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MULTIVARIATE NORMAL DISTRIBUTIONS AND THEIR APPLICATIONS TO THE DYNAMIC PROPERTIES OF MACRO-ECONOMETRIC MODELS

HIROYUKI KOSAKA

I. INTRODUCTION

The simulation results by Adelmans [1] in 1959 made a milestone in that they recognized the positive role of "random shocks" in econometric models during the course of cyclical processes in business cycles. And their conclusions remain valid till present in spite of the developments of economic theories and estimation techniques. (See Hickman [4]) In the 1960's, the applications of stationary stochastic process to the economic systems have been made theoretically and empirically to investigate the dynamic aspects of econometric models in conjunction with re-recognition of random shocks. In their analysis, instead of excluding the role of random shocks in the cyclical phenomena by removing the random parts in econometric models after estimation, they evaluated the contribution of random shocks in cyclical processes. The spectral theory looks into the cyclical property by the notion of spectral density under the stationarity assumption, and this short note also belongs to the same category in the sense that it assumes the stationarity.

Given a stochastic simultaneous linear difference equations, we can examine their dynamic properties of time path in terms of spectral density (spectral analysis) or autocovariances (equivalently autocorrelations) drawn from the system. In this note we make use of autocorrelations which have the same information amounts as the spectral density has, and from them extract useful informations about their time path.

In section II, we shall pick up some known propositions about multivariate normal distributions to prove Dodd's [3] formulas and to make some further extensions. In section III, we shall show that if the distrubances are normal in simultaneous autoregressive equations with moving average disturbances, then the endogenous variables are normal. Under normality and stationarity we shall propose a procedure to investigate the dynamic properties of time series. Finally, the crossing properties of Klein I-model [8] are studied in section IV.

II. MULTIVARIATE NORMAL DISTRIBUTIONS AND THEIR APPLICATIONS

In order to calculate the probabilities of multivariate normal distributions, I posit some propositions without proofs. The first one is simple, but it is im-

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portant to prove Dodd's formulas of crossing.

PROPOSTION 1 (Anderson [2]).

If

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}\right\},$$

then

$$P[x \geqq 0, y \geqq 0] = \frac{1}{4} + \frac{\sin^{-1}\rho_{xy}}{2\pi} = \frac{1}{2} - \frac{\cos^{-1}\rho_{xy}}{2\pi}. \tag{2.1}$$

The following two propositions are proved by Kendall [7] and used to prove another Dodd's formula.

PROPOSITION 2.

If

$$\begin{bmatrix} x \\ y \end{bmatrix} \sim N\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}\right\},$$

then

$$P\Big[x \geqq h_1, y \geqq h_2\Big] = \sum_{r=0}^{\infty} \rho_{xy}^r \tau_r(h_1)\tau_r(h_2) \tag{2.2}$$

where $\tau_r(\omega)$ is so-called "tetrachoric function" defined by

$$\tau_r(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{(-D)^r e^{-\omega^2/2}}{(r!)^{1/2}}$$

(*D: differential operator*)

This proposition is a general case of the first one and the next proposition is an extension to three-variable case of the above.

PROPOSITION 3.

If

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \sim N\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & \rho_{xy}\sigma_x\sigma_y & \rho_{xz}\sigma_x\sigma_z \\ \rho_{xy}\sigma_x\sigma_y & \sigma_y^2 & \rho_{yz}\sigma_y\sigma_z \\ \rho_{xz}\sigma_x\sigma_z & \rho_{yz}\sigma_y\sigma_z & \sigma_z^2 \end{bmatrix}\right\},$$

then

$$P[x \geqq h_1, y \geqq h_2, z \geqq h_3] = \sum_{j,k,l=0}^{\infty} \frac{\rho_{xy}^j \rho_{yz}^k \rho_{xz}^l}{j!\,k!\,l!} \tau_{j+k}(h_1)\tau_{j+l}(h_2)\tau_{k+l}(h_3). \tag{2.3}$$

Now let us prove Dodd's formulas using above propositions. Let x_t be real Gaussian weakly stationary process with $E(x_t) = 0$, $\gamma_k = E(x_t x_{t-k})$ and $\rho_k = \gamma_k/\gamma_0$. We put $y_t = x_{t+1} - x_t$ and $z_t = x_t - x_{t-1}$, then $E(y_t) = E(z_t) = 0$, $\sigma_{yz} = E(y_t z_t) = 2\gamma_1 - \gamma_0 - \gamma_2$, $\sigma_y^2 = E(y_t^2) = 2(\gamma_0 - \gamma_1)$ and $\sigma_z^2 = E(z_t^2) = 2(\gamma_0 - \gamma_1)$.

