separable, and if (a), (b) and (c) hold, then there exists a continuous utility function representing $\succsim$. Arrow and Hahn [1] try to give an alternative proof to a special case in which X is a convex subset of a finite-dimensional Euclidean space: If X is a convex subset of a finite-dimensional Euclidean space and the preference relation $\succsim$ satisfies (a), (b) and (c), then there exists a continuous utility function representing $\succsim$.

Their method of proof is intuitively simpler than that given by Debreu [2]. Their proof, however, is done with the following three additional assumptions:

(d) *Semi-strict convexity*:

$$\text{If } x^1 \succ x^2 \quad \text{and} \quad 0 \leqq \alpha < 1, \text{ then } (1-\alpha)x^1 + \alpha x^2 \succ x^2.$$

† Received September, 1977; Revised July, 1979.

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(e) *Non-satiation*:

$$\text{There exists no } x^* \text{ such that } x^* \gtrsim x \text{ for all } x.$$

(f) X is closed in the Euclidean space.

This fact will be known if one follows their proof with care (Arrow and Hahn [1], pp. 82–87). They need their Lemma 4.1 (a): Local non-satiation, which is implied by (d) and (e), in order to prove the continuity of the utility function. They also use the closedness of X here and need the full use of it in the last part of the proof.[1]

Interpretating the consumption set X, Arrow and Hahn consider that there is some limit on the amount of time in the period, so that the amount of consumption is constrained by time (see Arrow and Hahn [1], p. 76). If one follows their interpretation, X may be bounded in some cases, i.e., X may be compact by (f). Then one cannot apply their theorem to this case, since the compactness and (e) are incompatible under (a), (b) and (c). That is, the preference relation $\gtrsim$ is necessarily satiated. Hence it is desirable to prove the theorem without (e).

2. A THEOREM

LEMMA 2.1. *Let X be a convex subset in a finite-dimensional Euclidean space and let us assume $\gtrsim$ satisfies (a), (b) and (c). Let us define an upper contour set*

$$C(x^0) = \{x \mid x \gtrsim x^0\}$$

for any element $x^0 \in X$. Then a utility function u^0 on $C(x^0)$ can be constructed.

Proof. This is known from the first part of the proof of Arrow and Hahn.
Q.E.D.

LEMMA 2.2. *Let us add assumptions (d) and (f) to the hypotheses in Lemma 2.1. Then a continuous utility function $\bar{u}^0$ on each $C(x^0)$ is constructed.*

Proof. When the assumption (e) holds, Arrow and Hahn give the proof. Then we consider the case in which (e) does not hold. There is an $x^* \in X$ such that

$$x^* \gtrsim x \qquad \text{for all} \quad x \in X.$$

Then

$$C(x^*) = \{x \mid x \sim x^*\} \quad \text{and} \quad C(x^0) \supset C(x^*) \qquad \text{for any} \quad x^0 \in X.$$

If $C(x^*) = C(x^0)$, the assertion is trivial. Then we suppose $C(x^*) \neq C(x^0)$.

We define a real-valued function $\bar{u}^0$ on $C(x^0)$:

[1] See Arrow and Hahn [1], p. 85.

$$\bar{u}^0(x) = \begin{cases} u^0(x), & \text{if} \quad x \in C(x^0) - C(x^*), \\ \sup \{u^0(x) \mid x \in C(x^0) - C(x^*)\}, & \text{if} \quad x \in C(x^*), \end{cases}$$

where u^0 is the utility function constructed in Lemma 2.1. Since

$$u^0(x^0) \leqq u^0(x) < u^0(x^*) \qquad \text{for all} \quad x \in C(x^0) - C(x^*),$$

we get

$$\sup \{u^0(x) \mid x \in C(x^0) - C(x^*)\} \leqq u^0(x^*).$$

Then $\bar{u}^0$ is well-defined.

We need the following local non-satiation on $C(x^0) - C(x^*)$: For any $x \in C(x^0) - C(x^*)$ and any neighborhood $N(x)$ of x in $C(x^0) - C(x^*)$, there is an $x' \in N(x)$ such that $x' \succ x$.

Let $x \in C(x^0) - C(x^*)$ and let $N(x)$ be any neighborhood of x in $C(x^0) - C(x^*)$. Since $C(x^*)$ is closed in X from (c), $C(x^0) - C(x^*)$ is open relative to $C(x^0)$. Then there is a neighborhood $V \cap C(x^0)$ of x in $C(x^0)$ such that

$$V \cap C(x^0) \subset C(x^0) - C(x^*),$$

where V is a neighborhood of x in X. Let

$$N(x) = W \cap [C(x^0) - C(x^*)],$$

where W is a neighborhood of x in X.

