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Chapter 7

SOME STATISTICAL ASPECTS OF THE "COMPLETE DETERMINATION METHOD" FOR MEASURING PREFERENCE FUNCTIONS*

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I. INTRODUCTION

In a paper by Tsujimura and Sato [11], an unfamiliar method, instead of the Least Squares Method or Maximum Likelihood Method, was devised for measuring preference functions in the context of the theory of consumer behavior. Since then the method, which we call Complete Determination Method, has been modified and used intensively by their collaborators, [10] and [9].

The method is not necessarily based on the orthodox statistical principle but rather on the intuition develoved through the experience of actual estimation work. Reportedly, the method has been working quite well and providing satisfactory estimates, although obscurity remains concerning its statistical foundations.

On the Complete Determination Method, Tsujimura writes: The distribution of the Complete Determination Solution for the structural parameter forms the generalized Cauchy distribution, and it is almost certain that the median of the distribution coincides with the population regression coefficient.[1] This conjecture, although somewhat ambiguous, seems to contain several things:

(a) Each of the Complete Determination Solutions is distributed according to the generalized Cauchy distribution.

(b) The set of Complete Determination Solution can be regarded as random sampling from this generalized Cauchy distribution.

(c) Then the median of this sample is an appropriate estimate of the parameter (supposedly, the location parameter) of the Cauchy distribution.

(d) This parameter of the Cauchy population is considered to be the original parameter of the utility function, the final object to measure.

This paper, being in the nature of a somewhat tentative approach to a difficult problem, tries to deepen the understanding about Complete Determination Method with special reference to the following two points: (1) We derive the sampling distribution of the Complete Determination Solution and examine its statistical property. (2) We discuss the stochastic structure of a set of the Complete determination Solutions.

Through this analysis, the validity of Tsujimura's conjecture on the form of the

* This paper is based on an earlier Japanese version [7].

[1] The author's translation from Tsujimura [9, p. 5]. A similar argument appears in [9, p. 229].

distribution, as summarized in (a) above, will be confirmed.

Section II presents the structural equations to be estimated by the Complete determination Method. Sections III and IV provide the results of an analysis of the sampling distribution. Section V is a discussion about the nature of the set of Complete Determination Solutions.

II. MODEL AND THE COMPLETE DETERMINATION SOLUTION

The model consists of the balance equation and the tangency condition

$$\begin{bmatrix} 1 & 1 \\ 1 & -\beta \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ mp_1 \\ mp_2 \\ y \end{bmatrix}, \tag{2.1}$$

which is deduced by maximizing a utility indicator

$$I=(\alpha_{11}+\alpha_{12}m+q_1)^{\beta_1}(\alpha_{21}+\alpha_{22}m+q_2)^{\beta_2} \tag{2.2.}$$

subject to the balance equation

$$p_1q_1+p_2q_2=y. \tag{2.3}$$

Here E_g, p_g and q_g are expenditure, price and quantity of the g-th good, respectively, m number of family members and y total expenditure.

The parameters β and γ are combinations of those in the utility indicator,

$$\begin{aligned} &\beta=\beta_1/\beta_2, \\ &\gamma'=[\gamma_1, \cdots, \gamma_4]=[-\alpha_{11}, \beta\alpha_{21}, -\alpha_{21}, \beta\alpha_{22}]. \end{aligned} \tag{2.4}$$

The β and γ are estimated by the Complete Determination Method with time series budget data of sample size T,

$$\begin{aligned} &E'_{\cdot t}=[E_{1t}, E_{2t}], \\ &z'_{\cdot t}=[z_{1t}, \cdots, z_{5t}]=[p_{1t}, p_{2t}, m_tp_{1t}, m_tp_{2t}, y_t], \qquad t=1, \cdots, T. \end{aligned} \tag{2.5}$$

The reader may refer to [9] and [10] for full details of the Complete Determination Method. Here we simply define the Complete Determination Solution (CDS), which is our main concern.

