OPTIMAL POLICY ADJUSTMENT RULES, POLICY LAGS AND THE STABILITY OF THE SYSTEM
——The Application of Classical Automatic Control Theory to Stabilization Policy——

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1 INTRODUCTION

This paper has three purposes. The first is to discuss the three different policy adjustment rules, namely proportional, derivative and integral policies, in relation to the stability of the system. The second is to discuss both the length of policy lags and the strength of policy in relation to the stability of the system. The third is to obtain the optimal policy adjustment rules to acheive the system performance.

The model analysed in this paper is the dynamic multiplier-accelerator model and we assume the case where the deviation between current and desired level of the balance of international payments—measured by the balance on current account—is continually adjusted by the government expenditure. Let the desired level of the balance of international payments be $R^*(t)$, the current level of that be $R(t)$ and (a part of) the government expenditure be $g(t)$, then three types of policy adjustment rules are formulated as follows.

proportional policy : $g(t) = a_1 [R(t) - R^*(t)]$

derivative policy : $g(t) = a_2 D [R(t) - R^*(t)]$

integral policy : $g(t) = a_3 \int_0^t [R(\tau) - R^*(\tau)]d\tau$

where D is a differentiation operator d/dt.

We further assume the existence of the policy lags, which are specified as the simple exponential lag with the time constant $1/\delta$. Therefore each policy adjustment rule is written as follows.**

$$g(t) = \frac{a_1\delta}{D+\delta}[R(t) - R^*(t)]$$

$$g(t) = \frac{a_2\delta}{D+\delta}D[R(t) - R^*(t)]$$

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** Shinkai[8] analyses the similar problem taken up in this paper. However, the model analysed in his paper is quite different from this paper. Using final equation he discusses the magnitude of the proportional policy and the policy lags in connection with the stability of the system. Only the formulation of the adjustment rule for the balance of the international payments by government expenditure used in the present paper owes to Shinkai's paper.

$$g(t) = \frac{a_3\delta}{D+\delta}\int_0^t [R(\tau) - R^*(\tau)]d\tau \, .$$

In this study we discuss the value of parameters a_1, a_2, a_3 and δ in relation to the stability of the system. These parameters a_1, a_2 and a_3 show the strength of the policy action and δ shows the policy lags.

In order to obtain the optimal policy adjustment rules, the target to be optimized must exist and further it is necessary for the transitory process to the target value to be optimal for some reasons. "For some reasons" means that the transient time is the minimum, for example, or the transient time path is steadily declining.

However it seems that so far the transient process or the transient properties have been treated as a little matter of concern in discussing the proportional, derivative and integral policy. Phillips [7] analysed the case where the deviation between the desired and the actual output level is adjusted by the government expenditure and discussed the policy that can stabilize the fluctuations of the output. But his studies did not analyse the merits and demerits of each policy adjustment rule from the point of some optimal transient properties, but relied heavily on trial-and-error procedures. Phillips stated that if we have any successful stabilisation policy it must be composed of a proper combinations of proportional, integral and derivative elements. The bases of his argument were as follows. First, the major policy is based on the strong proportional policy, thus integral policy should be used to correct the error. The deviation between desired and actual value does not disappear in proper time. Finally an element of derivative correction is necessary to overcome the oscillation which may be introduced by the former two policies. Therefore, according to Phillips, general stabilisation policies are given by

$$\alpha_p e(t) + \alpha_i \int_0^t e(\tau)d\tau + \alpha_d De(t)$$

where $e(t)$ is the input signal (i.e. the deviation between desired and actual value) or the error above mentioned.

While Phillips gave the values of policy parameters α_p, α_i and α_d by trial-and-error method, in the present paper we have tried such a proceduce that we determine the policy adjustment rules and the values of policy parmeters having optimal transient properties. Then in this paper optimal policy adjustment rules are not given a priori as the combination of three elements, but the optimal adjustment rule is determined by examining the transient properties.

The method which we used in this paper is the one that optimises the system indirectly by optimising the transient properties of the system to a step response. The method is "a classical" approach so to speak in the field of the control theory. The quadratic criterion function subject to linear constraints is minimized in modern control theory. The reason why we did not use this "modern" approach in this paper is that we thought the transient properties approach is superior to "modern" method in studying the properties of such simple model as analysed in the present paper. We shall make some reference to the criterion function approach in Section 3 where we prove the existence of the optimal control.

In Section 2 the approach from automatic control theory is stated briefly. In Section 3 the multiplier-accelerator model is presented, dynamic properties of the model are analysed and the existence of the optimal control which minimizes a quadratic criterion function is discussed. In Section 4 some concept to analyse the properties of the system are given and propertional, derivative and integral policy are discussed in turn. In Section 5 it is shown that the derivative plus proportional policy is optimal if we are planning to make the unstable system stable, to make the steady-state error zero and to obtain a fast response.

2 THE METHOD OF AUTOMATIC CONTROL THEORY

The methods used in the present paper are the one used in the automatic control theory, which is widely used in controlling the rocket, the satelite, the petroleum refining plants and the elevators, etc. The control theory which has been developed in the field of engineering is also very useful to economic analysis that treats the feedback control. This feedback control consists of the target value (input signal) and the controlled variable (output signal) and the deviation between the two signals is compared and the actual level is manipulated in order to reduce the deviation.

The approach from automatic control theory can be divided into following five steps.

1. The observation of the controlled system—the building of the system.
2. The determinants of the system performance.
3. The analysis of the system (stability, transient properties)
4. The improvement of the system
5. The acheivement of optimal system performance

The first step—the observation of the controlled system—is not any special step in control theory, however, it is necessary to pay attention to the term of system rather model. The term of the system has a teleological meaning. It means that the model to be used must acheive a certain performance.

Now even when consensus to the observation of the system is obtained, it is not easy to determine the system performance. Since we cannot regard the economic society as the simple system having a single objective, it is difficult to obtain consensus to the following matters. What are controlled targets? What manipulations (policy measures) can be used to acheive that targets? Which transient process to the target value is good? What percents of the overshoot from the target value are allowed? On the other hand in the field of engineering the matter is not serious as in case of economics. None will allow the overshoot from the target floor in the elevator control and we shall not have disagreement on the allowable range of the fluctuations of the target temperature in controlling the temperature of electric furnace.

Furthermore, there may be more than one"optimal" responses. Here we specify optimal performance for the required system as following properties.

(1) There is a tendency to converge to the target value. (The system is required to be stable.)

(2) The steady-state error stays in allowable range.

(3) The time to reach the target value is not too long.
(4) The time path to approach the target value does not have a strong oscillation.

In modern control theory the criterion function representing the system performance is minimized or maximized.

Steps 3 through 5 are explained in the following example.

3 A MULTIPLIER-ACCELERATOR MODEL

3.1 Model

The model analysed in this paper is following multiplier-accelerator model (the time argument is abbreviated for ease of notation).

$$C = c(Y - T) \qquad 0 < c < 1 \tag{3.1}$$

$$I = \left(\frac{2\mu}{D+2\mu}\right)^2 vDY \qquad \mu > 0, \quad v > 0 \tag{3.2}$$

$$G = \bar{G} \tag{3.3}$$

$$X = \bar{X} \tag{3.4}$$

$$M = mY \qquad 0 < m < 1 \tag{3.5}$$

$$T = tY \qquad 0 < t < 1 \tag{3.6}$$

$$Y = C + I + G + X - M \tag{3.7}$$

$$R = \int_0^t (X - M)d\tau \tag{3.8}$$

where C = consumption expenditures
I = investment expenditures
G = government expenditures
X = exports
M = imports
T = taxes
Y = GNP
R = foreign currency reserves (measured by cumulated balance on current account).

Capital account is ignored in the foreign currency reserves R. D in equation (3.2) represents differentiation operator. The variables $\bar{X}$ and $\bar{G}$ are exogeneous.

In following analysis we call the actual world as the world of t function and the back world represented by the Laplace transforms of t functions as the world of s function. The notation $Y(s)$ is used to denote the Laplace transforms of $Y(t)$.

The signal flow graph for the model represented by equations (3.1) through (3.8) is shown by Fig. 3–1. The symbols C, X and Y etc. in Fig. 3–1 denote the s functions of $C(t)$, $X(t)$ and $Y(t)$ respectively.

Since we are interested in the balance of international payments and we consider the adjustment of it by the government expenditures, the signal flow graph shows the input and output signal as G and R respectively. The system in which the output R has no effect on the input G is called open-loop control system.

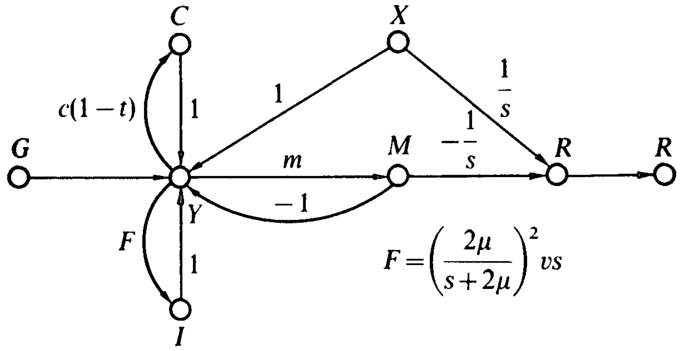

Fig. 3–1. Signal flow graph for the model of Eqs. (3.1) to (3.8)

Let us see the dynamic properties of this model. The model can be solved for the variable Y, thus we obtain the following relation between the variable Y and G.

(3.9) $$Y(s) = H(s)\,G(s)$$

where

$$H(s) = \frac{(s+2\mu)^2}{k\left\{s^2 + 4\mu\left(\frac{k-\mu v}{k}\right)s + 4\mu^2\right\}}$$

$$k = 1 - c(1-t) + m\,.$$

We shall determine the stability of the model by using Routh's criterion.* The roots of the characteristic equation are the values which make the denominator of $H(s)$ equal to zero. The Routhian array is

$$\begin{array}{ll} k & 4\mu^2 k \\ 4\mu(k-\mu v) & \\ 4\mu^2 k & \end{array}$$

Since k and μ are positive, the system is stable if $k-\mu v>0$ and unstable if $k-\mu v<0$. When the system is unstable, the real parts of the two roots are both positive. The system has the intrinsic oscillations if $2k>\mu v$ and stable (damped oscillations) if $k>\mu v$ but unstable (explosive oscillations) if $k<\mu v<2k$. When $k>\mu v$, then $2k>\mu v$ always holds, the system is stable but has no steady decline. The results obtained are shown in Fig. 3–2. The system has damped oscillation in region I ($k>\mu v$), explosive oscillation in region II ($k<\mu v<2k$) and steady growth in region III ($2k<\mu v$).

Now suppose that the time unit of the system is measured in 3-month periods and the following hypothetical structural perameters are set (Table 3–1).