From $x^* \succ x$ and (d), we get

$$x' = (1-\alpha)x^* + \alpha x \succ x \succsim x^0 \qquad \text{for all } \alpha \text{ with } 0 \leqq \alpha < 1.$$

Then if we let α be sufficiently close to 1, we get

(1) $x' \succ x$,
(2) $x' \in C(x^0)$,
(3) $x' \in V$,
(4) $x' \in W$.

From (2) and (3), we get $x' \in C(x^0) - C(x^*)$, and $x' \in N(x)$ with $x' \succ x$. Thus we have the local non-satiation on $C(x^0) - C(x^*)$.

Now we prove that $\bar{u}^0$ is a utility function. Let $x^1 \succsim x^2$. Then we have three cases:

(5) x^1 and $x^2 \in C(x^0) - C(x^*)$,
(6) $x^1 \in C(x^*)$ and $x^2 \in C(x^0) - C(x^*)$,
(7) x^1 and $x^2 \in C(x^*)$.

For each case, we easily know that $\bar{u}^0(x^1) \geqq \bar{u}^0(x^2)$. Conversely let $\bar{u}^0(x^1) \geqq \bar{u}^0(x^2)$. We have four cases:

(8) x^1 and $x^2 \in C(x^0) - C(x^*)$
(9) $x^1 \in C(x^*)$ and $x^2 \in C(x^0) - C(x^*)$,
(10) $x^1 \in C(x^0) - C(x^*)$ and $x^2 \in C(x^*)$,
(11) x^1 and $x^2 \in C(x^*)$.

It is easy to see that $x^1 \gtrsim x^2$ for (8), (9) and (11). We show that (10) is impossible. From the definition of $\bar{u}^0$ and $\bar{u}^0(x^1) \geqq \bar{u}^0(x^2)$, $\bar{u}^0(x^1) = \bar{u}^0(x^2)$. But, from the local non-satiation on $C(x^0) - C(x^*)$, there is an $x' \in C(x^0) - C(x^*)$ such that $x' \succ x^1$. So $\bar{u}^0(x') > \bar{u}^0(x^1) = \bar{u}^0(x^2)$; a contradiction.

Finally we prove the continuity of $\bar{u}^0$. We can use a part of the proof of Arrow and Hahn to show the continuity on $C(x^0) - C(x^*)$, since we have the local non-satiation on $C(x^0) - C(x^*)$.[2] Then it remains to prove the continuity on $C(x^*)$. Let $x \in C(x^*)$ and let $\{x^n\}$ in $C(x^0)$ be any sequence which converges to x. Since

$$\bar{u}^0(x^0) \leqq \bar{u}^0(x^n) \leqq \bar{u}^0(x),$$

there is a limit point k of $\{\bar{u}^0(x^n)\}$. Suppose $k < \bar{u}^0(x)$. There is a subsequence $\{x^{n_m}\}$ of $\{x^n\}$ for which $\{\bar{u}^0(x^{n_m})\}$ converges to k. From the definition of $\bar{u}^0(x)$, there is a $z \in C(x^0) - C(x^*)$ for which $k < \bar{u}^0(z) < \bar{u}^0(x)$. There is an m_0 such that

$$\bar{u}^0(x^{n_m}) \leqq \bar{u}^0(z) \qquad \text{for all} \quad m \geqq m_0 .$$

Then, since $\bar{u}^0$ is a utility function, we get $x^{n_m} \lesssim z$ for all $m \geqq m_0$. Then, from (c), $x \lesssim z$ which contradicts $x \in C(x^*)$ and $z \in C(x^0) - C(x^*)$. Thus any limit point of $\{\bar{u}^0(x^n)\}$ must be $\bar{u}^0(x)$. Hence $\{\bar{u}^0(x^n)\}$ converges to $\bar{u}^0(x)$. Q.E.D.

Once Lemma 2.2 is proved, we have the required theorem by following the last part of the proof of Arrow and Hahn.

THEOREM 2.3. *Let X be a closed and convex subset in a finite-dimensional Euclidean space and let us assume (a), (b), (c) and (d). Then there exists a continuous utility function on X.*

Finally we thank the referee of this journal for kind suggestions.

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[1] Arrow, K. J., and F. H. Hahn: *General Competitive Analysis*, Holden-Day, San Francisco, 1971.

[2] Debreu, G.: *Theory of Value*, Wiley, New York, 1959.

[2] See Arrow and Hahn [1], the 2nd and 3rd paragraphs on p. 84.