Five sample points at times $t=t_1, \cdots, t_5$ are needed to solve the equation in the five parameters,

$$E_{1t}=\beta E_{2t}+\gamma' z^*_{\cdot t}, \qquad t=t_1, \cdots, t_5, \tag{2.6}$$

where

$$z^{*\prime}_{\cdot t}=[z_{1t}, \cdots, z_{4t}]. \tag{2.7}$$

Let the solution (CDS) of (2.6) be $[b_{t_1 \cdots t_5}, c'_{t_1 \cdots t_5}]$ or $[b_\tau, c'_\tau]$ in short, then

$$\begin{bmatrix} b_\tau \\ \\ c_\tau \end{bmatrix} = \left[\begin{array}{c:c} E_{2t_1} & \\ \vdots & Z_\tau^* \\ E_{2t_5} & \end{array}\right]^{-1} \begin{bmatrix} E_{1t_1} \\ \vdots \\ E_{1t_5} \end{bmatrix}, \tag{2.8}$$

where

$$Z_\tau^* = Z_{t_1\cdots t_5}^* = \begin{bmatrix} z_{\bullet t_1}^{*\prime} \\ \vdots \\ z_{\bullet t_5}^{*\prime} \end{bmatrix}. \tag{2.9}$$

With T sample points in hand, we can set five-equation systems like (2.6) in $N = T!/5!(T-5)!$ different ways without replacement. And each system yields the CDS. So we have N CDS's, which are represented as $[b_1, c_1'], \cdots, [b_N, c_N']$ or as $[b_n, c_n']$, $n=1, \cdots, N$. It can be shown under the assumptions specified later that the CDS exists with probability one.

The quotation in Section I refers to the distribution of $[b_\tau, c_\tau']$ and the propetry of the median (vector) of $\{[b_1, c_1'] \cdots [b_N, c_N']\}$.

To investigate statistical properties of the CDS, we assume a simple shock model

$$\begin{bmatrix} 1 & 1 \\ 1 & -\beta \end{bmatrix} E_{\bullet t} = \begin{bmatrix} 0 & 1 \\ \gamma' & 0 \end{bmatrix} z_{\bullet t} + \begin{bmatrix} 0 \\ u_t \end{bmatrix}, \qquad t=1, \cdots, T, \tag{2.10}$$

where

$$\begin{aligned} & u_t \sim N(0, \sigma^2), \\ & u_t \text{ is independent of } u_s, \\ & Z = \begin{bmatrix} z_{\bullet 1}' \\ \vdots \\ z_{\bullet T} \end{bmatrix} \text{ is a fixed matrix of rank five.} \end{aligned} \tag{2.11}$$

Then it follows that[2]

$$\begin{aligned} & E_{\bullet t} \sim N\left(\begin{bmatrix} \pi_1' \\ \pi_2' \end{bmatrix} z_{\bullet t}, \ \begin{bmatrix} \omega & -\omega \\ -\omega & \omega \end{bmatrix}\right), \\ & E_{\bullet t} \text{ is independent of } E_{\bullet s}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} & \begin{bmatrix} \pi_1' \\ \pi_2' \end{bmatrix} = \frac{1}{1+\beta} \begin{bmatrix} \gamma' & \beta \\ -\gamma' & 1 \end{bmatrix}, \\ & \omega = \sigma^2/(1+\beta)^2. \end{aligned} \tag{2.13}$$

Since the covariance matrix of $E_{\bullet t}$ is singular, E_{1t} can be written, with propability one, as a linear function of E_{2t} (regression equation),

[2] We assume that $\beta > 0$ for the convexity of the indifference curve (2.2).

$$E_{1t}=\pi_1' z_{\cdot t}+\omega^{-1}(-\omega)(E_{2t}-\pi_2' z_{\cdot t}) \\ =z_{5t}-E_{2t}, \tag{2.14}$$

i.e., the balance equation.

From (2.12) and (2.14), we write out the distribution of the observed endogenous variables,

$$\begin{bmatrix} E_{1\cdot} \\ E_{2\cdot} \end{bmatrix} \equiv \begin{bmatrix} E_{11} \\ \vdots \\ E_{1T} \\ E_{21} \\ \vdots \\ E_{2T} \end{bmatrix} = \begin{bmatrix} z_{5\cdot}-E_{2\cdot} \\ \hline N(Z\pi_2,\ \omega I_T) \end{bmatrix}, \tag{2.15}$$

where

$$z_{5\cdot}'=[z_{51},\ \cdots,\ z_{5T}]. \tag{2.16}$$

III. SAMPLING DISTRIBUTION OF b_τ

For a specified value of $\tau=[t_1,\ \cdots,\ t_5]$, we derive the probability density function (pdf) of b_τ. In explicit form,

$$b_\tau=\sum_{i=1}^{5} d_\tau^{(t_i)} E_{1t_i} \Big/ \sum_{i=1}^{5} d_\tau^{(t_i)} E_{2t_i}, \tag{3.1}$$

where $d_\tau^{(t_i)}$ is the cofactor of E_{2t_i} in the matrix $\begin{bmatrix} E_{2t_1} & \vdots & \\ \vdots & \vdots & Z_\tau^* \\ E_{2t_5} & \vdots & \end{bmatrix}$. By (2.15) and (3.1), we have

