Title	COMPETITIVE EQUILIBRIUM WITH MARSHALLIAN EXTERNALITIES
Sub Title	
Author	長名, 寛明(OSANA, HIROAKI)
Publisher	Keio Economic Society, Keio University
Publication year	1977
Jtitle	Keio economic studies Vol.14, No.1 (1977.) ,p.31- 40
JaLC DOI	
Abstract	
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19770001-0031

COMPETITIVE EQUILIBRIUM WITH MARSHALLIAN EXTERNALITIES

HIROAKI OSANA

In this paper, we present a set of sufficient conditions for the existence of a competitive equilibrium of a private ownership economy with a class of externalities. The presence of externalities themselves is not incompatible with pure competition insofar as it does not cause perverse increasing returns. Moreover, as was noted by Marshall [7], the increasing returns in the aggregate production processes (more specifically, in an industry) may still be compatible with pure competition, provided they arise as a result of externalities caused by producers (more specifically, the firms in the industry) each of which is subject to non-increasing returns. Once we may expect the compatibility of pure competition with such externalities, it will be natural to ask a question of whether or not there exists a competitive equilibrium for an economy with those externalities.

The problem has been studied by McKenzie [8], Arrow and Hahn [1], and Laffont and Laroque [6] among others. McKenzie [8] allowed for externalities in preference relations only, while Arrow and Hahn [1] allowed for externalities in preference relations and in production possibilities. Laffont and Laroque [6] considered an economy with externalities in preference relations, in consumption possibilities, and in production possibilities. However, the formulations by Arrow and Hahn [1] and by Laffont and Laroque [6], as well as the formulation by Osana [10], seem to have a common difficulty, as will be explained in the first section of this paper. The purpose of this paper is to remedy the difficulty.

I. NOTATION AND DEFINITIONS

We consider an economy with l commodities, m consumers, and n producers. Denote by $H=\{1, 2, \ldots, l\}$, $I=\{1, 2, \ldots, m\}$, and $J=\{1, 2, \ldots, n\}$ the *set of commodities*, the *set of consumers*, and the *set of producers*, respectively. The l-dimensional Euclidean space R^l will be regarded as the *commodity space*. For each consumer i, his *consumption* is denoted by a point x_i of R^l, where positive components stand for *inputs* and negative components for *outputs*, while, for each producer j, his *production* is denoted by a point y_j of R^l, where positive components stand for *ouptuts* and negative components for *inputs*. An m-tuple $x=(x_1, x_2, \ldots, x_m)$ of consumptions is called a *consumption allocation*, and an n-tuple $y=(y_1, y_2, \ldots, y_n)$ of productions is called a *production allocation*. A *state* is a pair (x, y) of a consumption allocation and a production allocation. The $l(m+n)$-dimensional Euclidean space $R^{l(m+n)}$ will be regarded as the *state space*.

The purely technological relations determine the *set D of possible states* as a subset of the state space $R^{l(m+n)}$. The shape of D may be very complicated in the presence of externalities, though, in the absence of externalities, D may be written as $\prod_{i\in I}X^i \times \prod_{j\in J}Y^j$, where X^i is a subset of R^l called the *consumption set* of consumer i and Y^j is a subset of R^l called the *production set* of producer j.

Writing $x_{)i(}=(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$, we call $(x_{)i(}, y)$ a *circumstance* of consumer i. For each $i\in I$, the *set of possible circumstances* of consumer i is defined by

$$B_i=\{(x_{)i(}, y)\in R^{l(m-1+n)}\colon ((x_i, x_{)i(}), y)\in D \text{ for some } x_i\in R^l\},$$

where by $(x_i, x_{)i(})$ is meant the consumption allocation $x=(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$. Utilizing similar notation, we can define the *set of possible circumstances* of producer j by

$$C_j=\{(x, y_{)j(})\in R^{l(m+n-1)}\colon (x, (y_j, y_{)j(}))\in D \text{ for some } y_j\in R^l\}$$

for each $j\in J$. For each $i\in I$, let

$$X_i(x_{)i(}, y)=\{x_i\in R^l\colon ((x_i, x_{)i(}), y)\in D\} \text{ for every } (x_{)i(}, y)\in B_i$$

and, for each $j\in J$, let

$$Y_j(x, y_{)j(})=\{y_j\in R^l\colon (x, (y_j, y_{)j(}))\in D\} \text{ for every } (x, y_{)j(})\in C_j.$$

The correspondence X_i of B_i into R^l will be called the *consumption correspondence* of consumer i and the correspondence Y_j of C_j into R^l will be called the *production correspondence of producer j*. These are generalizations of consumption sets and of production sets.