The value of parameter k is supposed to be 0.46 and μ which represents the investment lags is supposed to be 0.25 (which means time constant of one year) in all cases of the structure I, II and III. The parameter δ shows the policy lags as mentioned in the introduction. These values of parameters will be used later.

Thus in all cases of the structure I, II and III the system is stable (damped oscillation) if $v<1.84$, oscillates explosively if $1.84<v<3.68$ and grows steadily if $v>3.68$.

* See Appendix A

Now we shall consider the response of Y to a unit-step input signal $G(s)=1/s$, which corresponds to a unit input $u(t)=1$ $(t>0)$ in the world of t function, that is, we shall consider the financial multiplier. Substituting $G(s)=1/s$ into equation (3.9), we have

$$Y(s)=\frac{(1/k)(s+2\mu)^2}{s(s-s_1)(s-s_2)} \tag{3.10}$$

where

$$\begin{aligned} s_{1,2} &= -\zeta\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2} \\ \omega_n &= 2\mu > 0 \\ \zeta &= 1-\mu v/k < 1 \end{aligned}$$

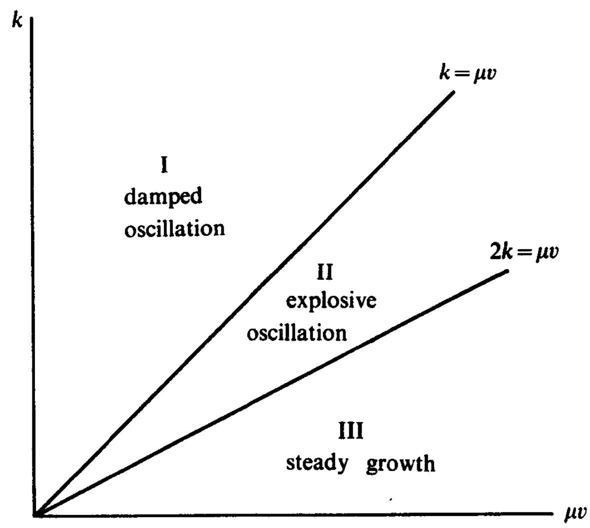

Fig. 3-2. The stability of the system

TABLE 3-1 THE HYPOTHETICAL STRUCTURAL PARAMETERS

	c	t	m	v	μ	δ	stability	policy lags	investment lags
structure I	0.8	0.2	0.1	1.6	0.25	0.5	stable	six-months	one year
structure II	0.8	0.2	0.1	1.6	0.25	0.25	stable	one year	one year
structure III	0.8	0.2	0.1	3.0	0.25	0.5	unstable	six-months	one year

and the symbol j shows the imaginary unit. Given initial conditions $Y(0) = 0$, $DY(0) = 0$, we obtain

$$(3.11) \qquad Y(t) = \frac{1}{k} - \frac{1}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \times \sin(\omega_n \sqrt{1-\zeta^2} + \cos^{-1}\zeta) .$$

If $\zeta\,\omega_n > 0$, that is, $k > \mu v$, then the system gives a damped oscillation for output $Y(t)$ and $Y(t)$ approaches the equilibrium level $1/k = 1/\{1-c(1-t)+m\}$. If $\zeta\,\omega_n < 0$, that is, $k < \mu v$, then the system gives an explosive oscillation for output $Y(t)$. The multiplier effect is equal to 2.2 for the structural parameters I and II giving stable system. Since the period of the cycle is given by $2\pi/\omega_n\sqrt{1-\zeta^2}$, it is equal to about three years for the structure I and II, about four years for the structure III.

Now let us consider the steady-state level Y^* of Y. The variables G and X are assumed to be constant in this system, so we obtain the relation $\bar{G} = [1-c(1-t)]\,Y^*$. This relation is obtained by using the equation

$$(3.12) \qquad [1 - c(1 - t) + m]\,Y^* = \bar{G} + \bar{X}$$

and using the fact that the relationship $\bar{X} = M^* = m\,Y^*$ must hold if there is no change in the foreign reserves. The relation $\bar{G} = [1 - c(1 - t)]\,Y^*$ is not a steady-state multiplier but represents the relation which should hold between given Y^* and $\bar{X}$.*

3.2 Controllability

We have discussed the properties of the model which is shown by equations (3.1) to (3.7). Next we shall consider the case where the adjustment of R to the desired foreign reserves R^* is made by the government expenditure. First we shall consider the controllability. The controllability explained here is the output controllability, which is defined as follows. A system is said to be completely output controllable if there exists an unconstrained piecewise continuous input $g(t)$ (the government expenditure in this system) which will drive the output $r(t)$ (the deviation between $R(t)$ and $R^*(t)$ in this system) from $t = t_0$ to any final output $r(t_f)$ for some finite time $t_f - t_0 \geqq 0$.

Let the constant parts of G be $\bar{G}$ and $g = G - \bar{G}$. If we let $R^*(t) = R$ (constant), $r = R - R^*$ and $X =$ constant, then from Eqs. (3.1) to (3.8), provided that all initial conditions are zero, we obtain

$$(3.12) \qquad Y(s) = \frac{(-m/k)(s + 2\mu)^2}{s\left\{s^2 + 4\mu\left(\dfrac{k - \mu v}{k}\right)s + 4\mu^2\right\}}\, g(s)$$

To describe the system by a set of dynamic equations which includes the state equations and output equations, let us define the state variables as follows.

$$(3.13) \qquad x_1(t) = r(t)$$

* See Shinkai[8], p. 18

(3.14) $$x_2(t) = \dot{r}(t) + \frac{m}{k} g(t)$$

(3.15) $$x_3(t) = \ddot{r}(t) + \frac{m}{k}\dot{g}(t) + \frac{4\mu^2 vm}{k^2} g(t)$$

Using Eqs. (3.13) to (3.15) we obtain the following state equations.

(3.16) $$\dot{x}(t) = Ax(t) + bg(t)$$

where

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_1 & -a_2 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$a_1 = 4\mu^2$$

$$a_2 = 4\mu\left(\frac{k-\mu v}{k}\right)$$

$$b_1 = -m/k$$

$$b_2 = -4\mu^2 vm/k$$

$$b_3 = 16\mu^3 vm(k-\mu v)/k^3$$

The output equation is simply

(3.17) $$r(t) = Dx(t) + Eg(t)$$

where

$$D = (1 \quad 0 \quad 0)$$
$$E = (0)$$

The necessary and sufficient condition that the system is completely output controllable is that the $p \times (n+1)m$ matrix T

$$T = [Db \;\; DAb \;\; DA^2b \cdots DA^{n-1}b \;\; E]$$

has a set of p linearly independent columns, where $p =$ number of outputs, $m =$ number of inputs and $n =$ number of state variables and matrices D, A, b and E are given by Eq. (3.16) and Eq. (3.17).

Since we have $T=(b_1\ b_2\ b_3\ 0)$ in this system, the rank of T is one and this system is completely output controllable. However even if we intend to control both R and Y by the government expenditure, we have the result rank $T=1<2$, therefore, we can similarly show that there does not exist an unconstrained piecewise continuous input $g(t)$ which will drive the output $\begin{pmatrix} R(t) \\ Y(t) \end{pmatrix}$ from $t=t_0$ to any final output $\begin{pmatrix} R(t_f) \\ Y(t_f) \end{pmatrix}$ for some finite time $t_f-t_0 \geqq 0$.

3.3 The existence of an optimal control

Now we shall show the existence and the uniqueness of an optimal control if the matrix which appears in quadratic loss function holds some certain conditions.

Suppose that the system performance is represented by the following loss function. (This assumption is used only to prove the existence of optimal control, therefore this loss function will not be used as criterion to discuss the system performance from the Section 4 on)

$$L=\frac{1}{2}\alpha r^2(t_f)+\int_{t_0}^{t_f}[\beta_1 r^2(t)+\beta_2 g^2(t)]dt \tag{3.18}$$

The first term in right-hand side in the cost function is called terminal cost, which evaluates the magnitude of the error $r(t_f)$ in terminal time t_f, and $\alpha \geqq 0$.

The meaning of the first term in the integrand is obvious and $\beta_1>0$. The second term in the integrand denotes the cost of changing the government expenditures, that is adjustment costs, and the more the costs are, the greater the loss of the system is, and $\beta_2>0$.

Using the state variables defined in Eqs. (3.14) to (3.15) the loss function (3.18) can be rewritten as follows.

$$L=\frac{1}{2}x'(t_f)F\,x(t_f)+\frac{1}{2}\int_{t_0}^{t_f}[x'(t)Qx(t)+\beta_2 g^2(t)]dt \tag{3.19}$$

where

$$F=\begin{bmatrix}\alpha & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad \text{and} \quad Q=\begin{bmatrix}\beta_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Since F and Q are positive semidefinite and $\beta_2 \neq 0$, then the optimal control uniquely exists and is given by*

$$g(t)=-\frac{1}{\beta_2}b'\,K(t)\,x(t)\,. \tag{3.20}$$

The 3×3 symmetric matrix $K(t)$ is the solution of the matrix Riccati equation

$$\dot{K}(t)=-K(t)A-A'K(t)+\frac{1}{\beta_2}K(t)bb'K(t)-Q \tag{3.21}$$

* See Athans and Falb [3], Control Law 9–1, p. 762

with the boundary condition

$$K(t) = F.^*$$

If we let $-(1/\beta_2)b'K(t) = [\theta_1(t), \theta_2(t), \theta_3(t)]$, then Eq. (3.20) can be rewritten as follows.

$$(3.22) \qquad g(t) = \theta_1(t)r(t) + \theta_2(t)[\dot{r}(t) + mg(t)/k] + \theta_3(t)[\ddot{r}(t) + m\dot{g}(t)/k + 4\mu^2 vmg(t)/k^2] .$$

The important points obtained from Eq. (3.20) which represents the existence and uniqueness of an optimal control are as follows.

(1) An optimal control is denoted by the linear function of the state variables $x(t)$.

(2) Even if both the system and the loss function are time-invariant, i.e., even if A, b, α, β_1 and β_2 are constant in time, the optimal control is time-varying as long as the control interval $[t_0, t_f]$ is finite.

It is difficult to control the time-varying system. If we let $F=0$ and $t_f \to \infty$, then we have a time-invariant optimal control for a time-invariant system and loss function.** Let us consider such a case. Given the time-invariant system, Eq. (3.16), and the loss function

$$(3.23) \qquad L_1 = \frac{1}{2}\int_0^\infty [\beta_1 r^2(t) + \beta_2 g^2(t)]dt ,$$

we have a unique optimal control, which is given by the equation.

$$(3.24) \qquad g(t) = -\frac{1}{\beta_2} b'Kx(t)$$

where K, 3×3 positive definite matrix, is the solution of the following quadratic equation.

$$(3.25) \qquad -KA - A'K + \frac{1}{\beta_2}Kbb'K - Q = 0 .$$

In this case θ_1, θ_2 and θ_3 become constant and given initial conditions $R(0) = G(0) = 0$, so we have the following optimal control.

$$(3.26) \qquad g(s) = \left(f_1 + f_2 s + \frac{f_3}{s - p_1}\right) r(s)$$

This equation (3.26) shows that the optimal control becomes a mixture of three different policy adjustment rules, that is, a mixture of proportional, derivative and integral policy.