Since

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} \sim N\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho_{yz}\sigma_y\sigma_z \\ \rho_{zy}\sigma_y\sigma_z & \sigma_z^2 \end{bmatrix}\right\},$$

we can apply the first proposition and get

$$(2.4) \qquad P[x_{t+1} \leqq x_t, x_t \geqq x_{t-1}] = P[x_{t+1} - x_t \leqq 0, x_t - x_{t-1} \geqq 0]$$
$$= P[y_t \leqq 0, z_t \geqq 0] = \frac{1}{2} - P[y_t \geqq 0, z_t \geqq 0]$$
$$= \frac{1}{2\pi} \cos^{-1} \rho_{yz} = \frac{1}{2\pi} \cos^{-1}\left[\frac{2\gamma_1 - \gamma_0 - \gamma_2}{2(\gamma_0 - \gamma_1)}\right]$$
$$= \frac{1}{2\pi} \cos^{-1}\left[\frac{2\rho_1 - 1 - \rho_2}{2(1 - \rho_1)}\right].$$

Similarly $\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \sim N\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{bmatrix}\right\}$ and then we obatin

$$(2.5) \qquad P[x_{t-1} \leqq 0, x_t \geqq 0] = \frac{1}{2} - P[x_{t-1} \geqq 0, x_t \geqq 0]$$
$$= \frac{1}{2\pi} \cos^{-1} \frac{\gamma_1}{\gamma_0} = \frac{1}{2\pi} \cos^{-1} \rho_1.$$

Therefore, the mean distance between peaks (troughs) and the mean distance between upcrosses (downcrosses) are easily obtained.

Mean distance between peaks (MDP)

$$(2.6) \qquad = \frac{1}{P[x_{t+1} \leqq x_t, x_t \geqq x_{t-1}]} = \frac{2\pi}{\cos^{-1}\left[\frac{2\rho_1 - 1 - \rho_2}{2(1 - \rho_1)}\right]}.$$

Mean distance between upcrosses (MDU)

$$(2.7) \qquad = \frac{1}{P[x_{t-1} \leqq 0, x_t \geqq 0]} = \frac{2\pi}{\cos^{-1} \rho_1}.$$

Another formula by Dodd is the mean distance betweenpeaks without ripples. Let

$$y_t = x_{t+1} - x_t, z_t = x_t - x_{t-1} \quad \text{and} \quad w_t = x_t - x_{t-k},$$

then from the assumptions

$$E(y_t) = E(z_t) = E(w_t) = 0, \quad \sigma_y^2 = E(y_t^2) = 2(\gamma_0 - \gamma_1), \quad \sigma_z^2 = E(\sigma_z^2) = 2(\gamma_0 - \gamma_1),$$
$$\sigma_w^2 = E(w_t^2) = 2(\gamma_0 - \gamma_k), \quad \rho_{yz} = E(y_t z_t) = 2\gamma_1 - \gamma_0 - \gamma_2, \sigma_{zw} = E(z_t w_t)$$
$$= \gamma_0 - \gamma_1 + \gamma_{k-1} - \gamma_k \quad \text{and} \quad \sigma_{zw} = E(y_t w_t) = \gamma_1 - \gamma_0 + \gamma_k - \gamma_{k+1}.$$

Since (y_t, z_t, w_t) distributes as multivariate normal distribution dependent upon the above parameters, we calculate the following probability using the first and the third propositions.

$$
\begin{aligned}
(2.8)\quad & P[x_{t+1} \leqq x_t, x_t \geqq x_{t-1}, x_t \geqq x_{t-k}] \\
& = P[x_{t+1} - x_t \leqq 0, x_t - x_{t-1} \geqq 0, x_t - x_{t-k} \geqq 0] \\
& = P[y_t \leqq 0, z_t \geqq 0, w_t \geqq 0] = P[z_t \geqq 0, w_t \geqq 0] \\
& \quad - P[y_t \geqq 0, z_t, \geqq 0, w_t \geqq 0] \\
& = \frac{1}{2}\left[1 - \frac{1}{\pi}\cos^{-1}\rho_{zw}\right] - \sum_{j,k,l=0}^{\infty} \frac{\rho_{yz}^j \rho_{zw}^k \rho_{yw}^l}{j!\,k!\,l!} \tau_{j+k}(0)\tau_{j+l}(0)\tau_{k+l}(0).
\end{aligned}
$$

So the mean distance between peaks without ripples is expressed by the inverse number of the probability (2.8).