$$\sum d_\tau^{(t_i)} E_{2t_i}=\sum d_\tau^{(t_i)} z_{5t_i}/(b_\tau+1) \\ =|Z_\tau|/(b_\tau+1), \tag{3.2}$$

where

$$\sum d_\tau^{(t_i)} E_{2t_i} \sim N\left(\sum d_\tau^{(t_i)} z_{\cdot t_i}'\pi_2,\ \frac{\sigma^2}{(\beta+1)^2} \sum d^{(t_i)2}\right), \\ Z_\tau \equiv Z_{t_1\cdots t_5}=\begin{bmatrix} z_{\cdot t_1}' \\ \vdots \\ z_{\cdot t_5}' \end{bmatrix}. \tag{3.3}$$

After some matrix and determinant calculations, we have

$$\sum d_\tau^{(t_i)} z_{\cdot t_i}'\pi_2=|Z_\tau|/(\beta+1), \\ \sigma^2 \sum d_\tau^{(t_i)2}=\mu_\tau^2 |Z_\tau|^2, \tag{3.4}$$

where

$$\mu_\tau^2 \equiv \mu_{t_1 \cdots t_5}^2 = \sigma^2 \times \text{ the (5, 5)th element of } (Z_\tau' Z_\tau)^{-1}. \tag{3.5}$$

Therefore, we get

$$\sum d_\tau^{(t_i)} E_{2t_i} \sim N\left(\frac{1}{\beta+1}|Z_\tau|, \frac{1}{(\beta+1)^2}\mu_\tau^2 |Z_\tau|^2\right), \tag{3.6}$$

The Jacobian of the transformation (3.2) is

$$J(\sum d_\tau^{(t_i)} E_{2t_i} : b_\tau) = |Z_\tau|/(b_\tau+1)^2, \qquad b_\tau \neq -1. \tag{3.7}$$

We then obtain the pdf of b_τ,

$$f_1(b_\tau) = \frac{1}{\sqrt{2\pi}} \frac{(\beta+1)}{(b_\tau+1)^2 \mu_\tau} \exp -\frac{1}{2\mu_\tau^2}\left(\frac{\beta - b_\tau}{b_\tau+1}\right)^2, \qquad b_\tau \neq -1, \tag{3.8}$$

which depends on β, μ_τ^2 but not on γ.[3] The f_1 is essentially a distribution of a ratio of two normal variables with non-zero mean values and singular covariance matrix (perfect correlation).

To obtain the moment of f_1 of order ν, we must calculate

$$\int_{-\infty}^{\infty} \frac{b_\tau^\nu}{(b_\tau+1)^2} \exp -\frac{1}{2\mu_\tau^2}\left(\frac{\beta+1}{b_\tau+1} - 1\right)^2 db_\tau. \tag{3.9}$$

But this integral does not exist for $\nu \geqq 1$. Therefore the sampling distribution of b_τ does not have any finite moments. Henceforth we pursue some other features of the distribution.

The equation

$$\frac{df_2}{db_\tau} = 0 \tag{3.10}$$

has two different solutions

$$b_\tau = -\left\{1 + \frac{1}{4\mu_\tau^2}(\beta+1)\right\} \pm \frac{1}{2\mu_\tau^2}|\beta+1|\sqrt{1+\frac{1}{8\mu_\tau^2}}. \tag{3.11}$$

Therefore the pdf is bimodal.[4] And it is seen that a distance between the modes is a decreasing function of μ_τ^2 and an increasing function of β (>-1).

Let λ_τ or $\lambda_{t_1 \cdots t_5}$ be the median of f_1, then the equation

$$\int_{\lambda_\tau}^{\infty} f_1(b_\tau) db_\tau = \frac{1}{2} \tag{3.12}$$

must be satisfied. After transformation, this is shown to be equivalent to

$$\int_{-1/\mu_\tau}^{-(1/\mu_\tau)[1-|\beta+1|/(\lambda_\tau+1)]} \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}x^2 dx = \frac{1}{2}, \tag{3.13}$$

[3] The pdf (3.8) is already obtained in another context, [3], [8].

[4] These properties are found in [8].

i.e., the integral of the standardized normal density. Since the lower limit of the integral is negative, we have a restriction on λ_τ,

$$(3.14)\qquad 0<\frac{1}{\mu_\tau}\left(\frac{|\beta+1|}{\lambda_\tau+1}-1\right)<\infty$$

or

$$(3.15)\qquad -1<\lambda_\tau<\beta,$$

that is, the median of the distribution of b_τ lies between -1 and β. The median is equal to β only in the trivial case with $\sigma^2=0$.