It should be noticed that our procedure adopted here to describe the technological possibility is reverse to that adopted by Arrow and Hahn [1], Laffont and Laroque [6], and Osana [10]. The latter uses as primitive notions the consumption correspondences and the production correspondences defined on the whole space $R^{l(m+n-1)}$. This amounts to requiring that the set of possible circumstances of each agent should coincide with $R^{l(m+n-1)}$, a rather stringent requirement which is unlikely to be fulfilled. Furthermore, we may be unable to extend the correspondences to the whole space in case they are defined on proper subsets of $R^{l(m+n-1)}$. This is the first point of the difficulties lying in the latter procedure. There is another aspect of the difficulties. To make it clear, let us introduce the consumption correspondences $\bar{X}_i$ and the production correspondences $\bar{Y}_j$ which are defined on the whole space $R^{l(m+n-1)}$. According to the procedure used in [10], we define the set of possible states by

$$D=\{(x, y)\in R^{l(m+n)}\colon (x, y)\in \textstyle\prod_{i\in I}\bar{X}_i(x_{)i(}, y)\times \prod_{j\in J}\bar{Y}_j(x, y_{)j(})\}.$$

On the basis of the set D thus defined, we can define the consumption correspondences X_i and the production correspondences Y_j in the way adopted by the present paper. Then X_i does not necessarily coincide with $\bar{X}_i$. In general, for each $(x_{)i(}, y)\in B_i$, $X_i(x_{)i(}, y)$ is a subset of $\bar{X}_i(x_{)i(}, y)$. All points of $\bar{X}_i(x_{)i(}, y)$ not belonging to $X_i(x_{)i(}, y)$ turn out to be impossible under the circumstance $(x_{)i(}, y)$. This implies that agents' knowledge is incomplete. So, if we assume complete knowledge on the part of agents, it seems better to start from D than from consumption and production correspondences.

It will be useful to introduce the following notation:

$$X=\{x\in R^{lm}\colon (x, y)\in D \text{ for some } y\in R^{ln}\},$$
$$Y=\{y\in R^{ln}\colon (x, y)\in D \text{ for some } x\in R^{lm}\},$$
$$X^i=\bigcup\{X_i(x_{)i(}, y)\colon (x_{)i(}, y)\in B_i\} \text{ for each } i\in I,$$
$$G=\{(x, y)\in R^{l(m+n)}\colon (x_{)i(}, y)\in B_i \text{ for every } i\in I \text{ and } (x, y_{)j(})\in C_j \text{ for every } j\in J\}.$$

X may be called the *set of possible consumption allocations*, Y the *set of possible production allocations*, and X^i the *set of possible consumptions* of consumer i. G may be called the *set of jointly possible circumstances*, which evidently contains D.

For each $i\in I$, a complete preordering $\succsim_i$, called the *preference relation* of consumer i, is defined on D. $(x, y)\succsim_i (x', y')$ is interpreted to mean that consumer i desires (x, y) at least as much as (x', y'). We write $(x, y)\sim_i(x', y')$ if and only if $(x, y)\succsim_i(x', y')$ and $(x', y')\succsim_i(x, y)$, and also write $(x, y)\succ_i(x', y')$ if and only if $(x, y)\succsim_i(x', y')$ and not $(x', y')\succsim_i(x, y)$.

For each $i\in I$ and each $(x, y)\in D$, let

$$M_i(x, y)=\{x_i'\in X_i(x_{)i(}, y)\colon ((x_i', x_{)i(}), y)\succsim_i(x, y)\},$$
$$M_i^0(x, y)=\{x_i'\in X_i(x_{)i(}, y)\colon ((x_i', x_{)i(}), y)\succ_i(x, y)\},$$
$$L_i(x, y)=\{x_i'\in X_i(x_{)i(}, y)\colon ((x_i', x_{)i(}), y)\precsim_i(x, y)\}.$$

For each $i\in I$, the *resource endowment* of consumer i is specified by a point ω_i of R^l and his *claim to the share of the profit* of each producer j is specified by a non-negative number θ_{ij}. All the profits of each producer are assumed to be distributed among consumers: $\sum_{i\in I}\theta_{ij}=1$ for every $j\in J$. Let $\theta_i=(\theta_{i1}, \theta_{i2}, \ldots, \theta_{in})$ for each $i\in I$.

The description of a private ownership economy has now been completed. Formally, we have

DEFINITION. $E=(H, I, J, D, (\succsim_i, \omega_i, \theta_i)_{i\in I})$ is called a *private ownership economy* if

(i) $H=\{1, 2, \ldots, l\}$, $I=\{1, 2, \ldots, m\}$, and $J=\{1, 2, \ldots, n\}$,
(ii) D is a subset of $R^{l(m+n)}$,
(iii) $\succsim_i$ is a complete preordering on D for each $i\in I$,
(iv) ω_i is a point of R^l for each $i\in I$,
(v) $\theta_i=(\theta_{i1}, \theta_{i2}, \ldots, \theta_{in})$ is a point of R^n_+ for each $i\in I$ and $\sum_{i\in I}\theta_{ij}=1$ for each $j\in J$.

We conclude this section by setting forth a formal definition of competitive equilibrium for the private ownership economy E.