Above observations show that the relationships among different time points of time series are described by its autocorrelations with lags of less and equal order. Probabilities associated with multivariate normal distributions can be obtained analytically in special cases as pointed out above. Further more, let us consider some interesting probabilities as Dodd's extensions that are meaningful in economic phenomena. If these probabilities are known, the mean distance or period that these events will occur can be obtained easily.

1) Probability of h-level upcrossing.
   When "h" equals zero, it is Dodd's case. We can calculate the probability $P(x_{t-1} \leqq h, x_t \geqq h] = P[x_t \geqq h] - P[x_{t-1} \geqq h, x_t \geqq h]$.
2) Probability that the height of peak is over h-level.
   The probability $P[x_{t-1} \leqq x_t, x_t \geqq x_{t-t}, x_t \geqq h] = P(x_t - x_{t+1} \geqq 0, x_t - x_{t-t} — 0, x_t \geqq h]$ is the special case of the third proposition with $h_1 = 0$, $h_2 = 0$ and $h_3 = h$.
3) Probability that the changing rate is over h-level.
   $P[(x_t - x_{t-1})/x_{t-1} \geqq h] = P[x_t - (1 + h)x_{t-1} \geqq 0] = P[y_t \geqq 0]$ is a univariate probability with mean zero and variance $(1 + h)^2 \gamma_0 + \gamma_0 - 2(1 + h)\gamma_1$.
4) Probability of h_1-level upcrossing with its rising rate over h_2-level.
   $P[x_t \geqq h_1, x_{t-1} \leqq h_1, (x_t - x_{t-1})/x_{t-1} \geqq h_2] = P[x_t \geqq h_1, x_t - (1+h_2)x_{t-1} \geqq 0] - P[x_t \geqq h_t, x_{t-1} \leqq h_1, x_t - (1 + h_2)x_{t-1} \geqq 0]$ is a special case of (2.2) and (2.3)

Taking some examples of interesting probabilities, we can observe that these are combined results of finite or infinite number of autocorrelations. Generally, according to the aim of analysis we may adopt appropriate probabilities, and then compute them in terms of autocorrelations that have the full information about various second order properties of time series.

III. APPLICATIONS TO ECONOMETRIC MODELS

Normality of time series generated by autoregressive model plays an important role in the application of the methods in section II. Here we show the normality only in the case of simultaneous autoregressive model with moving average disturbances, in a similar way as with white noise disturbances.

Given the following equation system

$$
\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & 1 & \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{mt} \end{pmatrix} + \begin{pmatrix} a_{11}^{(1)} a_{12}^{(1)} & \cdots & a_{1m}^{(1)} \\ a_{21}^{(1)} a_{22}^{(1)} & \cdots & a_{2m}^{(1)} \\ & & \\ a_{m1}^{(1)} a_{m2}^{(1)} & \cdots & a_{mm}^{(1)} \end{pmatrix} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \\ \vdots \\ x_{mt-t} \end{pmatrix} + \cdots \tag{3.1}
$$

$$
+ \begin{pmatrix} a_{11}^{(n)} a_{12}^{(n)} & \cdots & a_{1m}^{(n)} \\ a_{21}^{(n)} a_{22}^{(n)} & \cdots & a_{2m}^{(n)} \\ \vdots & & \\ a_{m1}^{(n)} a_{m2}^{(n)} & \cdots & a_{mm}^{(n)} \end{pmatrix} \begin{pmatrix} x_{1t-n} \\ x_{2t-n} \\ \vdots \\ x_{mt-n} \end{pmatrix} = \begin{pmatrix} u_{1t} \\ u_{2t} \\ \vdots \\ u_{mt} \end{pmatrix}
$$