3.4 The introduction of the adjustment policy to the balance of international payments

As mentioned in the preceding section, if the linear time-invariant system and the

* See Athans and Falb [3], p. 767

** See Athans and Falb [3], Control Law 9–2, p. 771

quadratic criterion function satisfy certain conditions, then the optimal control exists for the adjustment of $R(t)-R^*$ by $g(t)$. However even if the structural parameters are known the values of policy parameters f_1, f_2, f_3 and p_1 in Eq. (3.26) are not obtained unless the parameters β_1 and β_2 of loss function are known.

Therefore we shall adopt the transient properties approach rather than the criterion function approach in the following analysis. First we shall discuss three different policy adjustment rules in turn.

A Proportional policy

Suppose that the proportional policy is introduced to adjust $R(t)-R^*$ by $g(t)$ and policy lags is represented by simple exponential lag with time-constant $1/\delta$. Then the system is written by

(3.27) $C = c(Y - T) \qquad 0<c<1$

(3.28) $I = \left(\dfrac{2\mu}{D+2\mu}\right)^2 vDY \qquad \mu>0,\ v>0$

(3.29) $G = \bar{G} + g$

(3.30) $g = \dfrac{\delta}{D+\delta} a_1(R - R^*) \qquad \delta>0,\ a_1>0$

(3.31) $R = \int_0^t (X - M)d\tau$

(3.32) $X = \bar{X}$

(3.33) $M = mY \qquad 0<m<1$

(3.34) $T = tY \qquad 0<t<1$

(3.35) $Y = C + I + G + X - M$

The signal flow graph of this system is shown in Fig. 3–3. In Fig. 3–3, E is given by the following equation

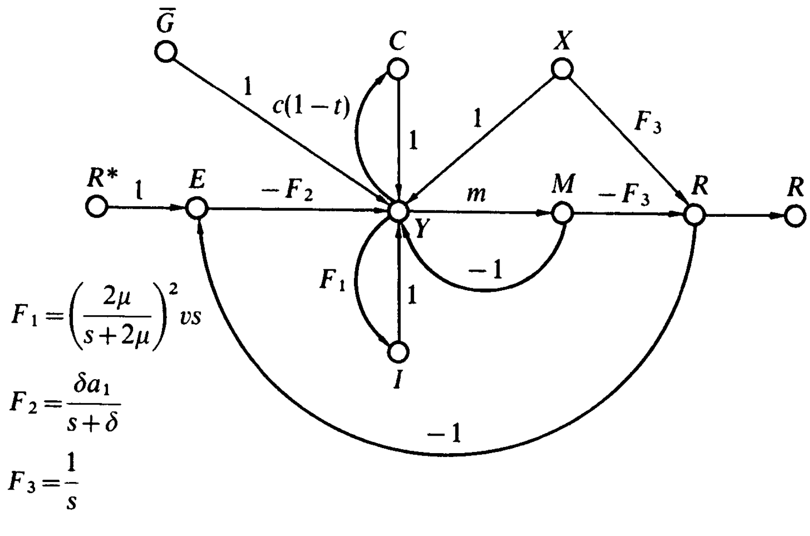

Fig. 3–3. Signal flow graph of Eqs. (3.27) to (3.35)

(3.35) $$E(s) = R^*(s) - R(s)$$

and this value operates as input. The system as shown in Fig. 3–3, i.e., the system which the controlled signal $R(t)$ effects the actuating signal $E(t)$ in order to maintain the output $R(t)$ at the desired level R^*, is called a closed system. Thus, the output $R(t)$ is controlled so as to approach the desired value in the system with the feedback path.

Using Mason's rule* the following equation is obtained from Fig. 3–3.

(3.37) $$M(s) = \frac{R(s)}{R^*(s)} = \frac{mF_2 F_3}{1 - c(1-t) - F_1 + m + mF_2 F_3}$$

or

(3.38) $$M(s) = \frac{(m\delta a_1/k)(s+2\mu)^2}{s(s+\delta)\left\{s^2 + 4\mu\left(\frac{k-\mu v}{k}\right)s + 4\mu^2\right\}}$$

Let

(3.39) $$G_1(s) = \frac{K_1(s+2\mu)^2}{s(s+\delta)\left\{s^2 + 4\mu\left(\frac{k-\mu v}{k}\right)s + 4\mu^2\right\}}$$

then (3.38) can be written as

(3.40) $$M(s) = \frac{G_1(s)}{1 + G_1(s)}$$

where

(3.41) $$K_1 = m\delta a_1/k\ .$$

Thus the block diagram of this feedback control system can be denoted as Fig. 3–4. Let us define some important terminology used in the block diagrams with reference to Fig. 3–4.

$R^*(s)$ = input (target value or final value)
$R(s)$ = output (controlled variable)
$B(s)$ = $R(s)$ = feedback signal
$E(s)$ = actuating input
$G_1(s)$ = $R(s)/E(s)$ = forward transfer function or open loop transfer function
$M(s)$ = $R(s)/R^*(s)$ = closed loop transfer function.

Eq. (3.40) is obtained by using these relations. The closed loop transfer function $M(s)$ is common to the three different policy adjustment rules except the forward trânsfer function G_1. The forward transfer function of the proportional policy is given by Eq. (3.39).

* See D'Azzo J. J. and C. H. Houpis [4], p. 165

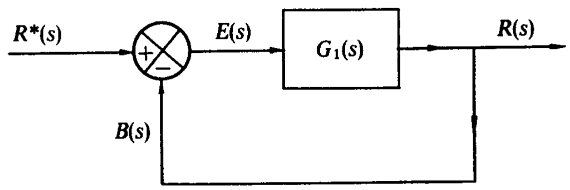

Fig. 3–4. Block diagram representation of Eqs. (3.27) to (3.35)

B Derivative policy

Derivative policy can be formulated by

$$g = \frac{\delta}{D+\delta} a_2 D(R - R^*) . \tag{3.42}$$

Then in the signal flow graph shown in Fig. 3–3 all but F_2 is common to the proportional policy. Since F_2 for the derivative policy is given by

$$F_2 = \frac{\delta a_2 s}{s+\delta} , \tag{3.43}$$

the forward transfer function becomes as follows.

$$G_2(s) = \frac{K_0(s+2\mu)^2}{(s+\delta)\left\{s^2 + 4\mu\left(\frac{k-\mu v}{k}\right)s + 4\mu^2\right\}} \tag{3.44}$$

where K_0 is given by

$$K_0 = m\delta a_2/k . \tag{3.45}$$

C Integral policy

Integral policy is of the form

$$g = \frac{\delta a_3}{D+\delta} \int_0^t (R - R^*) d\tau \tag{3.46}$$

Then substituting F_2 given by Eq. (3.47) for F_2 in Fig. 3–3 we obtain the forward transfer function G_3 of the integral policy.

$$F_2 = \frac{\delta a_3}{s(s+\delta)} \tag{3.47}$$

$$G_3(s) = \frac{K_2(s+2\mu)^2}{s^2(s+\delta)\left\{s^2 + 4\mu\left(\frac{k-\mu v}{k}\right)s + 4\mu^2\right\}} \tag{3.48}$$

where K_2 is given by

$$K_2 = m\delta a_3/k . \tag{3.49}$$

If we substitute each forward transfer function G_i ($i=1, 2, 3$) into Eq. (3.40), then the closed-loop transfer function of each policy can be obtained. Thus we shall investigate the properties of this closed-loop transfer functions to know the feedback effect of each policy.

By introducing the adjustment policy as mentioned above for the balance of the international payments, the output $R(s)$ has a feedback effect. This feedback effect would not only have an effect of reducing the deviation E between R and R^*, but also have an effect on the properties of the system, such as the stability and the transient responses.

It may occur that the policy introduced to adjust the balance of international payments cannot acheive the original objective, i.e., the policy cannot make the error zero. Furthermore, the strength of the policy action and/or the length of the policy lags may make the stable system unstable or, on the contrary, may have an effect of making the unstable system stable.

And the transient process to the target value, i.e., the time path to the target value depends on the policy adjustment rules and the values of policy parameters.

The purpose of the following section is to discuss these matters.

4 THREE POLICY ADJUSTMENT RULES

It can be seen from the forward transfer function given by Eqs. (3.44), (3.39) and (3.48) that the derivative, proportional and integral policy give the system order 0, 1 and 2 respectively. (See Appendix C) Since, in general, the higher the system order, the more the system tends to become unstable, we see that the derivative policy is the best policy of three policies in respect to the stability of the system analysed in this paper. But three policies are discussed in turn in this section.

4.1 Proportional policy — for the stable system

First we shall discuss the proportional policy in connection with the stability of the system, namely, given structural parameters c, v, m, t, μ, and δ, we shall consider the stability of the closed-loop system as a function of the policy parmeter a_1, which shows the strength of the proportional policy action. The root-locus method* is useful to analyse such a case.

We shall first discuss whether the stable system becomes unstable by the magnitude of the value of the policy parameter a_1. Therefore we shall use the structure I in Table 3–1.

Given the parameters in the structure I, if we substitute these values into Eq. (3.39), then we obtain the following forward transfer function.

$$G_1(s) = \frac{K_1(s+0.5)}{s(s+0.065-j0.495)(s+0.065+j0.495)} \tag{4.1}$$

The characteristic equation of the closed-loop system, namely, $1+G_1(s)=0$, has three roots. The loci of these three roots when K_1 varies from 0 to ∞ are given in Fig.

* See Appendix B

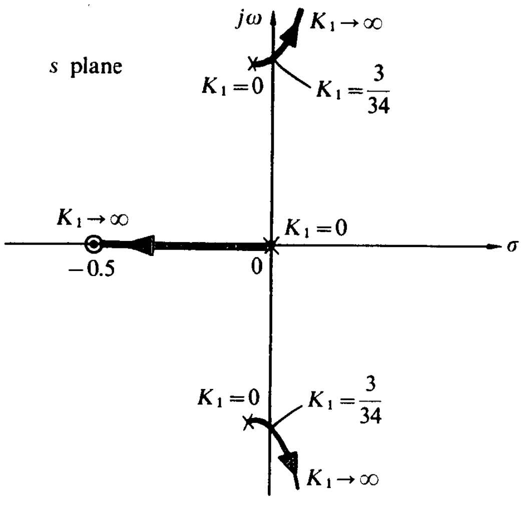

Fig. 4–1. Root locus for Eq. (4–1)

4–1. In Fig. 4–1 real component is represented in the horizontal axis, and imaginary component is measured on the vertical axis in the complex s-plane. Zero, the root which makes the numerator of G_1 equal to zero, is represented by the notation $\odot$ and poles, the roots which make the denominator of G_1 zero, are represented by the notation $\times$ in root locus diagrams.