DEFINITION. A pair (x^*, y^*, p^*) of a state and of a point of R^l is called a *competitive equilibrium* for a private ownership economy E if

(i) $(x^*, y^*)\in D$, $x_I^*\leqq y_J^*+\omega_I$, $p^*\geqq 0$, and $p^*\cdot(x_I^*-y_J^*-\omega_I)=0$,
(ii) for each $i\in I$, $p^*\cdot x_i^*\leqq p^*\cdot\omega_i+\theta_i\cdot(p^*\cdot y^*)$ and $(x^*, y^*)\succsim_i ((x_i, x^*_{)i(}), y^*)$ for each $x_i\in X_i(x^*_{)i(}, y^*)$ such that $p^*\cdot x_i\leqq p^*\cdot\omega_i+\theta_i\cdot(p^*\cdot y^*)$, and
(iii) for each $j\in J$, $p^*\cdot y_j^*\geqq p^*\cdot y_j$ for each $y_j\in Y_j(x^*, y^*_{)j(})$,

where $x_I^*=\sum_{i\in I}x_i^*$, $y_J^*=\sum_{j\in J}y_j^*$, $\omega_I=\sum_{i\in I}\omega_i$, and $p^*\cdot y^*=(p^*\cdot y_1^*, p^*\cdot y_2^*, \ldots, p^*\cdot y_n^*)$.[1]

II. STATEMENT OF THE THEOREM

THEOREM. *The private ownership economy E has a competitive equilibrium if:*
(T.1) *D is connected and closed in* $R^{l(m+n)}$,
(T.2) *G is closed in* $R^{l(m+n)}$ *and star-shaped with some center* $(\bar{x}, \bar{y})\in G$;
for every $i\in I$,
(C.1) X_i *is lower semi-continuous on* B_i,
(C.2) $X_i(x_{)i(}, y)$ *is convex for every* $(x_{)i(}, y)\in B_i$,
(C.3) *for every* $(x_{)i(}, y)\in B_i$ *there is* $x_i^0\in X_i(x_{)i(}, y)$ *such that* $x_i^0<\omega_i$,
(C.4) X^i *is bounded from below*,
(C.5) $\{(x', y')\in D: (x', y')\succsim_i(x, y)\}$ *and* $\{(x', y')\in D: (x', y')\precsim_i(x, y)\}$ *are closed in D for every* $(x, y)\in D$,
(C.6) $(((1-t)x_i^1+tx_i^2, x_{)i(}), y)\succ_i((x_i^1, x_{)i(}), y)$ *for every* $t\in\,]0, 1[$, *every* $(x_{)i(}, y)\in B_i$, *and every* $x_i^1, x_i^2\in X_i(x_{)i(}, y)$ *such that* $((x_i^2, x_{)i(}), y)\succ_i((x_i^1, x_{)i(}), y)$,
(C.7) $M_i^0(x, y)\neq\varnothing$ *for every* $(x, y)\in D$;
for every $j\in J$,
(P.1) Y_j *is lower semi-continuous on* C_j,
(P.2) $Y_j(x, y_{)j(})$ *is convex for every* $(x, y_{)j(})\in C_j$,
(P.3) $0\in Y_j(x, y_{)j(})$ *for every* $(x, y_{)j(})\in C_j$,
(P.4) *for every* $y\in A(Y)$, *if* $y_J\geqq 0$ *then* $y=0$.[2]

Conditions (C.1), (P.1), and the second half of (T.1) correspond to the usual assumption of closedness of consumption sets and of production sets. In fact, our conditions are equivalent to the usual ones, in the absence of externalities, as can be readily seen.

Conditions (C.2) and (P.2) correspond to the usual assumption of convexity of consumption sets and of production sets. In conjunction with (P.3), (P.2) implies that each producer operates subject to non-increasing returns to scale from his individual point of view. It should be noticed, however, that these do not necessarily imply the convexity of D and hence that increasing returns to scale may prevail from a social viewpoint. The increasing returns to scale are due to external effects caused by individual agents. These external effects may be referred to as Marshallian

[1] Given two points a and b of R^l, we write $a\geqq b$ if and only if $a_h\geqq b_h$ for every $h\in H$; $a\geq b$ if and only if $a\geqq b$ and $a\neq b$; $a>b$ if and only if $a_h>b_h$ for every $h\in H$; moreover, we denote by $a\cdot b$ the inner product of a and b: that is, $a\cdot b=\sum_{h\in H}a_hb_h$.

[2] Given a subset S of R^{ln}, we denote by $A(S)$ its asymptotic cone with vertex 0. For the definition of an asymptotic cone, see Debreu [3, 1.9.n] or Osana [10].

externalities (cf. Osana [11]). In the absence of externalities, (C.2) and (P.2) imply the convexity and hence connectedness of D. But, in the presence of externalities, this may not be the case. So we stipulate the connectedness of D in (T.1) without much loss of realism.

Condition (C.3) is a stringent assumption requiring that the resource endoment of each consumer should be rich enough to make him survive even if some positive amount of each commodity is subtracted from his endowment. The possibility of weakening (C.3) will be considered in Section IV.

Condition (C.4) corresponds to the lower boundedness of consumption sets. Conditions (C.5), (C.6), and (C.7) mean the continuity, convexity, and non-satiability of consumers' preference relations, respectively. Condition (P.4) says that no free production of indefinitely large scale is possible and that any aggregative production process of indefinitely large scale is irreversible (cf. Osana [10]).