In matrix notation

$$
A(L)X_t = [I + A_1(L) + A_2(L) + \dots + A_n(L)]\, X_t = U_t \tag{3.2}
$$

The characteristic roots of

$$
\mathrm{A}(\rho) = 0 \tag{3.3}
$$

are assumed to lie outside the unit circle and U_t is finite moving average of non-autocorrelated variables ε_t that identically distributes as $N(\mathbf{0}, \Sigma_\varepsilon)$ (Σ_ε: positive definite);

$$
U_t = D(L)\varepsilon_t = \begin{pmatrix} d_1(L) & & & \\ & d_2(L) & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & d_m(L) \end{pmatrix} \varepsilon_t \tag{3.4}
$$

where $d_i(L) = d_i^{(0)} + d_i^{(1)}L + d_i^{(2)}L^2 + \dots + d_i^{(r_i)}L^{r_i}$ $(i = 1, 2, \dots, m)$ and $|d_i(\rho)| = 0$ $(i = 1, 2, \dots, m)$ have the roots of modulus larger than unity. Then the stationary solution is expressed by

$$
X_t = \sum_{s=0}^{\infty} H_s U_{t-s} = \sum_{s=0}^{\infty} H_s D(L)\varepsilon_{t-s} = \sum_{s=0}^{\infty} H_s^* \varepsilon_{t-s} \tag{3.5}
$$

Σ_ε is a positive definite matrix and there exists a orthogonal matrix such that

$$
T\Sigma_\varepsilon T' = \Lambda = \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \lambda_2 & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_m \end{bmatrix} \tag{3.6}
$$

where $\lambda_i (i = 1, 2, \dots, m)$ are the characteristic roots of Σ_ε. We transform H_s^* and ε_t in the following way:

$$
\begin{aligned} \tilde{H}_s^* &= H_s^* T' \Lambda^{-1/2} \\ \tilde{\varepsilon}_t &= \Lambda^{1/2} T \varepsilon_t \end{aligned} \tag{3.7}
$$

$$\Lambda^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & 0 \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_m} \end{pmatrix} \quad \Lambda^{-1/2} = \begin{pmatrix} 1/\sqrt{\lambda_1} & & & \\ & 1/\sqrt{\lambda_2} & & 0 \\ & & \ddots & \\ 0 & & & 1/\sqrt{\lambda_m} \end{pmatrix}$$

where we see

$$E(\tilde{\varepsilon}_t) = 0 \qquad (3.8)$$
$$E(\tilde{\varepsilon}_t\tilde{\varepsilon}_t') = I$$

The moving average form (3.5) can be transformed as,

$$X_t = \sum_{s=0}^{\infty} H_s^*\varepsilon_{t-s} = \sum_{s=0}^{\infty} (H_s^*T'\Lambda^{-1/2})(\Lambda^{1/2}T\varepsilon_{t-s}) = \sum_{s=0}^{\infty} \tilde{H}_s^*\tilde{\varepsilon}_{t-s} \qquad (3.9)$$

Hence each element of X_t is a infinite linear combination of independent normal variables and the normality is guaranteed with $E(X_t) = 0$.

The autoregressive model with normal moving average residuals generates normal variables under certain assumptions as seen above. So if we are interested in the dynamic properties of autoregressive model, we must first obtain its sequence of autocorrelations. The Yule-Walker equation gives autocorrelations in the single-autoregressive case and the methods are suggested in Otsuki [9] in the multivariate case. Various probabilities of events such as in section II are computed exactly or approximately in terms of finite number of autocorrelations. And the inverse number of probability is the mean distance or period of the event.

It should be noticed that such mean distance methods reveal the time pattern or time shape of normal time series rather than cycle itself, some of which would be examined in the next section using a simple famous model.

The above process of analyzing the dynamic properties of time series is not limited to model analysis, but we can directly calculate the autocorrelations from the original data and analyze them.