Let x_τ be the upper limit of the integral (3.13), then

$$(3.16)\qquad \lambda_\tau=\frac{1}{1+\mu_\tau x_\tau}\beta-\frac{\mu_\tau x_\tau}{1+\mu_\tau x_\tau}.$$

Given a value of μ_τ^2, we can evaluate x_τ by making use of the normal distribution table. Substituting these values of μ_τ^2 and x_τ into the right hand side of (3.16), we get Table 1 which presents the relation of λ_τ and β for a given μ_τ. It is observed that for a fixed β the deviation of λ_τ from β becomes large as μ_τ increases. Note that the increase of μ_τ^2 implies an increase of σ^2 and/or the (5, 5)th element of $(Z_\tau'Z_\tau)^{-1}$.

TABLE 1.

μ_τ	λ
4	$0.162\beta-0.838$
3	$0.229\beta-0.771$
2	$0.364\beta-0.636$
1	$0.710\beta-0.290$
1/2	$0.972\beta-0.028$
1/3	$0.999\beta-0.001$
1/4	$\approx\beta$

Figures 1 and 2 illustrate the shape of f_1 for several combinations of the parameter value as given by Table 2; e.g. the $f^{(1)}$ has the parameters $\beta=0.67$ and $\mu_\tau^2=1/18$.[5] The median of $f^{(j)}$ is pointed by the arrow (j). The $f^{(1)}$, $f^{(2)}$ and $f^{(3)}$ on the left of the coordinate $b_\tau=-1$ are negligible and are omitted in Figure 1.

Recall that we have N CDS's each of which corresponds to a five-equation system and the exogenous variable Z_τ. Therefore the N pdf's of the CDS's have common β and σ^2 and different Z_τ's. And Figure 1 shows how the N pdf's change their shapes corresponding to Z_τ's. This is essential when considering the stochastic structure of the N CDS's as a whole.

[5] In Fig. 2, the vertical scales on the right side and the left side of the coordinate $b_\tau=-1$ are different.

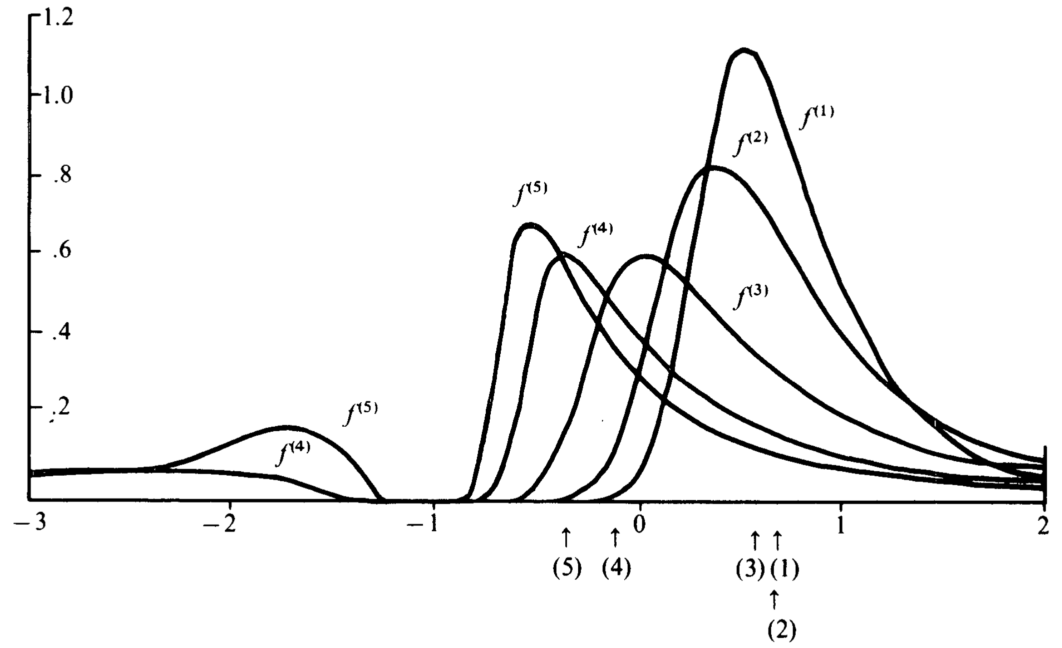
1.2
1.0
.8
.6
.4
.2
f(1)
f(2)
f(3)
f(4)
f(5)
f(5)
f(4)
−3
−2
−1
0
1
2
(5)
(4)
(3)
(1)
(2)

Fig. 1.