Unlike the conditions discussed above, condition (T.2) may appear strange. The closedness of G is only a technical requirement. It should also be noticed that the star-shapedness of G is trivially satisfied if G is assumed to be the whole space, as in Arrow and Hahn [1], Laffont and Laroque [6], and Osana [10]. Even though G does not coincide with the whole space, it will not be so restrictive to assume that G is star-shaped. As we shall see below, G might be assumed to be even convex. Let (x, y) and (x', y') be any points of G. Let $i \in I$. Then $(x_{)i(}, y)$ and $(x'_{)i(}, y')$ are possible circumstances of consumer i. One may not be able to find any plausible reason why some weighted average of these two circumstances should be an impossible circumstance for him. These considerations will lead us to think G to be convex. But we do not need the convexity of G, so we simply assume that G is star-shaped with some center.

III. PROOF OF THE THEOREM

By (C.4), for every $i \in I$ there is $a_i \in R^l$ such that $a_i < x_i$ for every $x_i \in X^i$. Let

$$S_i = \{x_i \in R^l\colon\ a_i < x_i < \omega_i\} \text{ for each } i \in I.$$

It follows from (C.4) that X is bounded from below, so that, by (C.4), the *set of attainable states*

$$A = \{(x, y) \in D\colon\ x_I \leqq y_J + \omega_I\}$$

is bounded (cf. Osana [10, Theorem 1]). Hence there is a closed cube K of R^l with center 0 such that

$$A \subset \operatorname{int} K^{m+n},\ (\bar{x}, \bar{y}) \in \operatorname{int} K^{m+n}, \text{ and } S_i \subset \operatorname{int} K \text{ for every } i \in I.$$

Let

$$T = G \cap K^{m+n}.$$

Then T is compact and star-shaped with center $(\bar{x}, \bar{y})$.

LEMMA 1. (i) *For every $i \in I$, the correspondence X'_i of T into R^l defined by $X'_i(x, y) = X_i(x_{)i(}, y) \cap K$ for every $(x, y) \in T$ is continuous on T and $X'_i(x, y)$ is non-empty and convex for every $(x, y) \in T$.*

(ii) *For every $j \in J$, the correspondence Y'_j of T into R^l defined by $Y'_j(x, y) = Y_j(x, y_{)j(}) \cap K$ for every $(x, y) \in T$ is continuous on T and $Y'_j(x, y)$ is non-empty and convex for every $(x, y) \in T$.*

Proof. We shall prove the continuity of the correspondence X'_i only. K is clearly upper semi-continuous on T as a constant correspondence of T into R^l. By (T.1), the graph D of the correspondence X_i is closed in $R^{l(m+n)}$, so that X_i is closed on B_i. Hence X_i is closed on T, when it is looked upon as a correspondence of T into R^l. Therefore X'_i is upper semi-continuous on T as the intersection of a closed correspondence and an upper semi-continuous correspondence (cf. Berge [2, Theorem 7 of VI.1]).

To prove lower semi-continuity, let $(x^0, y^0) \in T$ and let Z be any open subset of R^l such that $X'_i(x^0, y^0) \cap Z \neq \emptyset$. Then there is $x_i^1 \in X_i(x^0_{)i(}, y^0) \cap K \cap Z$. By (C.3) and (C.4), $X_i(x^0_{)i(}, y^0) \cap S_i \neq \emptyset$ and hence there is $x_i^2 \in X_i(x^0_{)i(}, y^0) \cap S_i$. Let $x_i(t) = (1-t)x_i^1 + tx_i^2$. Then $x_i(t) \in X_i(x^0_{)i(}, y^0)$ for every $t \in [0, 1]$ by (C.2), $x_i(t) \in Z$ for every t close enough to 0, and $x_i(t) \in \operatorname{int} K$ for every $t \in]0, 1]$ because of $x_i^2 \in S_i \subset \operatorname{int} K$. So $x_i(t) \in X_i(x^0_{)i(}, y^0) \cap (Z \cap \operatorname{int} K)$ for every $t \in]0, 1]$ close enough to 0. Let $C = Z \cap \operatorname{int} K$. Then $X_i(x^0_{)i(}, y^0) \cap C \neq \emptyset$ and C is an open subset of R^l. Since X_i is lower semi-continuous, there is a neighborhood U of $(x^0_{)i(}, y^0)$ such that $(x_{)i(}, y) \in U$ implies that $\emptyset \neq X_i(x_{)i(}, y) \cap C$. Let $V = \{((x_i, x_{)i(}), y) \in R^{l(m+n)}: x_i \in R^l \text{ and } (x_{)i(}, y) \in U\}$. Then $\emptyset \neq X_i(x_{)i(}, y) \cap C \subset X'_i(x, y) \cap Z$ for every $(x, y) \in V$. Therefore X'_i is lower semi-continuous at (x^0, y^0). This completes the proof of the lemma.