Since $K_1 = m\delta a_1/k$ and m, δ and k are given, the variation of K_1 equals to that of a_1. The bold solid lines in Fig. 4–1 denote all possible roots of the characteristic equation of this closed-loop system for all values of K_1, therefore for all values of a_1, from zero to infinity. The necessary and sufficient condition for the system to be stable is that the real component of all roots of the characteristic equation is negative, that is, all roots of the equation $1 + G_1(s) = 0$, i.e., poles, are in the left-half s plane. However we see from Fig. 4–1 that there may be two imaginary roots with positive real component. Thus the system may well be unstable. Using the Routhian array for the characteristic equation $1 + G_1(s) = 0$, we can see what value of K_1 makes the system unstable.

The Routhian array for the denominator of $M(s)$, which is equal to the characteristic equation of the system, is

$$
\begin{array}{lll}
s^3 & 1 & 0.25 + K_1 \\
s^2 & 0.13 & 0.5K_1 \\
 & 1 & 3.846K_1 \\
s^1 & \dfrac{1}{4} - \dfrac{17}{6}K_1 & \\
s^0 & 3.846K_1 &
\end{array}
$$

Therefore, when $1/4 - (17/6)K_1 < 0$, i.e., $K_1 > 3/34$, the system becomes unstable. $K_1 > 3/34$ means $a_1 > 0.81$, because we are discussing the parameters of the structure I.

Similarly the forward transfer function for a set of parameters of the structure II

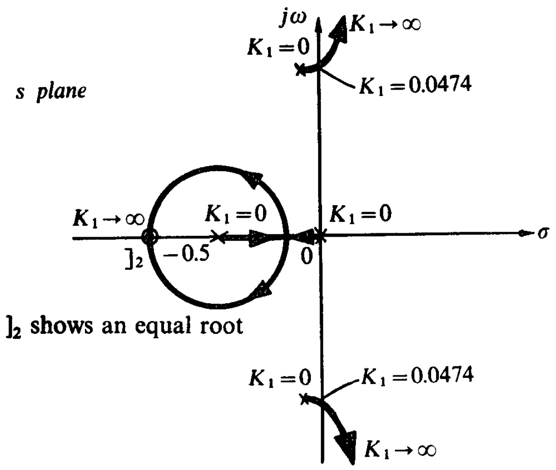

Fig. 4–2. Root locus for Eq. (4.2)

which gives a stable system is

$$G_1(s) = \frac{K_1(s+0.5)^2}{s(s+0.25)(s+0.065-j0.495)(s+0.065+j0.495)} \tag{4.2}$$

The root locus for the equation $1+G_1(s)=0$ is given by the bold solid line in Fig. 4–2. In this case for values of $K_1 > 0.0474$, i.e., $a_1 > 0.87$ two of the four roots lie in the right half of the s plane, resulting in an unstable system.

Now we shall consider the case where all parameters except δ equal to that of the structure I (or structure II) in Table 3–1 and δ equals to unity. These parameters also give the stable system and it is the case where policy lag is short, that is, one quarter. Then $G_1(s)$ is

$$G_1(s) = \frac{K_1(s+0.5)^2}{s(s+1)(s+0.065-j0.495)(s+0.065+j0.495)} \tag{4.3}$$

The root locus for Eq. (4.3) is shown in Fig. 4–3. Interestingly the system does not become unstable whatever the value of K_1, that is, a_1 is.

Three cases discussed so far, that is the case where δ equals to 0.25, 0.5 and 1, and a_1 equals to 0.85, are summarized in Table 4–1.

We can summarize the results obtained so far as follows.

[1] The stable system may become unstable by strong proportional policy (for example, consider the case where a_1 equals to unity in the structure I or II).

[2] The stable system may become unstable unless the strength of the proportional policy becomes weaker as a policy lag becomes shorter (for example, if the proportional policy with fixed value of $a_1 = 0.85$ is introduced, the system is stable for one year policy lag ($\delta = 0.25$) but the system becomes unstable for a half year policy lag ($\delta = 0.5$)).

This second conclusion, however, exceedingly depends on the value of δ, so we cannot generalize this result.

Now let us consider the steady-state error.

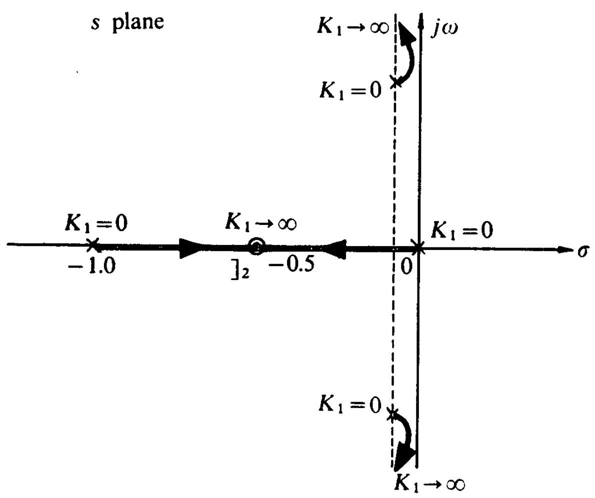

Fig. 4–3. Root locus for Eq. (4.3)

TABLE 4–1 THE STABILITY OF THE SYSTEM FOR THREE POSSIBLE VALUES OF δ AND $a_1 = 0.85$

δ	a_1	system	unstable region
0.25	0.85	stable	$a_1 > 0.87$
0.5	0.85	unstable	$a_1 > 0.81$
1.0	0.85	stable	—

The error is

$$e(t) = R^* - R(t)$$

and the steady-state error $e(t)_{ss}$ is defined by

$$e(t)_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\,E(s)\,. \tag{4.4}$$

Since the forward transfer function for the proportional policy is a Type 1 system, there is zero steady-state error for a unit-step input signal $R^*(t) = R^*u(t)$, where R^* is constant and

$$u(t) = \begin{cases} 1 & \text{if } t > 0^* \\ 0 & \text{if } t \leqq 0 \end{cases}$$

It also can be stated that the proportional policy which gives a Type 1 system produces a constant output of value R identical with the constant input R^* (of cource this is the case where the system is stable). This means that in steady-state we

* See Apendix C

have $X = mY^*$, where Y^* is the steady-state level of Y, that is, exports coincides with imports and the adjustment of the balance of international payments is completely finished and the variation of the foreign reserves does not occur.

4.2 Proportional policy — for the unstable system

Next, let us consider whether the proportional policy makes the unstable system stable. We shall use the values of the parameters of the structure III in Table 3–1 as an example. By substituting these values of parameters into Eq. (3.39), we obtain

$$G(s) = \frac{K_1(s+0.5)}{s(s-0.315-j0.388)(s-0.315+j0.388)} \tag{4.5}$$

The root locus for the forward transfer function given in Eq. (4.5) is shown in Fig. 4–4. We see from this root locus that the proportional policy does not make the unstable system stable.

4.3 Proportional policy — policy lags and the strength of the action

So far we have considered the case where the roots of the characteristic equation of the closed-loop system are represented as the function of the policy parmeter a_1 given the value of δ, which denotes policy lags. Here we consider both δ and a_1 to see the effect of both policy lags and the strength of the action on the stability of the system. Thus we consider the case where the values of δ and a_1 are not specified but all other parameters are given the following values, $c=0.8$, $v=1.6$, $m=0.1$, $t=0.2$, $\mu=0.25$. The characteristic equation $1+G_1(s)=0$ for these values is as follows.

$$\begin{aligned} &s^4 + (0.1304 + \delta)s^3 + (0.25 + 0.1304\delta + 0.2174\delta a_1)s^2 \\ &\quad + \delta(0.25+0.2174a_1)s + 0.05435\delta a_1 = 0\,. \end{aligned} \tag{4.6}$$

Considering the Routhian array for this equation we see that if $0<a_1<0.4$, whatever the value of $\delta(>0)$ is, then the system is stable. Therefore the strength of

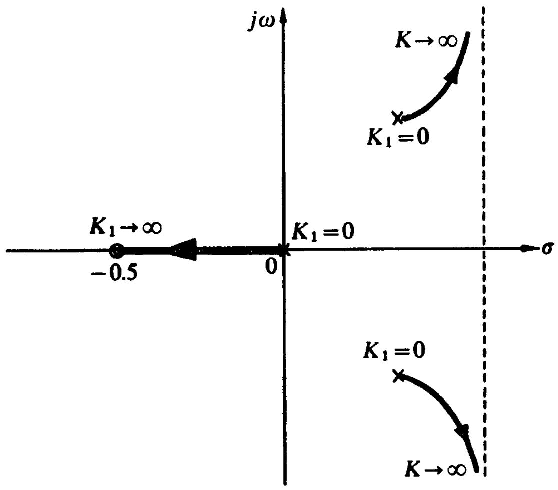

Fig. 4–4. Root locus for Eq. (4.5)

the policy action has a limit to keep the system stable and a strong action may make the system unstable. Thus if we have no information about the policy lags, strong proportional policy should not be introduced.

4.4 Derivative policy — the strength of the action and the stability

We mentioned at the beginning of this section that since the derivative policy gives the Type 0 system, it is the most favorable policy for stabilizing the system. Let us assure this fact by considering both the stable and the unstable system.