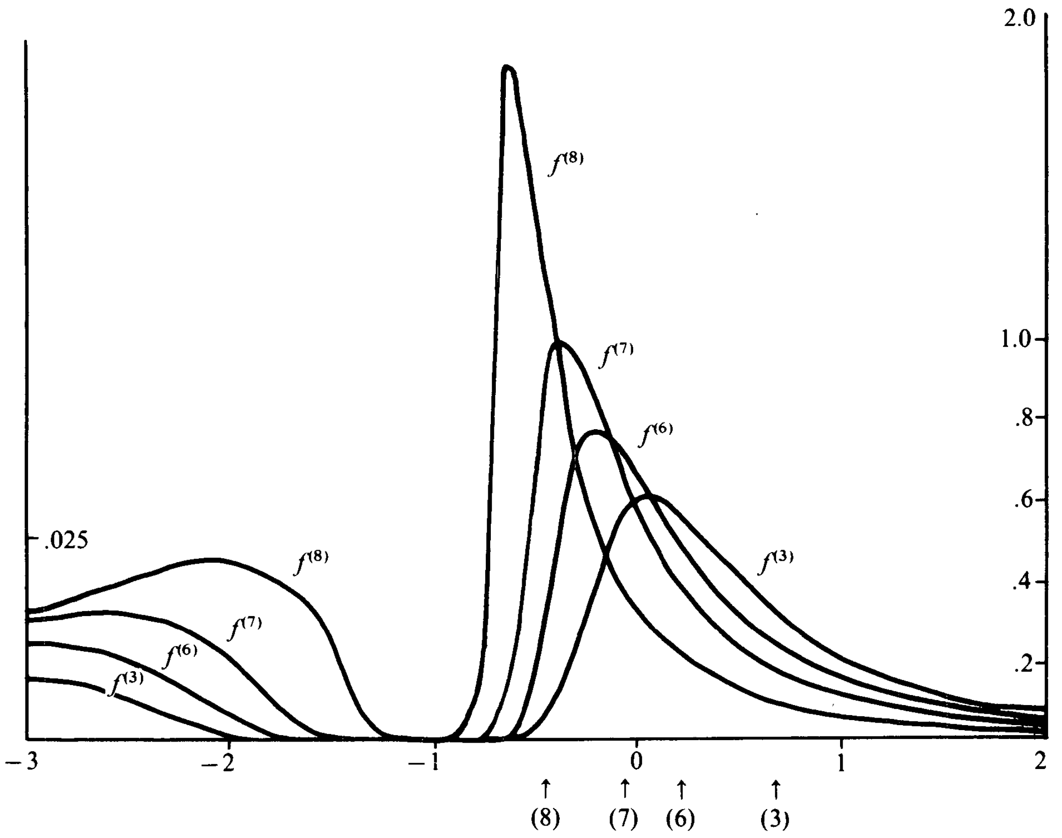
2.0
1.0
.8
.6
.4
.2
.025
f(8)
f(7)
f(6)
f(3)
f(8)
f(7)
f(6)
f(3)
−3
−2
−1
0
1
2
(8)
(7)
(6)
(3)

Fig. 2.

TABLE 2.

β	μ_τ^2				
	1/18	1/8	1/2	2	9/2
0.67	$f^{(1)}$	$f^{(2)}$	$f^{(3)}$	$f^{(4)}$	$f^{(5)}$
0.3	—	—	$f^{(6)}$	—	—
0	—	—	$f^{(7)}$	—	—
−0.4	—	—	$f^{(8)}$	—	—

IV. SAMPLING DISTRIBUTION OF $c_{m\tau}$

We derive the pdf of an element of the $c_\tau'=[c_{1\tau}, \cdots, c_{4\tau}]$, first element $c_{1\tau}$ for instance. First we obtain the joint pdf of $[b_\tau, c_\tau']$. In view of (2.8) and (2.14), we write

$$\begin{bmatrix} E_{2t_1} \\ \vdots \\ E_{2t_5} \end{bmatrix} = Z_\tau \begin{bmatrix} -c_\tau \\ \\ 1 \end{bmatrix} \frac{1}{b_\tau+1}, \tag{4.1}$$

where, from (2.15),

$$\begin{bmatrix} E_{2t_1} \\ \vdots \\ E_{2t_5} \end{bmatrix} \sim N\left(Z_\tau \pi_2, \frac{\sigma^2}{(\beta+1)^2} I_5\right). \tag{4.2}$$