IV. EXAMPLES

Let us illustrate numerically the crossing interval properties of Klein's six equations model (See [8]) as an application of preceeding analysis. The Klein's I model is trnsformed to a single autoregressive form for national income as follows;

$$\begin{aligned} (4.1) \quad & Y_t + a_1Y_{t-1} + a_2Y_{t-2} + a_3Y_{t-3} \\ & = b_0 + b_1(t-1931) + b_2W_{2t} + b_3W_{2t-1} + b_4W_{2t-2} + b_5W_{2t-3} \\ & \quad + b_6T_t + b_7T_{t-1} + b_8T_{t-2} + b_9T_{t-3} + b_{10}G_t + b_{11}G_{t-1} \\ & \quad + c_1u_{1t} + c_2u_{1t-1} + c_3u_{2t} + c_4u_{2t-1} \\ & \quad + c_5u_{3t} + c_6u_{3t-1} + c_7u_{3t-2}\,. \end{aligned}$$

The exogenous variables (T_t, G_t, W_{2t}) are approximated by linear trend as,

$$T_t = 6.8048 + 0.2286\ (t - 1931) + w_{1t}$$
(4.2)
$$G_t = 9.9143 + 0.5477\ (t - 1931) + w_{2t}$$
$$W_{2t} = 5.1190 + 0.3091\ (t - 1931) + w_{3t}\,.$$

Then the unhomogeneous form (4.1) becomes homogeneous one;

(4.3)
$$\begin{aligned} y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \alpha_3 y_{t-3} &= \beta_1 w_{1t} + \beta_2 w_{1t-1} + \beta_3 w_{1t-2} + \beta_4 w_{1t-3} \\ &+ \beta_5 w_{2t} + \beta_6 w_{2t-1} \\ &+ \beta_7 w_{3t} + \beta_8 w_{3t-1} + \beta_9 w_{3t-2} + \beta_{10} w_{3t-3} \\ &+ \gamma_1 u_{1t} + \gamma_2 u_{1t-1} \\ &+ \gamma_3 u_{2t} + \gamma_4 u_{2t-1} \\ &+ \gamma_5 u_{3t} + \gamma_6 u_{3t-1} + \gamma_7 u_{3t-2} \end{aligned}$$

The coefficients of (4.3), recuced from several estimates (ordinary least squares estimator*, two stage least squares estimator*, limited information maximum likelihood estimator* and Sawa's [10] combined estimator) are tabulated in Table I The residual term u_t is computed from the above identity, and its autocovariance estimates $\hat{\gamma}_u(k)(k = 0, 1, 2, \ldots, T-1)$ lead to the spectrum through the usual formula:

TABLE 1. ESTIMATES OF COEFFICIENTS

	OLS	2SLS	LIML	COMB
α_1	−1.7121	−1.8377	−1.8627	−1.8485
α_2	1.1037	1.1732	1.1765	1.1783
α_3	−0.2209	−0.2127	−0.2103	−0.2115
β_1	−3.4628	−1.3043	−0.7972	−1.0830
β_2	2.5471	0.6252	0.2297	0.4301
β_3	0.4081	0.2766	0.2135	0.2623
β_4	0.2209	0.2127	0.2103	0.2115
β_5	3.6618	1.8167	1.3765	1.6274
β_6	−3.2519	−1.5301	−1.1449	−1.3534
β_7	0.2538	0.6552	0.7557	0.6963
β_8	−1.0494	−1.5472	−1.6595	−1.5960
β_9	1.1037	1.1732	1.1765	1.1783
β_{10}	−0.2209	−0.2127	−0.2103	−0.2115
γ_1	3.6618	1.8167	1.3765	1.6274
γ_2	−3.2519	−1.5301	−1.1449	−1.3534
γ_3	3.6618	1.8167	1.3765	1.6274
γ_4	−3.6618	−1.8167	−1.3765	−1.6274
γ_5	0.4528	1.1675	1.3351	1.2408
γ_6	−1.7541	−2.4521	−2.5748	−2.5193
γ_7	1.5118	1.4498	1.3900	1.4406

OLS . . . Ordinary Least Squares LIML . . . Limited Information Maximum Likelihood
2SLS . . . Two Stage Least Squares COMB . . . Combined estimater

* The exact estimates were provided by Prof. K. Mori.

(4.4) $$\hat{f}_u(\lambda) = \frac{1}{2\pi}\left[\hat{\gamma}_u(0) + 2\sum_{k=1}^{m-1}\hat{\gamma}_u(k)\lambda_k \cos \lambda k\right]$$

where m is the truncation point (we choose a few points) and $\lambda_k(k = 1, 2, \ldots, m - 1)$ is the covariance averaging kernel (we use Parzen's and Tucky-Hanning's method [6]). The spectrum of y_t is determined by

(4.5) $$\hat{f}_y(\lambda) = \frac{1}{|1 + \alpha_1 e^{i\lambda} + \alpha_2 e^{2i\lambda} + \alpha_3 e^{3i\lambda}|^2}\hat{f}_u(\lambda).$$

Its autocovariances are calculated from

(4.6) $$\hat{\gamma}_y(k) = \int_{-\pi}^{\pi} \cos\, k\lambda \hat{f}_y(\lambda)\, d\lambda.$$

Mean distance between peaks and mean distance between upcrosses are computed from the above estimates $\hat{\gamma}_y(k)$ $(k = 1, 2, \ldots)$ and are presented in Tables II and III respectively. As time series moves upward and downward about its mean curve, mean distance between upcrosses is apt to have longer interval than mean distance between peaks in general.