The derivative policy for the stable system, i.e. the structure I, gives the forward transfer function

$$G_2(s) = \frac{K_0(s+0.5)}{(s+0.065-j0.495)(s+0.065+j0.495)} \tag{4.7}$$

The root locus for Eq. (4.7) is shown in Fig. 4–5, it can be seen from this figure that the stable system does not become unstable. This conclusion is always true in the structure I, whatever the value of δ takes. The Routhian array made by the following characteristic equation can make this clear. The characteristic equation in the case where the values of δ and a_2 are not specified is

$$\begin{aligned} s^3 &+ [0.1304 + 0.2174a_2)]s^2 \\ &+ [0.25 + \delta(0.1304 + 0.2174a_2)]s + \delta(0.25 + 0.05435a_2) \end{aligned} \tag{4.8}$$

Next let us consider whether the unstable system may become stable or not by derivative policy. The forward transfer function for the structure III is

$$G_2(s) = \frac{K_0(s+0.5)}{(s-0.315+j0.388)(s-0.315-j0.388)} \tag{4.9}$$

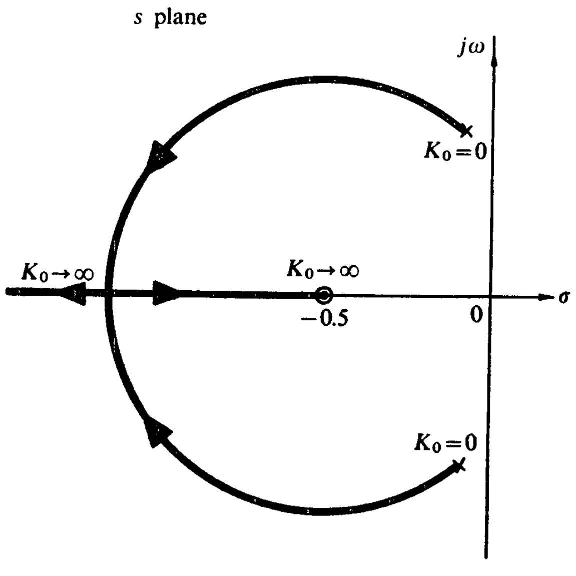

Fig. 4–5. Root locus for Eq. (4.7)

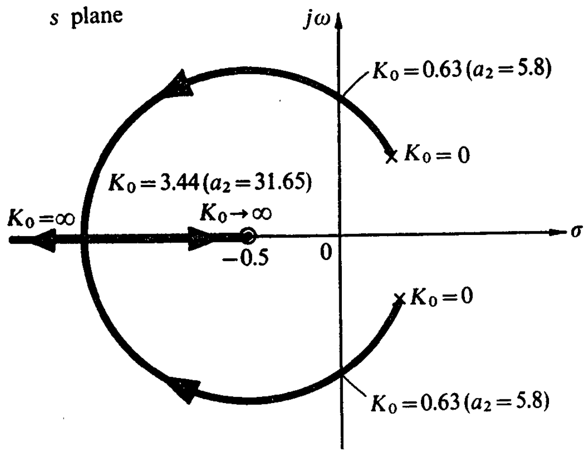

Fig. 4–6. Root locus for Eq. (4.9)

and the root locus for Eq. (4.9) is shown in Fig. 4–6. We can see from the figure that the unstable system may become stable, in this example the system becomes stable if $a_2 > 5.8$.

The derivative policy has such a desirable properties from the stability point of view, but the transfer function $G_2(s)$ given in Eq. (3.44) is a Type 0 system, so the steady-state error remains. The steady-state error for a unit-step input R^* is given by*

$$(4.10) \qquad e(t)_{ss} = \frac{R^*}{1 + ma_2/k} = \frac{R^*}{1 + a_2 \dfrac{m}{1 - c(1-t) + m}} = \frac{R^*}{1 + 0.2174a_2}$$

Then the steady-state error will be the smaller:

1. The larger the value of a_2.
2. The larger the multiplier effect.
3. The larger the propensity to import.

In the numerical example of the structure III, $a_2 \geqq 41.4$ must be satisfied to make the steady-state error less than or equal to 10 per cent of R^*.

The remaining of the steady-state error in the derivative policy means that the adjustment of R to R^* is not completely finished and this can be seen from the fact that in steady state, we have

$$(4.11) \qquad k/a_2 = X - mY^*$$

and $X \neq mY^*$. We can see from Eq. (4.11) that the larger the value of a_2 is, we shall have $X \approx mY^*$.

* See Appendix C

The remaining of the steady-state error, however, may not be quite fatal to the system. Since we are not controlling the rocket or the elevator but the economic system, it is all right that the system has a tendency to converge to the target value rather than the zero steady-state error, provided that the deviation between the target and the actual value does not exceed the allowable range (for example 5% or 10%).

Suppose the case where using the derivative policy we can make the unstable system stable and the steady-state error of about 10 per cent of R^* is allowable. Then the next problem is the time path of the adjustment. The policy which causes such results that the path approaching the target value has a violent oscillations or the settling time is too long will not be considered to be good.

Suppose that the derivative policy is introduced to the unstable system (i.e., the structure III) and the government expenditure making the steady-state error less than 10%, that is, $a_2 = 41.4$ ($K_0 = 4.5$) is spended. Then the characteristic equation of the closed-loop system is

$$s^2 + 3.87s + 2.5 = (s + 0.82)(s + 3.05) \tag{4.12}$$

therefore the closed-loop transfer function $M(s) = R(s)/R^*(s)$ is

$$\frac{R(s)}{R^*(s)} = \frac{4.5(s+0.5)}{(s+0.82)(s+3.05)}. \tag{4.13}$$

Thus for the unit-step input $R^*(s) = 1/s$, $R(s)$ is as follows.

$$\begin{aligned} R(s) &= \frac{4.5(s+0.5)}{s(s+0.82)(s+3.05)} \\ &= \frac{0.9}{s} + \frac{0.79}{s+0.82} - \frac{1.69}{s+3.05} \end{aligned} \tag{4.14}$$

Finally we have the following output equation using the inverse Laplace transformation.

$$R(t) = 0.9 + 0.79\, e^{-0.82t} - 1.69\, e^{-3.05t} \qquad t \geqq 0 \tag{4.15}$$

The time path obtained from Eq. (4.15) is shown in Fig. 4–7. $R(t)$ does not reach a prescribed percentage of the final value, that is 0.9, until $t = 0.34$, and reach the maximum overshoot when $t = 0.93$, and the magnitude of this maximum overshoot is about 17 per cent of the final value. $R(t) \to 0.9$ if $t \to \infty$.

Let us summarize the main results on the derivative policy obtained so far.

[1] The derivative policy gives a Type 0 system in this model.

[2] The derivative policy can make the unstable system stable. In the numerical example of the structure III (marginal propensity to consume = 0.8, marginal propensity to import = 0.1, marginal tax rate = 0.2, accelerator coefficient = 3, investment lags = one year, policy lags = a half year), the system is unstable but the derivative policy to the balance of international payments makes this system stable.

[3] The steady-state error remains, i.e. the actual output R does not converge to the desired level R^* even if the target value R^* is constant over time. It is because of Type 0 system. For example, if a_2 equals 6, then the unstable system becomes stable

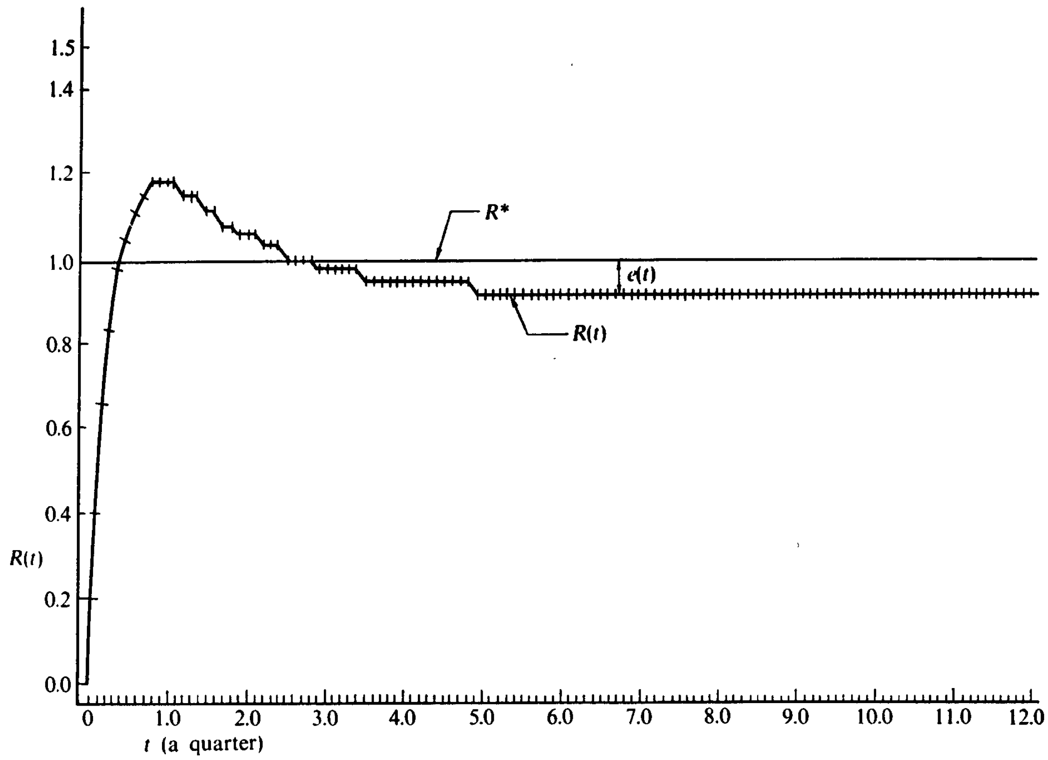

Fig. 4–7. Time path of Eq. (4.15), the derivative policy ($a_2 = 41.4$)

but the steady-state error remains more than 40 per cent of the target value and the oscillation does not disappear. The value of a_2 must be greater than 41.4 to keep the steady-state error remain within ten per cent of the target value. When $a_2 = 41.4$ the unstable system becomes stable, the steady-state error remains about 10 per cent and the explosive oscillation disappears.

[4] The time response of the system is very fast. The time path of $R(t)$ is as follows. Time required for $R(t)$ to reach 90 per cent of the target value is about one month and after about one and a half months $R(t)$ first reaches the target value. The first overshoot occurs after about one and a half quarters and its magnitude is about 17 per cent of the target value. After the first overshoot, $R(t)$ declines and reaches the target value again after about two and a half quarters and thereafter converges 90 per cent level of the target value.

4.5 Derivative policy and policy lags

We shall consider how policy lags affect the stability of the system in the derivative policy. Let us first consider the stable system which all parameters but δ are given by the structure I. And the case where the derivative policy parameter a_2 is fixed at 41.4, which makes the steady-state error 10 per cent of the final value. Since the characteristic equation of the closed-loop system is given by

$$(s + \delta)\left\{s^2 + 4\mu\left(\frac{k - \mu v}{k}\right)s + 4\mu^2\right\} + (m\delta a_2/k)(s + 2\mu)^2 = 0 \ ,$$

dividing both sides of above equation by the terms which do not contain δ gives

$$1+\frac{\delta\left\{\left(1+\frac{ma_2}{k}\right)s^2+\frac{4\mu}{k}(k-\mu v+ma_2)s+4\mu^2\left(1+\frac{ma_2}{k}\right)\right\}}{s\left\{s^2+4\mu\left(\frac{k-\mu v}{k}\right)s+4\mu^2\right\}}=0$$

Therefore we let

$$H_2(s)=\frac{\delta\left(1+\frac{ma_2}{k}\right)\left(s^2+\frac{4\mu(k-\mu v+ma_2)}{k+ma_2}s+4\mu^2\right)}{s\left\{s^2+4\mu\left(\frac{k-\mu v}{k}\right)s+4\mu^2\right\}}$$

then the characteristic equation of the system can also be shown by $1+H_2(s)=0$. Substituting the parameters but δ of the structure I into $H_2(s)$ we have

$$H_2(s)=\frac{10\delta(s+0.4565-j0.204)(s+0.4565+j0.204)}{s(s+0.065-j0.495)(s+0.065+j0.495)}\ . \tag{4.16}$$

Therefore the roots of the characteristic equation of the system can be shown by the root locus as the function of δ. This is shown in Fig. 4–8. It can be seen from this figure that the derivative policy which makes the steady-state error 10 per cent of the final value does not make the system unstable whatever the length of policy lags.