The transformation from $[E_{2t_1}, \cdots, E_{2t_5}]$ to $[b_\tau, c_\tau]$ with

$$J([E_{2t_1}, \cdots, E_{2t_5}]: [b_\tau, c_\tau']) = |Z_\tau|/(b_\tau+1)^6, \qquad b_\tau \neq -1, \tag{4.3}$$

gives the joint pdf

$$\begin{aligned} f_2(b_\tau, c_\tau') = {} & (2\pi)^{-5/2} \left| \frac{\sigma^2}{(\beta+1)^2} (Z_\tau' Z_\tau)^{-1} \right|^{-1/2} (b_\tau+1)^{-6} \\ & \exp -\frac{1}{2} \left\{ [c_\tau', -1] - \frac{b_\tau+1}{\beta+1} [\gamma', -1] \right\} \left\{ \frac{\sigma^2}{(\beta+1)^2} (Z_\tau' Z_\tau)^{-1} \right\}^{-1} \\ & \times \left\{ \begin{bmatrix} c_\tau \\ -1 \end{bmatrix} - \frac{b+1}{\beta+1} \begin{bmatrix} \gamma \\ -1 \end{bmatrix} \right\}, \qquad b_\tau \neq -1 \end{aligned} \tag{4.4}$$

Rewriting (3.8) as

$$\begin{aligned} f_1(b_\tau) = {} & \frac{1}{\sqrt{2\pi}} \left\{ \frac{\sigma^2}{(\beta+1)^2} l'(Z_\tau' Z_\tau)^{-1} l \right\}^{-1/2} (b_\tau+1)^{-2} \\ & \exp -\frac{1}{2} \left\{ \left(l' - \frac{b_\tau+1}{\beta+1} l' \right) \frac{\sigma^2 (b_\tau+1)^2}{(\beta+1)^2} (Z_\tau' Z_\tau)^{-1} \right\}^{-1} \left(l - \frac{b_\tau+1}{\beta+1} l \right), \end{aligned} \tag{4.5}$$

where $l'=[0,0,0,0,1]$, and using (4.4), we get the conditional pdf of c_τ given b_τ,

$$f_3(c_\tau'|b_\tau)=(2\pi)^{-2}\left(\frac{\sigma^2(b_\tau+1)}{(\beta+1)^2}\right)^{-2}\left(\frac{|(Z_\tau'Z_\tau)^{-1}|}{l'(Z_\tau'Z_\tau)^{-1}l}\right)^{-1/2}$$
$$\exp -\frac{1}{2}\left(c_\tau'-\frac{b_\tau+1}{\beta+1}l'\right)\left\{\frac{\sigma^2(b_\tau+1)^2}{(\beta+1)^2}(Z_\tau^{*\prime}Z_\tau^*)^{-1}\right\}^{-1}$$
$$\times\left(c_\tau-\frac{b_\tau+1}{\beta+1}\gamma\right),\qquad b_\tau\neq-1. \tag{4.6}$$

Noting the relation

$$|Z_\tau'Z_\tau|=|Z_\tau^{*\prime}Z_\tau^*|\cdot l'(Z_\tau'Z_\tau)^{-1}l \tag{4.7}$$

we have

$$f_3=(2\pi)^{-2}\left|\frac{\sigma^2(b_\tau+1)^2}{(\beta+1)^2}(Z_\tau^{*\prime}Z_\tau^*)^{-1}\right|^{-1/2} \tag{4.8}$$
$$\exp -\frac{1}{2}\left(c_\tau'-\frac{b_\tau+1}{\beta+1}\gamma'\right)\left\{\frac{\sigma^2(b_\tau+1)^2}{(\beta+1)^2}(Z_\tau^{*\prime}Z_\tau^*)^{-1}\right\}^{-1}\left(c_\tau-\frac{b_\tau+1}{\beta+1}\gamma\right),$$

which is the multivariate normal distribution with mean vector $(b_\tau+1)/(\beta+1)\cdot\gamma$ and covariance matrix $[\sigma^2(b_\tau+1)^2/(\beta+1)^2](Z_\tau^{*\prime}Z_\tau^*)^{-1}$. Therefore the conditional pdf of $c_{1\tau}$ given b_τ is

$$f_4(c_{1\tau}|b_\tau)=N\left(\frac{b_\tau+1}{\beta+1}\gamma_1,\ \frac{\sigma^2(b_\tau+1)^2}{(\beta+1)^2}\varphi_{1\tau}^2\right), \tag{4.9}$$

where

$$\varphi_{1\tau}^2=\sigma^2\times \text{ the (1, 1)th element of } (Z_\tau^{*\prime}Z_\tau^*)^{-1}. \tag{4.10}$$