Let

$$P = \{p \in R^l: p \geqq 0 \text{ and } \textstyle\sum_{h \in H} p^h = 1\}.$$

For each $j \in J$, we define the supply correspondence and profit function of producer j by

$$\eta_j(x, y, p) = \{y_j^* \in Y'_j(x, y): p \cdot y_j^* \geqq p \cdot y_j \text{ for every } y_j \in Y'_j(x, y)\},$$

$$\pi_j(x, y, p) = \max p \cdot Y'_j(x, y)$$

for every $(x, y, p) \in T \times P$, respectively.

LEMMA 2. *For every $j \in J$, the function π_j is continuous on $T \times P$, the correspondence η_j is upper semi-continuous on $T \times P$, and $\eta_j(x, y, p)$ is non-empty and convex for every $(x, y, p) \in T \times P$.*

Proof. Immediate from the maximum theorem of Berge [2, VI.3].

For each $i \in I$, we define the wealth constraint of consumer i by

$$\gamma_i(x, y, p) = \{x'_i \in X'_i(x, y): p \cdot x'_i \leqq p \cdot \omega_i + \theta_i \pi(x, y, p)\}$$

for every $(x, y, p) \in T \times P$, where

$$\pi(x, y, p) = (\pi_1(x, y, p), \pi_2(x, y, p), \ldots, \pi_n(x, y, p)).$$

LEMMA 3. *For every* $i \in I$, *the correspondence* γ_i *is continuous on* $T \times P$ *and* $\gamma_i(x, y, p)$ *is non-empty, compact, and convex for every* $(x, y, p) \in T \times P$.

Proof. Let $W_i(x, y, p) = \{x_i' \in R^l\colon\ p \cdot x_i' \leqq p \cdot \omega_i + \theta_i \cdot \pi(x, y, p)\}$ for each $(x, y, p) \in T \times P$. Then the correspondence W_i is closed on $T \times P$, since the function π is continuous on $T \times P$ by Lemma 2. On the other hand, by Lemma 1, X_i' is upper semi-continuous on $T \times P$, when it is looked upon as a correspondence defined on $T \times P$. Since $\gamma_i(x, y, p) = W_i(x, y, p) \cap X_i'(x, y)$ for every $(x, y, p) \in T \times P$, the correspondence γ_i is upper semi-continuous on $T \times P$ as the intersection of a closed correspondence and an upper semi-continuous correspondence.

To prove lower semi-continuity, let $(x^0, y^0, p^0) \in T \times P$ and let Z be any open subset of R^l such that $Z \cap \gamma_i(x^0, y^0, p^0) \neq \varnothing$. Then there is $x_i^1 \in Z \cap W_i(x^0, y^0, p^0) \cap X_i'(x^0, y^0)$. On the other hand, by (C.3) and (C.4), there is $x_i^2 \in X_i(x^0_{)i(}, y^0)$ such that $a_i \leqq x_i^2 < \omega_i$. Since $\pi(x^0, y^0, p^0) \geqq 0$ by (P.3), it follows from the choice of K that $x_i^2 \in X_i'(x^0, y^0) \cap \operatorname{int} W_i(x^0, y^0, p^0)$. Let $x_i(t) = (1-t)x_i^1 + tx_i^2$. Since $X_i'(x^0, y^0)$ is convex by (C.2), $x_i(t) \in X_i'(x^0, y^0)$ for every $t \in [0, 1]$. Since $W_i(x^0, y^0, p^0)$ is clearly convex, $x_i(t) \in \operatorname{int} W_i(x^0, y^0, p^0)$ for every $t \in]0, 1]$. Furthermore, $x_i(t) \in Z$ for every t close enough to 0. It follows that $Z \cap \operatorname{int} W_i(x^0, y^0, p^0) \cap X_i'(x^0, y^0) \neq \varnothing$. Let $C = Z \cap \operatorname{int} W_i(x^0, y^0, p^0)$. Then $C \cap X_i'(x^0, y^0) \neq \varnothing$ and C is an open subset of R^l. Let $x_i^3 \in C \cap X_i'(x^0, y^0)$. Since C is open, there is $\delta > 0$ such that $U(x_i^3; \delta) \subset C$, where $U(x_i^3; \delta)$ is the δ-neighborhood of x_i^3. Hence $U(x_i^3; \delta) \subset \operatorname{int} W_i(x^0, y^0, p^0)$. Let

$$f(x_i', x, y, p) = p \cdot \omega_i + \theta_i \cdot \pi(x, y, p) - p \cdot x_i'.$$

Then f is continuous on $R^l \times T \times P$. Let M be a compact subset of $U(x_i^3; \delta)$ such that $x_i^3 \in \operatorname{int} M$. Then we can define

$$g(x, y, p) = \min f(M, x, y, p).$$

By the maximum theorem of Berge, the function g is continuous on $T \times P$. Since $g(x^0, y^0, p) > 0$, there is a neighborhood V_1 of (x^0, y^0, p^0) such that $(x, y, p) \in V_1$ implies $g(x, y, p) > 0$, i.e., $M \subset \operatorname{int} W_i(x, y, p)$. On the other hand, $X_i'(x^0, y^0) \cap \operatorname{int} M \neq \varnothing$ and, by Lemma 1, X_i' is lower semi-continuous at (x^0, y^0), so that there is a neighborhood V_2 of (x^0, y^0) such that $(x, y) \in V_2$ implies $X_i'(x, y) \cap \operatorname{int} M \neq \varnothing$. Thus, if $(x, y, p) \in V_1 \cap (V_2 \times R^l)$ then $W_i(x, y, p) \cap X_i'(x, y) \cap \operatorname{int} M \neq \varnothing$ and therefore $Z \cap \gamma_i(x, y, p) \neq \varnothing$. Hence γ_i is lower semi-continuous at (x^0, y^0, p^0).