Now let us consider the unstable system which all parameters but δ are given by the structure III and the case where the value of a_2 is 41.4, which makes the unstable system stable and the steady-state error 10 per cent. This case can be discussed in the same way as before. The root locus for

$$\frac{10\delta(s+0.4185-j0.2736)(s+0.4185+j0.2736)}{s(s-0.315-j0.388)(s-0.315+j0.388)} \tag{4.17}$$

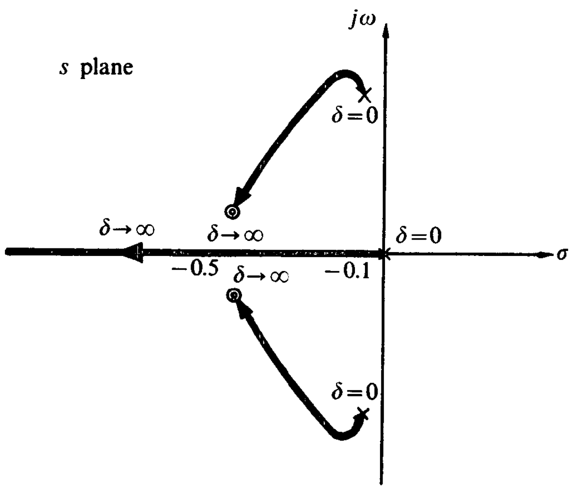

Fig. 4–8. Root locus for Eq. (4.16)

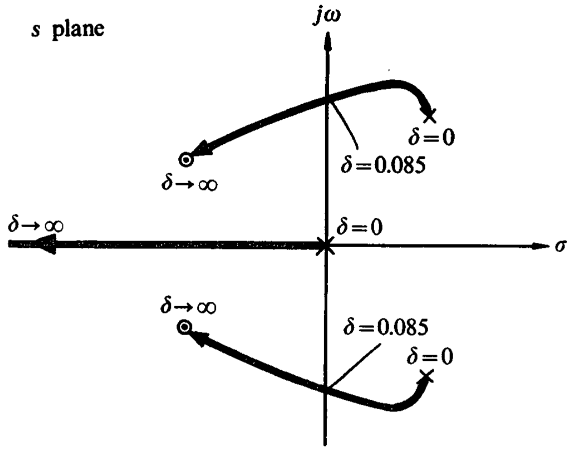

Fig. 4–9. Root locus for Eq. (4.17)

is shown in Fig. 4–9 and this gives the roots of the characteristic equation of the system.

We can see from Fig. 4–9 that the value of $a_2=41.4$ makes the unstable system stable and makes the steady-state error 10 per cent if $\delta>0.085$, that is, if the policy lags are less than about three years. But the system remains unstable if the policy lags are longer than three years.

The results obtained so far show that the length of the policy lags in the derivative policy does not have much effect on the stability of the system.

4.6 Integral policy

As we noted at the beginning of this section, the integral policy gives the Type 2 system. Paying attention to this fact it will be possible to understand the following results.

[1] There is a very strong possibility that the integral policy makes the stable system unstable. (The range of a_3 which keeps the system stable is $0<a_3<0.22$ for the structure I and $0<a_3<0.54$ for the structure II).

[2] It is impossible for the integral policy to make the unstable system stable.

[3] If the policy lags becomes much longer, the range of the policy parameter a_3 which keeps the system stable becomes wider.

Let us summarize the results obtained so far on three policy adjustment rules.

[1] If the system is stable and the length of the policy lags is given, then the integral policy has a strong possibility to make the system unstable. The proportional policy makes the system unstable when the value of the policy parameter exceeds a certain limit but on the other hand there are no cases where the derivative policy makes the stable system unstable. (Recall the system type of each policy).

[2] For a unit-step input R^*, the steady-state error is zero for both the proportional and the integral policy but it remains for the derivative policy, thus the adjustment is not finished in the derivative policy. Strong action is necessary to make the steady-state error small in the derivative policy. (These results also have relationship with the system order).

[3] If the system is unstable and the length of the policy lags are given, then both the proportional and the integral policy do not have an effect of making the system stable. But the derivative policy has an effect of making the system stable when the policy parameter exceeds a certain value. Strong derivative policy makes the steady-state error small and further lets the intrinsic oscillations of the system disappear.

4.7 Policy lags

The relation between policy lags and the stability of the system has been discussed on various occasions. But let us give some supplementary explanation here.

The policy lags play a remarkably important role in the proportional policy. There occurs a possibility that the stable system becomes unstable because of the existence of the policy lags. Let us see this fact. The forward transfer function of the proportional policy without policy lags is give by

$$G_1(s) = \frac{K'_1(s+2\mu)^2}{s\left\{s^2 + 4\mu\left(\frac{k-\mu v}{k}\right)s + 4\mu^2\right\}} \tag{4.18}$$

where

$$K'_1 = ma_1/k\ . \tag{4.19}$$

Therefore the root locus for Eq. (4.18) is shown in Fig. 4–10. It can be seen from Fig. 4–10 that if there are no policy lags, then the stable system does not become unstable by any proportional policy. We have discussed the case where there was a half year policy lag. Thus the unstable effect according to the existence of the policy lags is quite obvious by comparing Fig. 4–10 with Fig. 4–1, which shows the root locus for the proportional policy with a half year policy lag.

Furthermore if there are no policy lags in the proportional policy, then the unstable system may become stable. This is shown in Fig. 4–11, which gives the root locus for the proportional policy with no policy lags and parameters of structure III bringing unstable system.

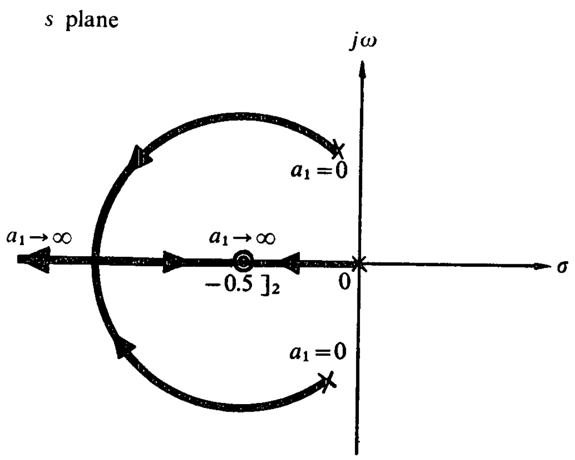

Fig. 4–10. Root locus for Eq.(4.18), the proportional policy (the structure I without policy lag)

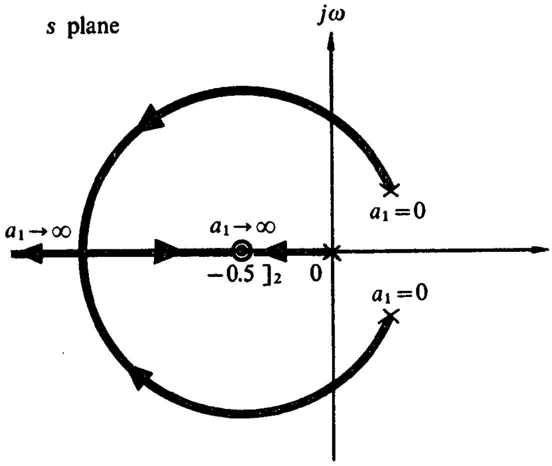

Fig. 4 11. Root locus for Eq. (4.18), proportional policy (the structure III without policy lags)

If the forward transfer function of the proportional policy without policy lags, given in Eq. (4.18), is compared with that of the original system with a half year policy lag, given in Eq. (3.39), then it is seen that policy lags add a pole to the transfer function shown in Eq. (4.18). The addition of a pole, in general, has the effect of pulling the locus to the right,* therefore yielding a less stable system. Comparing Fig. 4–4 with Fig. 4–11 clarifies this fact.

The effects of policy lags on the derivative and the integral policy can be analysed similarly. The results show that the existence of policy lags does not play an important role. The policy lags have a little effect on the stability of the system in the derivative and the integral policy.

5 OPTIMAL POLICY

As we discussed in the previous section, if there are no policy lags, then the proportional policy has the effect of stabilizing the unstable system and further makes the steady-state error zero. Therefore the main policy is based on the proportional policy and the policy having a desirable transient properties (mainly consisting of the magnitude of a peak overshoot and the time required for output $R(t)$ first to reach and thereafter remain within allowable range—that is, settling time) should be added to the main policy. However we cannot ignore the policy lags,** therefore the existence of the policy lags is supposed as before. Then the proportional policy is not necessarily main policy in this case.

Furthermore the choice of optimal policy adjustment rules crucially depends on whether the system is stable or not. If the system is stable, then we may obtain the optimal adjustment rules which are mainly based on the proportional policy. However, here we suppose that the system is unstable,*** and then consider the choice of optimal adjustment rules under the assumption of the structure III.

* See Appendix B
** See Shinkai[8]
*** Almost all Japanese macro-model have the explosive oscillation.

The unstable system can be stabilized by the derivative policy. As we have discussed in the Section 4.4, the transient properties of the derivative policy with the value of policy parameter $a_2 = 41.4$ is satisfactory except the remaining of the ten per cent's steady-state error. Therefore it will be natural to consider such a derivative plus proportional policy that the proportional policy which makes the steady-state error zero is added to the derivative policy. Thus we shall consider the following policy adjustment rules.

(5.1) $$g = \frac{\delta}{D+\delta}(b_1 + b_2 D)(R - R^*)$$

Substituting $F_2 = \delta(b_1 + b_2 s)/(s+\delta)$ for F_2 in Eq. (3.37) we obtain the following forward transfer function for this combined policy.

(5.2) $$G_0(s) = \frac{T_1\left(s + \frac{b_1}{b_2}\right)(s + 2\mu)^2}{s(s+\delta)\left\{s^2 + 4\mu\left(\frac{k - \mu v}{k}\right)s + 4\mu^2\right\}}$$

where

(5.3) $$T_1 = m\delta b_2/k\ .$$

The forward transfer function given by Eq. (5.2) has a Type 1 system, which should be compared with the derivative policy with the system type 0. Thus the steady-state error becomes zero for the unit step function $R^*(t) = R^*$ $(t \geqq 0)$ in this derivative plus proportional policy.