From the marginal pdf (4.5) and the conditional pdf (4.9), we obtain the joint pdf of $c_{1\tau}$ and b_τ. After some rearrangements, the joint pdf is

$$f_5(c_{1\tau}, b_\tau)=k(b_\tau+1)^{-3}e^{-A[1/(b\tau+1)]^2}e^{B[(1/b\tau+1)]}, \tag{4.11}$$

where

$$k=(\beta+1)^2\exp -\frac{1}{2}\left(\frac{\gamma_1^2}{\varphi_{1\tau}^2}+\frac{1}{\mu_\tau^2}\right)\Big/2\pi(\varphi_{1\tau}\mu_\tau),$$
$$A=\frac{1}{2}(\beta+1)^2\left(\frac{c_{1\tau}^2}{\varphi_{1\tau}^2}+\frac{1}{\mu_\tau^2}\right), \tag{4.12}$$
$$B=(\beta+1)\left(\frac{\gamma_1c_{1\tau}}{\varphi_{1\tau}^2}+\frac{1}{\mu_\tau^2}\right).$$

Integrating out b_τ, we have the pdf of c_τ,

$$f_6(c_{1\tau})=\frac{1}{\pi\varphi_{1\tau}\mu_\tau}\left(\frac{c_{1\tau}^2}{\varphi_{1\tau}^2}+\frac{1}{\mu_\tau^2}\right)^{-1}\exp -\frac{1}{2}\left(\frac{\gamma_1^2}{\varphi_{1\tau}^2}+\frac{1}{\mu_\tau^2}\right) \tag{4.13}$$
$$\times\sum_{i=0}^{\infty}\frac{\Gamma(1+i)\Gamma(\frac{1}{2})}{\Gamma(1)\Gamma(\frac{1}{2}+i)!}\left\{\frac{(\gamma_1c_{1\tau}/\varphi_{1\tau}^2+1/\mu_\tau^2)^2}{2(c_{1\tau}^2/\varphi_{1\tau}^2+1/\mu_\tau^2)}\right\}^i.$$

It is seen that the derivation process applies to the pdf of every element of c_τ and the pdf of $c_{m\tau}$ can be obtained by replacing γ_1 and $\varphi_{1\tau}^2$ of (4.13) by γ_m and $\varphi_{m\tau}^2 \equiv \sigma^2 \times$ the (m, m)th element of $(Z_\tau^{*\prime}Z^*)^{-1}$, respectively.

To obtain the moment of f_6 of order ν, we must calculate

$$\int_{-\infty}^{\infty} \frac{c_{1\tau}^2(\gamma_1 c_{1\tau}/\varphi_{1\tau}^2 + 1/\mu_\nu^2)^{2i}}{(c_{1\tau}^2/\varphi_{1\tau}^2 + 1/\mu_\tau^2)^{i+1}} dc_{1\tau}. \tag{4.14}$$

Since the integral does not exist for $\nu \geqq 1$, the sampling distribution of $c_{1\tau}$ does not have any finite moments.

It can be shown[6] that the pdf f_6 itself is obtained for the variable

$$c_{1\tau} = \frac{\varphi_{1\tau}}{\mu_\nu}\left(\frac{\gamma_1/\varphi_{1\tau} + x}{1/\mu_\tau + y}\right), \tag{4.15}$$

when the random variables are distributed as

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right). \tag{4.16}$$

Marsaglia [6] considers the distribution of the variable $t = (\mu_\tau/\varphi_{1\tau})c_{1\tau}$ and gives comprehensive graphs of the distribution. Since our $c_{1\tau}$ is the multiple of the t, his graphs, except for scaling, illustrates the shape of f_6. And it is observed that the asymmetricity and bimodality of the distribution appear depending on the parameter value. Here we give Figure 3 which includes the pdf's having $\gamma_1=0$, 1, 2 and 3, respectively, with $\varphi_{1\tau}^2 = \mu_\tau^2 = 1$.