The non-emptiness, compactness, and convexity of the image sets of γ_i are obvious. Thus the lemma has been established.

For each $i \in I$, we define the demand correspondence of consumer i by

$$\xi_i(x, y, p) = \{x_i^* \in \gamma_i(x, y, p)\colon\ ((x_i^*, x_{)i(}), y) \succsim_i ((x_i, x_{)i(}), y) \quad \text{for every } x_i \in \gamma_i(x, y, p)\}$$

for every $(x, y, p) \in T \times P$.

LEMMA 4. *For every* $i \in I$, *the correspondence* ξ_i *is upper semi-continuous on* $T \times P$ *and* $\xi_i(x, y, p)$ *is non-empty and convex for every* $(x, y, p) \in T \times P$.

Proof. By (T.1) and (C.5), there is a continuous utility function u_i on D (cf. Debreu [3, (1) of 4.6]). Since D is closed in $R^{l(m+n)}$ by (T.1), it follows from Tietze's extension theorem that there is a continuous extension of u_i to $R^{l(m+n)}$ (cf. Kuratowski [5, p. 127]). Without causing any confusion, we may denote this extension by u_i again. Let

$$u_i'(w, y, p, x_i) = u_i(x, y),$$

where u_i' is considered to be independent of the component x_i in x. Then u_i' is continuous on $R^{l(m+n)} \times P \times R^l$. Since $\gamma_i(x, y, p)$ is non-empty and compact for every $(x, y, p) \in T \times P$, we can define a real-valued function v_i on $T \times P$ by

$$v_i(x, y, p) = \max\{u_i'(x, y, p, x_i'): x_i' \in \gamma_i(x, y, p)\}.$$

Then $\xi_i(x, y, p) = \{x_i^* \in \gamma_i(x, y, p): u_i'(x, y, p, x_i^*) = v_i(x, y, p)\}$. Hence use can be made of the maximum theorem of Berge to establish the upper semi-continuity of ξ_i. The non-emptiness of the image sets of ξ_i follows from the non-emptiness and compactness of the image sets of γ_i. On the other hand, the convexity of the image sets of ξ_i follows from (C.2), (C.5), and (C.6).

LEMMA 5. *For every $i \in I$ and every $(x, y, p) \in T \times P$, if $x_i' \in \xi_i(x, y, p)$ then $p \cdot x_i' = p \cdot \omega_i + \theta_i \cdot \pi(x, y, p)$.*

Proof. Let $w_i = p \cdot \omega_i + \theta_i \cdot \pi(x, y, p)$. Since $x_i' \in \xi_i(x, y, p)$, it follows that $p \cdot x_i' \leqq w_i$. On the other hand, by the definition of ξ_i, $x_i'' \in X_i'(x, y)$ and $p \cdot x_i'' \leqq w_i$ imply $((x_i'', x_{)i(}), y) \precsim_i ((x_i', x_{)i(}), y)$, or equivalently, $x_i'' \in X_i'(x, y)$ and $((x_i'', x_{)i(}), y) \succ_i ((x_i', x_{)i(}), y)$ imply $p \cdot x_i'' > w_i$. By (C.5), this implies that if $x_i'' \in X_i'(x, y)$ then $((x_i'', x_{)i(}), y) \succsim_i ((x_i', x_{)i(}), y)$ implies $p \cdot x_i'' \geqq w_i$. Trivially, $x_i' \in X_i'(x, y)$ and $((x_i', x_{)i(}), y) \succsim_i ((x_i', x_{)i(}), y)$, so that $p \cdot x_i' \geqq w_i$ and therefore $p \cdot x_i' = w_i$. This completes the proof of the lemma.

We are now ready to go into the final step of our proof of the theorem. For each $(x, y) \in T$, let

$$\mu(x, y) = \{p \in P: p \cdot (x_I - y_J - \omega_I) \geqq q \cdot (x_I - y_J - \omega_I) \text{ for every } q \in P\}.$$

Then the correspondence μ of T into P is upper semi-continuous on T and $\mu(x, y)$ is non-empty and convex for every $(x, y) \in T$. Let

$$F(x, y, p) = \left(\prod_{i \in I} \xi_i(x, y, p)\right) \times \left(\prod_{j \in J} \eta_j(x, y, p)\right) \times \mu(x, y)$$

for every $(x, y, p) \in T \times P$. By Lemmas 2 and 4, the correspondence F of $T \times P$ into itself is upper semi-continuous on $T \times P$ (cf. Berge [2, Theorem 4′ of VI.2]), and $F(x, y, p)$ is non-empty and convex for every $(x, y, p) \in T \times P$. Clearly $T \times P$ is non-empty and compact. Furthermore, by (T.2), $T \times P$ is star-shaped and therefore contractible. It follows from the fixed-point theorem of Eilenberg and Montgomery [4] that there is $(x^*, y^*, p^*) \in T \times P$ such that $(x^*, y^*, p^*) \in F(x^*, y^*, p^*)$.