This new trasfer function $G_0(s)$ is obtained by adding the compensator transfer function

(5.3) $$G_c(s) = \frac{b_1 + b_2 s}{a_2 s} = \frac{b_2}{a_2}\cdot\frac{s + (b_1/b_2)}{s}$$

to the following original forward transfer function (i.e., the forward transfer function for the derivative policy)

(5.4) $$G_2(s) = \frac{(m\delta a_2/k)(s + 2\mu)^2}{(s+\delta)\left\{s^2 + 4\mu\left(\frac{k - \mu v}{k}\right)s + 4\mu^2\right\}}\ .$$

That is, adding $G_c(s)$ to the closed loop we obtain

(5.6) $$G_0(s) = G_2(s)\cdot G_c(s)$$

This is shown in Fig. 5–1.

The system having the forward transfer function $G_2(s)$ and the policy parameter $a_2 = 41.4$ was stable and its transient response was satisfactory as discussed before except the remaining of the steady-state error. Therefore if we let $b_1/b_2 \approx 0$, that is, we make the value of b_2, the parameter of the derivative policy, larger than the value of b_1, the parameter of the proportional policy, then the effect of adding a pole and a

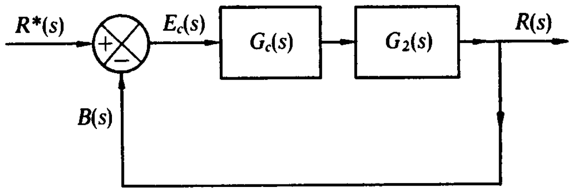

Fig. 5–1. Block diagram for the derivative plus the proportional policy

zero contained in $G_c(s)$ to the closed loop system is canceled out and the root locus for the derivative policy actually remains unchanged.

Suppose the case where a ratio $b_1/b_2 = 0.01$ is selected. Then the forward transfer function for the parameters of the structure III is

$$G_0(s) = \frac{T_1(s + 0.01)(s + 0.5)}{s(s^2 - 0.6304s + 0.25)}\,. \tag{5.7}$$

The root locus for Eq. (5.7) is shown in Fig. 5–2. It will be seen that Fig. 5–2 and Fig. 4–6 are almost the same.

Let the value of b_2 be 41.4, which makes the steady-state error ten per cent of the target value in the derivative policy. This value was such that made the unstable system stable and damped the intrinsic oscillation of the system. Since b_1/b_2 is supposed to have a value 0.01, $b_2 = 41.4$ means $b_1 = 0.414$.

Thus, if $b_1 = 0.414$, $b_2 = 41.4$, then we have the following closed-loop transfer function.

$$\frac{R(s)}{R^*(s)} = \frac{4.5(s + 0.01)(s + 0.5)}{(s + 0.009)(s + 3.0336)(s + 0.8274)} \tag{5.8}$$

Therefore, for a unit-step input $R^*(s) = 1/s$, output is

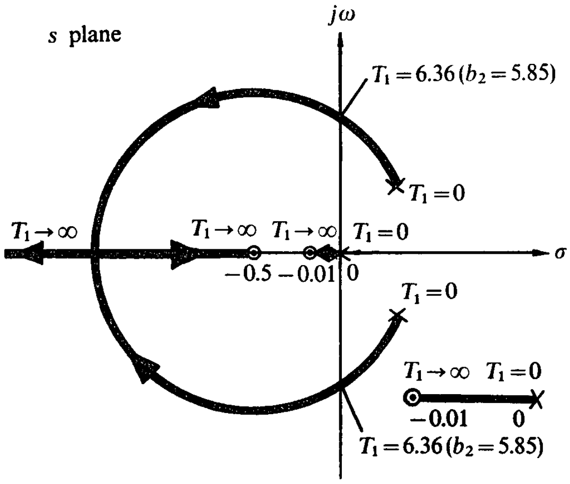

Fig. 5–2. Root locus for Eq. (5.7)

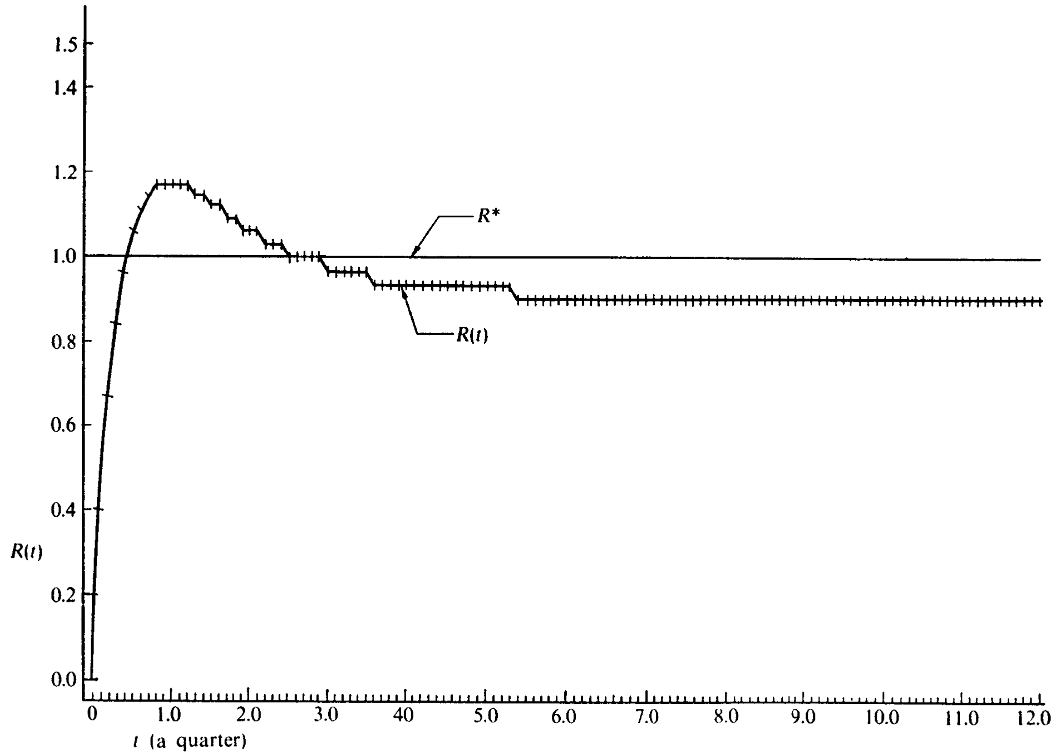

Fig. 5–3. Time path of Eq. (5.9)

$$R(t) = 1.0 - 0.0992\,e^{-0.009t} + 0.80612\,e^{-0.8274t} - 1.70692\,e^{-3.0336t} \qquad t \geqq 0 \tag{5.9}$$

The graph for Eq. (5.9) is shown in Fig. 5–3. The output $R(t)$ reaches the target value after about 1.5 months, then overshoot occurs and reaches the maximum overshoot of about 17 per cent after about 1 quarter, decreases to the target value once more after 2.5 quarter, reaches the trough (about 90.9 per cent of the target value) on about the second year, thereafter again increases and slowly converges to the target value. Thus this system has almost the same transient properties as the derivative policy. The main difference between the two systems is that this new system eliminates the steady-state error. This derivative plus proportional policy will be optimal for the system with the parameters of the structure III. If we do not adhere to the remaining of the steady-state error so much, then we can think that the adjustment rule consists of the derivative policy only as the optimal policy.

6 CONCLUSIONS

The analyses as in the present paper have some limitations in the applicability to the real world. The first limitation is that the model analysed in this paper is very simple. The second is that the model is specified in linear form, thus the model only generates the linear cyclical behavior. The third is that the economic system does not pursue the single purpose of the improvement of the balance of international payments as supposed in this paper and further the government expenditure is not spent only for such a purpose. Our economic system pursues multiple purposes and has various policy instruments to acheive them. The fourth is that it has been assumed in this paper that the government expenditure can be adjusted as much as desired at zero costs. But the level of government expenditure will not be so freely adaptable.

Though there are these limitations, considering that the feed-back effect will be built in the econometric system in near future,* even analysis using the simple model like this paper will give some suggestions to the feed-back effects, policy adjustment rules and optimal policy and so on.

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* See Klein [5]

APPENDIX A

Routh's Stability Criterion

The stability criterion of the linear time-invariant systems stated in this Appendix A is that the roots of the characteristic equation must all have negative real parts.

Suppose that the characteristic equation of a linear system is

$$A(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0$$

where all a's are real numbers.

All powers of s from s^n to s^0 present in the characteristic equation. If the following conditions (1) or (2) are not satisfied, then there are roots with positive real parts

(1) All the coefficients have the same sign.

(2) None of the coefficients other than a_0 vanish.

Now let us form the following Routhian array by using the coefficients of the characteristic equation.

$$\begin{array}{lllllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & a_{n-8} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & a_{n-9} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots & \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots & & \\
s^{n-4} & d_1 & d_2 & \cdots & & & \\
\vdots & \vdots & & & & & \\
s^1 & k_1 & & & & & \\
s^0 & l_1 & & & & &
\end{array}$$

The constants b_1, b_2, b_3, b_4, etc., in the third row are made as follows.

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_3 = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

$$b_4 = \frac{a_{n-1}a_{n-8} - a_n a_{n-9}}{a_{n-1}}$$

The constants c_1, c_2, c_3, etc. are formed as follows by using the s^{n-1} and s^{n-2} row.

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

$$c_3 = \frac{b_1 a_{n-7} - a_{n-1} b_4}{b_1}$$

This Appendix A, B and C owe much to D'Azzo and Houpis [4] and Kuo [6]

This is continued until no more c terms are present. Similarly d row is formed by using the s^{n-2} and s^{n-3} row.

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

Routh's criterion notices the first column in the last tabulation. The criterion states: The roots of the characteristic equation have all the negative real parts if all the elements of the first column have the same sign. The number of changes of sign of the coefficients in the first column is equal to the nunber of roots with positive real parts.

APPENDIX B

Root Locus

Let us consider the roots of the characteristic equation $s^2+4s+K=0$. The roots vary with the value of K. The two roots s_1 and s_2 for a number of values of K are shown in Table A–1. The roots s_1 and s_2 are given by

$$s_{1.2} = -2 \pm j\sqrt{K-4}$$

All possible roots when K varies from 0 to ∞ are given as the points on the curves in Fig. A–1. These curves are defined as the root-locus plot of Eq. $s^2+4s+K=0$. The procedure for constructing the root-locus is stated in [4].