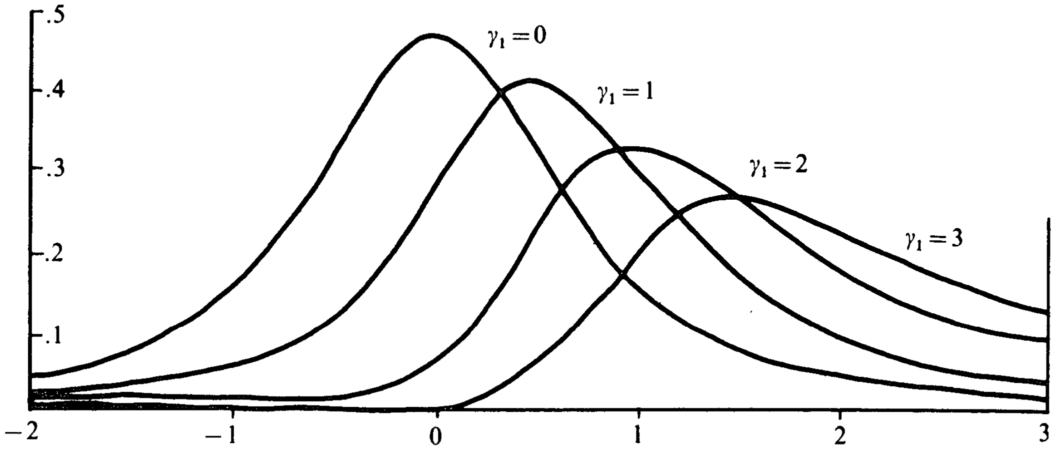

Fig. 3.

It can be shown that in the null case, $\gamma_1=0$, the pdf f_6 is symmetric about the origin and unimodal. This fact is also observed in Figure 3.

[6] This is a formulation by Marsaglia [6]. The pdf (4.13) is also obtained for $c_{1\tau} = x/y$ when $\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left(\begin{bmatrix} \gamma_1 \\ 1 \end{bmatrix}\begin{bmatrix} \varphi_{1\tau}^2 & 0 \\ 0 & \mu_\tau^2 \end{bmatrix}\right)$. See also [1], [2], [4], [5], for a discussion of a ratio of two normal variables.

V. STOCHASTIC STRUCTURE OF THE SET $\{b_1 \cdots b_N\}$

We turn to discuss the stochastic structure ruling the behavior of the set of N CDS's. It is shown that the *sample* $\{b_1 \cdots b_N\}$ is stochastically and functionally dependent drawing from varying populations.

It was shown in Section III that the b_n, $n=1, \cdots, N$, is distributed as $f_1(b_n)$, $n=1, \cdots, N$. The N distributions have the common functional form and the same parameters value β and σ^2. But the N parameters $l'(Z_n^1 Z_n)^{-1}l$ which also characterize the distributions are different with each other. Therefore the population (or distribution), from which the b_n is drawn, changes with n in the way shown in Figure 1.

It is easily expected that the b_n's are stochastically and functionally dependent. To show this, let us consider two CDS's, say b_{12345} and b_{23456}. They are caluculated from the sample at times $t=1, \cdots, 5$ and $t=2, \cdots, 6$, respectively. Since the sample points at times $t=2, \cdots, 4$ are common to the both, the b_{12345} and b_{23456} are stochastically dependent even if the $u_1, \cdots, u_6$ are independent.

The $N(>T)$ CDS's are calculated from $E_{11} \cdots E_{2T}$, which are esentially composed of $E_{21} \cdots E_{2T}$. Therefore there should be a singular transformation in the process from $E_{21} \cdots E_{2T}$ to $b_1 \cdots b_N$. In fact, there are T functionally independent CDS's, say $b_1 \cdots b_T$, and the rest, say $b_{T+1} \cdots b_N$, are functions of $b_1 \cdots b_T$.

Thus the usual random sampling theory for the sample $\{b_1 \cdots b_N\}$, like the term (b) in Section I, does not hold without further amplification.

VI. CONCLUDING REMARKS

We have obtained the sampling distributions of the Complete Determination Solution and investigated their statistical properties. The conjecture of the generalized Cauchy distribution is now verified and further implications of the conjecture are provided.

We arrived at the Geary-Fieller distribution for a ratio of two normal variables having non-zero mean values and covariance matrices, which are singular in one case and non-singular in another. While the ordinary Cauchy distribution is for a ratio of two normal variables with zero mean values.

It was shown, however, that the reasoning of the Complete Determination Method for reducing the set of the Complete Determination Solutions to a single statistic can not be assured.

Finally, it should be mentioned that the present analysis employs an overly simplified model instead of Tsujimura's original dynamic model which hypothesizes the habit formation of the consumer behavior. And it may be thought that the conjecture does not hold for this dynamic model.

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