It remains to show that (x^*, y^*, p^*) is a competitive equilibrium. Since $x_i^* \in \xi_i(x^*, y^*, p^*)$ for every $i \in I$, it follows from Lemma 5 that $p^* \cdot x_i^* = p^* \cdot \omega_i + \theta_i \cdot (p^* \cdot y^*)$ for every $i \in I$ and therefore $p^* \cdot (x_I^* - y_J^* - \omega_I) = 0$. Since $p^* \in \mu(x^*, y^*)$, it follows that $p^* \cdot (x_I^* - y_J^* - \omega_I) \geqq p \cdot (x_I^* - y_J^* - \omega_I)$ for every $p \in P$, so that $p \cdot (x_I^* - y_J^* - \omega_I) \leqq 0$ for every $p \in P$ and therefore $x_I^* \leqq y_J^* + \omega_I$. On the other hand, since $x_i^* \in \xi_i(x^*, y^*, p^*)$ for every $i \in I$, it is obvious that $(x^*, y^*) \in D$. Hence, condition (i) of competitive equilibrium is satisfied.

To prove that condition (ii) holds, suppose to the contrary. Then, for some $i \in I$, there would be $x_i' \in X_i(x_{)i(}^*, y^*)$ such that $p^* \cdot x_i' \leqq p^* \cdot \omega_i + \theta_i \cdot (p^* \cdot y^*)$ and $((x_i', x_{)i(}^*), y^*) \succ_i (x^*, y^*)$. Let $x_i(t) = (1-t)x_i^* + tx_i'$. Then, for every $t \in]0, 1[$, clearly $p^* \cdot x_i(t) \leqq p^* \cdot \omega_i + \theta_i \cdot (p^* \cdot y^*)$ and by (C.6), $((x_i(t), x_{)i(}^*), y^*) \succ_i (x^*, y^*)$. But, since $x_i^* \in \operatorname{int} K$, $x_i(t) \in X_i'(x^*, y^*)$ for every $t \in]0, 1[$ close enough to 0. This contradicts the fact that $x_i^* \in \xi_i(x^*, y^*, p^*)$. Hence condition (ii) is satisfied.

A reasoning similar to the last paragraph will verify condition (iii). Thus (x^*, y^*, p^*) is a competitive equilibrium, and the proof is complete.

IV. REMARK

In Section 2, we noticed that condition (C.3) is unsatisfactory. This section will be devoted to weakening (C.3) slightly. We make the following assumption:

(C.3′) *for every* $(x, y) \in G$ *there is* $x^0 \in \prod_{i \in I} X_i(x_{)i(}, y)$ *such that* $x_I^0 < \omega_I$ *and* $x_i^0 \leqq \omega_i$ *for every* $i \in I$.

In order to prove the existence of a competitive equilibrium under this assumption, we need the following assumption:

(C.7′) *for every* $(x_{)i(}, y) \in B_i$, *every* $x_i' \in X_i(x_{)i(}, y)$, *and every* $h \in H$, *there is a positive real number* λ *such that* $x_i(\lambda, h) \in X_i(x_{)i(}, y)$ *and* $((x_i(\lambda, h), x_{)i(}), y) \succ_i ((x_i', x_{)i(}), y)$, *where* $x_i(\lambda, h) = (x_{i1}, \ldots, x_{i,h-1}, x_{ih} + \lambda, x_{i,h+1}, \ldots, x_{il})$.

Condition (C.3′) requires that the total amount of the endowment of each commodity should exceed the amount needed to make all consumers survive, but not that, for each consumer, his endowment of each commodity should exceed the amount needed to make him survive. For each consumer, it suffices to have initially the amount of each commodity at least as much as needed to make him survive and to have some commodity in excess of his subsistence level. This condition is clearly weaker than (C.3).

Condition (C.7′), in conjunction with (C.6), implies the monotonicity of preference relations, and hence is much stronger than (C.7).

With (C.3′) and (C.7′) substituted for (C.3) and (C.7), we can prove that the private ownership ecnomy has a competitive equilibrium. We follow the technique due to Nikaido [9, Theorem 16.2].