Consider a feedback control system whose closed-loop transfer function is

$$M(s) = \frac{R(s)}{R^*(s)} = \frac{G(s)}{1+G(s)}$$

TABLE A–1 THE ROOTS FOR THE EQUATION $s^2+4s+K=0$ FOR A NUMBER OF VALUES OF K

K	s_1	s_2
0	$0 \quad +j0$	$-4 \quad -j0$
1.0	$-0.268+j0$	$-3.732-j0$
2.0	$-0.586+j0$	$-3.414-j0$
3.0	$-1.0 \quad +j0$	$-3.0 \quad -j0$
4.0	$-2.0 \quad +j0$	$-4.0 \quad -j0$
5.0	$-2.0 \quad +j1.0$	$-2.0 \quad -j1.0$
6.0	$-2.0 \quad +j1.414$	$-2.0 \quad -j1.414$
7.0	$-2.0 \quad +j1.732$	$-2.0 \quad -j1.732$

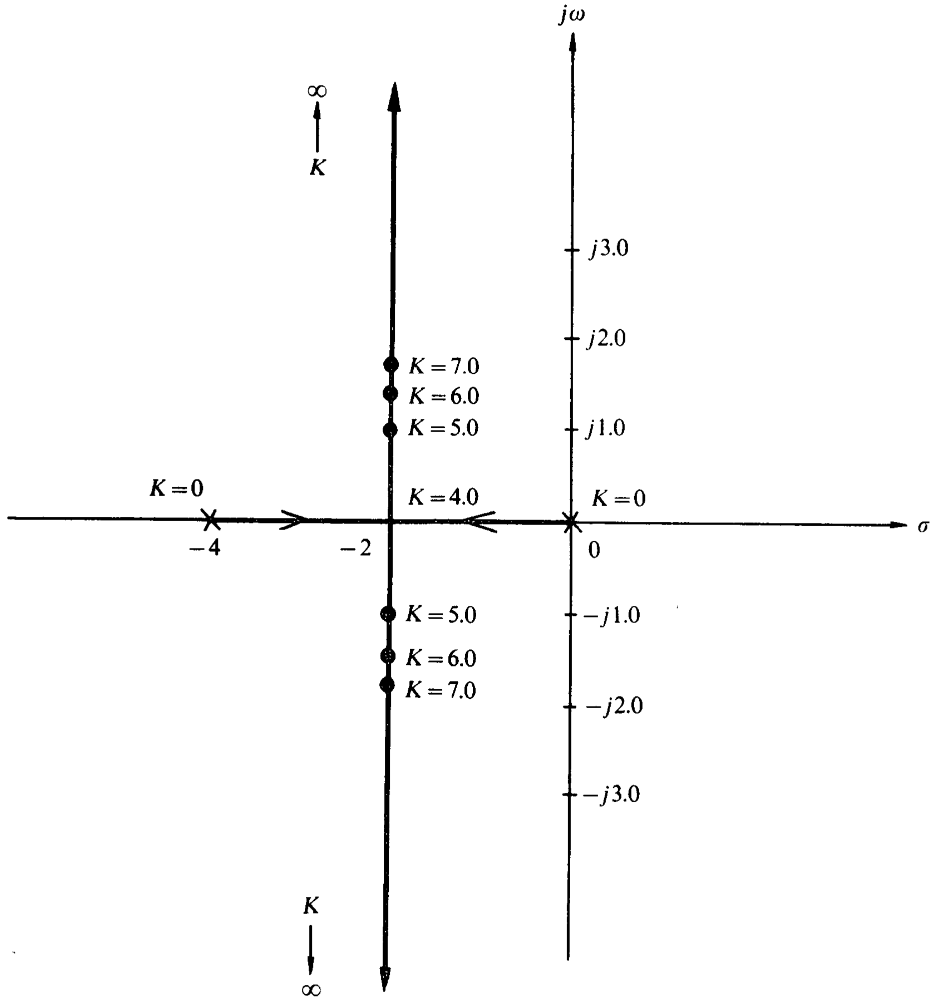

Fig. A-1. Root locus of the equation $s^2+4s+K=0$

where

$$G(s) = \begin{cases} \dfrac{K_0(s+2\mu)^2}{(s+\delta)(s^2+\beta s+4\mu^2)} & \text{for the derivative policy} \\[2ex] \dfrac{K_1(s+2\mu)^2}{s(s+\delta)(s^2+\beta s+4\mu^2)} & \text{for the proportional policy} \\[2ex] \dfrac{K_2(s+2\mu)^2}{s^2(s+\delta)(s^2+\beta s+4\mu^2)} & \text{for the integral policy} \end{cases}$$

and where

$$\begin{aligned} \beta &= 4\mu\left(\frac{k-\mu v}{k}\right) \\ K_0 &= m\delta a_2/k \\ K_1 &= m\delta a_1/k \\ K_2 &= m\delta a_3/k\,. \end{aligned}$$

The roots of the characteristic equation of the system is determined from

$$1 + G(s) = 0$$

and these roots vary with K_i if δ, β, and μ are given. Thus the root-locus method is useful in this case.

Closed-loop transfer function can be rewritten with both numerator and denominator factored,

$$M(s) = \frac{K(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

The values $p_1, p_2, \cdots, p_n$ that make the denominator equal to zero are called poles of $M(s)$ and the values $z_1, z_2, \cdots, z_m$ that make the numerator equal to zero are called zeros of $M(s)$.

The following general conclusions can be drawn from the addition of poles or zeros to the original system (See [4] p. 204).

(1) The addition of poles to $G(s)$ has the effect of pulling the root locus to the right, tending to reduce the relative stability and making the system slower respondent.

(2) The addition of zeros to $G(s)$ has the effect of pulling the root locus to the left, tending to make the system more stable and making the system faster respondent.

Thus if we consider the forward transfer function $G_2(s)$ for the derivative policy as the original system, then the proportional and integral policy can be thought as the system adding zeros to the original system.

APPENDIX C

Types of Feedback Systems

Generally the forward transfer function is of the form

$$G(s) = \frac{R(s)}{E(s)} = \frac{K_n M(s)}{s^n N(s)}$$

where the denominator $N(s)$ does not have the same factor as $M(s)$ and does not include the factors of s.

When $n=0$ the system having this forward transfer function is called a Type 0 system; when $n=1$ it is called a Type 1 system; when $n=2$ it is called a Type 2 system; etc. Thus we see that the derivative, the proportional and the integral policy give Type 0, Type 1, Type 2 systems respectively.

$E(s)$ can be expressed in terms of the $R^*(s)$ as follows.

$$E(s) = \frac{R(s)}{G(s)} = \frac{1}{G(s)} \cdot \frac{G(s) \cdot R^*(s)}{1 + G(s)} = \frac{R^*(s)}{1 + G(s)}$$

Then the steady-state error, using the final-value theorem, is given by

$$e(t)_{ss} = \lim_{s \to 0} [sE(s)]$$

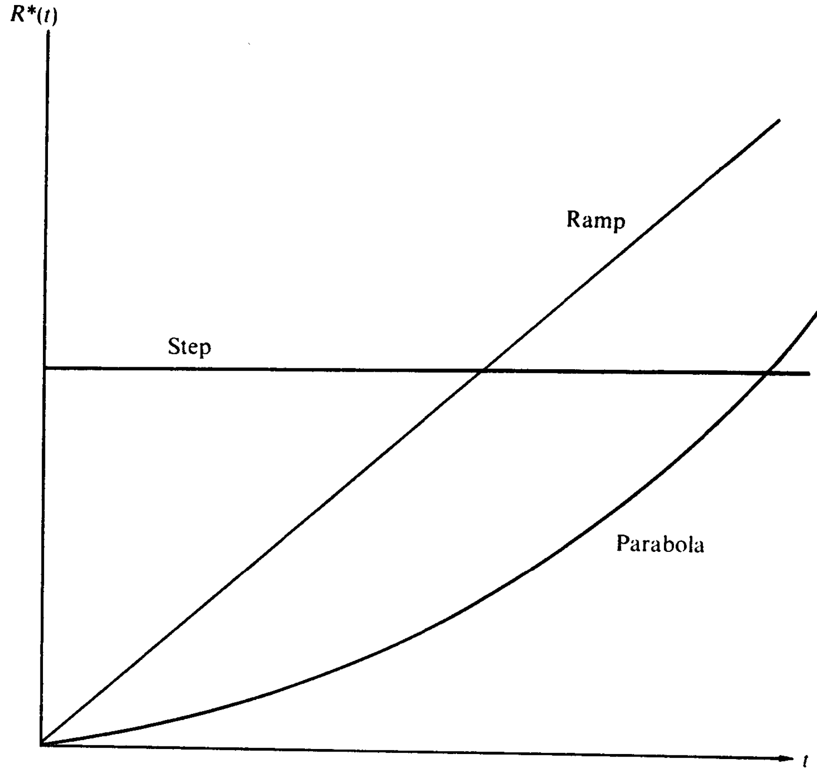

Fig. A–2. Three basic types of input

$$= \lim_{s \to 0} s \left[\frac{R^*(s)}{1 + G(s)} \right]$$

$$= \lim_{s \to 0} \frac{s^{n+1}}{s^n + K_n \dfrac{M(s)}{N(s)}} R^*(s)$$

which shows that whether the steady-state error vanishes or not will depend on the types of the system and the input $R^*(s)$. Three basic types of input are shown in Fig. A–2. Step, ramp and parabola inputs are denoted as follows.

Step : $R_0^* u(t)$
Ramp : $R_1^* tu(t)$
Parabola : $R_2^* t^2 u(t)$

where R_1^*, R_2^* and R_3^* are constant and $u(t)$, unit step input function, is given by

$$u(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leqq 0 \end{cases}$$

The steady-state response characteristics for stable feedback systems are shown in Table A–2. It can be seen from this table that for a Type 1 system, for exanple, the steady-state error becomes zero for a step input, remains for a ramp input and increases with time for a parabolic input. Furthermore we see that there is a trade-off

TABLE A–2 STEADY-STATE RESPONSE CHARACTERISTICS FOR STABLE SYSTEM

System type n	Input $R^*(t)$	Steady-state error $e(t)_{ss}$
	$R_0^*\, u(t)$	$\dfrac{R_0^*}{1 + K_0\dfrac{M(0)}{N(0)}}$
0	$R_1^*\, tu(t)$	∞
	$R_2^*\, t^2u(t)$	∞
	$R_0^*\, u(t)$	0
1	$R_1^*\, tu(t)$	$\dfrac{R_1^*}{K_1\dfrac{M(0)}{N(0)}}$
	$R_2^*\, t^2u(t)$	∞
	$R_0^*\, u(t)$	0
2	$R_1^*\, tu(t)$	0
	$R_2^*\, t^2u(t)$	$\dfrac{2R_2^*}{K_2\dfrac{M(0)}{N(0)}}$

relation between the steady-state error and the stability of the system. Because the higher the system order, the less the steady-state error becomes but the more the system tends to become unstable.