Let

$$P^0 = \{p \in P\colon\ p > 0\}$$

and

$$Q_i = \{(x, y, p) \in T \times P\colon\ w_i(x, y, p) > \min p \cdot X_i'(x, y)\} \text{ for every } i \in I,$$

where

$$w_i(x, y, p) = p \cdot \omega_i + \theta_i \cdot \pi(x, y, p).$$

As in the proof of Lemma 3, we can show that, by (C.3′), $T \times P^0 \subset Q_i$ for every $i \in I$. By the proof of Lemma 4, for each $i \in I$, ξ_i is upper semi-continuous on Q_i and $\xi_i(x, y, p)$ is non-empty and convex for every $(x, y, p) \in Q_i$. Since $T \times P^0$ is dense in $T \times P$, it follows that Q_i is dense in $T \times P$ for every $i \in I$. Hence, for every $i \in I$, the correspondence ξ_i of Q_i into R^l can be extended to an upper semi-continuous correspondence ξ_i' of $T \times P$ into R^l (cf. Nikaido [9, Lemma 4.4 and Theorem 4.7]). For each $i \in I$ and each $(x, y, p) \in T \times P$, let $\xi_i''(x, y, p)$ be the convex hull of $\xi_i'(x, y, p)$. Then, for every $i \in I$, the correspondence ξ_i'' of $T \times P$ into R^l is upper semi-continuous on $T \times P$ and $\xi_i''(x, y, p)$ is non-empty and convex for every $(x, y, p) \in T \times P$ (cf. Nikaido [9, Theorem 4.8]). For each $(x, y, p) \in T \times P$, let

$$F'(x, y, p) = (\prod_{i \in I} \xi_i''(x, y, p)) \times (\prod_{j \in J} \eta_j(x, y, p)) \times \mu(x, y).$$

Then the correspondence F' of $T \times P$ into itself is upper semi-continuous on $T \times P$ and $F'(x, y, p)$ is non-empty and convex for every $(x, y, p) \in T \times P$. By the fixed-point theorem of Eilenberg and Montgomery, F' has a fixed-point (x^*, y^*, p^*).

By (C.3′), there is $x^0 \in \prod_{i \in I} X_i(x^*_{)i(}, y^*)$ such that $\sum_{i \in I} w_i(x^*, y^*, p^*) = p^* \cdot \omega_I + p^* \cdot \sum_{j \in J} \pi_j(x^*, y^*, p^*) \geqq p^* \cdot \omega_I > p^* \cdot x_I^0$, and therefore $w_k(x^*, y^*, p^*) > p^* \cdot x_k^0$ for some $k \in I$. It follows from the choice of K that $w_k(x^*, y^*, p^*) > \min p^* \cdot X_k'(x^*, y^*)$, i.e. $(x^*, y^*, p^*) \in Q_k$. Since ξ_k coincides with ξ_k'' on Q_k, it follows that $x_k^* \in \xi_k(x^*, y^*, p^*)$.

Suppose $p_h^* = 0$ for some $h \in H$. By (C.7′), there is a positive real number λ such that $x_k(\lambda, h) \in X_k(x^*_{)k(}, y^*)$ and $((x_k(\lambda, h), x^*_{)k(}), y^*) \succ_k (x^*, y^*)$. Let $x_k(t) = (1-t)x_k^* + t x_k(\lambda, h)$. Then, for every $t \in]0, 1[$, $((x_k(t), x^*_{)k(}), y^*) \succ_k (x^*, y^*)$ by (C.6) and clearly $p^* \cdot x_k(t) \leqq p^* \cdot \omega_k + \theta_k \cdot (p^* \cdot y^*)$. Since $x_k^* \in \text{int } K$, it follows that $x_k(t) \in X_k'(x^*, y^*)$ for every $t \in]0, 1[$ close enough to 0. This contradicts the fact that $x_k'' \in \xi_k(x^*, y^*, p^*)$. Consequently $p^* > 0$.

Therefore $(x^*, y^*, p^*) \in T \times P^0 \subset Q_i$ for every $i \in I$, so that $x^* \in \prod_{i \in I} \xi_i(x^*, y^*, p^*)$. The remaining part of the proof is identical with that of Section III. Hence (x^*, y^*, p^*) is a competitive equilibrium.

Keio University

REFERENCES

[1] Arrow, K. J. and Hahn, F. H., *General Competitive Analysis* (Holden-Day, 1971).

[2] Berge, C., *Topological Spaces* (Macmillan, 1963).

[3] Debreu, G., *Theory of Value* (John Wiley and Sons, 1959).

[4] Eilenberg, S. and Montgomery, D., "Fixed Point Theorems for Multi-Valued Transformations," *American Journal of Mathematics*, 68 (1946), 214–222.

[5] Kuratowski, K., *Topology*, Vol. 1 (Academic Press, 1966).

[6] Laffont, J. J. and Laroque, G., "Effets externes et theorie de l'equilibre general," *Cahiers du Seminaire d'Econometrie* (Centre National de la Recherche Scientifique, 1972).

[7] Marshall, A., *Principles of Economics* (Macmillan, 8th edition, 1920).

[8] McKenzie, L. W., "Competitive Equilibrium with Dependent Consumer Preferences," H. A. Antosiewicz, ed., *Proceedings of the Second Symposium in Linear Programming* (National Bureau of Standards, 1955), 277–294.

[9] Nikaido, H., *Convex Structures and Economic Theory* (Academic Press, 1968).

[10] Osana, H., "On the Boundedness of an Economy with Externalities," *Review of Economic Studies*, 40 (1973), 321–331.

[11] Osana, H., "Optimal Tax-Subsidy System for an Economy with Marshallian Externalities," *Econometrica*, 45 (1977), 329–340.