Here in this example the former is about twice as long as the latter. Table IV shows the quotient (mean distance between peaks)/(mean distance between upcrosses), which diplicts the time pattern or time shape of this time series as in Figure 1. About the truncation point, the smaller is the value we take (i.e. the more we neglect the informations that the autocovariances have), the longer are the mean intervals. Between kernels, Parzen kernel yields longer intervals. Among estimators, we can divide them into two groups (ordinary least squares estimator group and two stage least squares-limited information maximum likelihood-Sawa's combined estimators group), and the former gives shorter intervals.

TABLE 2. MEAN DISTANCE BETWEEN PEAKS

Kernel	m	OLS	2SLS	LIML	COMB
	4	6.70	7.88	8.24	8.04
PARZEN	6	6.32	7.28	7.40	7.40
	8	6.03	6.85	6.98	6.96
	4	6.41	7.42	7.54	7.54
TUCKY-	6	6.01	6.79	6.91	6.91
HANNING	8	5.67	6.38	6.54	6.50

TABLE 3. MEAN DISTANCE BETWEEN UPCROSSES

Kernel	m	OLS	2SLS	LIML	COMB
	4	11.44	14.18	15.52	14.53
PARZEN	6	9.99	12.64	13.94	13.02
	8	8.97	11.51	12.77	11.91
	4	10.33	13.03	14.35	13.40
TUCKY-	6	8.72	11.18	12.42	11.58
HANNING	8	7.95	10.41	11.67	10.83

TABLE 4. MEAN DISTANCE BETWEEN PEAKS/MEAN DISTANCE BETWEEN UPCROSS

Kernel	m	OLS	2SLS	LIML	COMB
	4	0.586	0.556	0.518	0.552
PARZEN	6	0.633	0.576	0.531	0.568
	8	0.672	0.595	0.547	0.584
	4	0.621	0.570	0.525	0.562
TUCKY-	6	0.689	0.607	0.556	0.597
HANNING	8	0.713	0.613	0.560	0.600

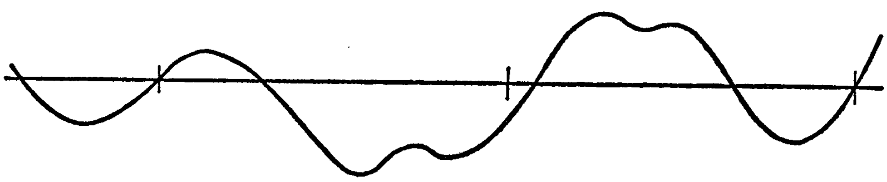

Figure 1. Time shape of Klein I model.

Now let us compare the above results with the other investigations by different methods. In conventional method from difference equation with ignoring disturbance terms, the period is 14.75 year, and in Howrey and Kalejian work [5] (spectral method with accepting disturbance terms) it is 13.33 year. And Otsuki [9] derived shorter cycle (7.30) supposing y_t is subject to a stationary stochastic process. Therefore, the former two measurements belong to the results of the mean distance between upcrosses and Otsuki's to those of the mean distance between peaks. In comparison with the quotients in Table IV, the ratio of Otsuki-cycle and Howrey-Kalejian-cycle (0.5476) is suggestive of the fact that both cycles have wide difference.

In conclusion possibly we can interprete that Howrey and Kalejian have extracted longer cycle while Otsuki has measured the shorter cycle in Figure 1.

Concerning the applications to large scale models, the process of reducing a model to so-called "solved from" as in (4.1) will be the most complicated problem. Even in Klein's I model obtaining solved from by manual operation is troublesome. However the merit of the solved form is that we can see how the policy and exogenous variables influence the determination of the endogenous variable, for instance national income, dynamically as seen in (4.1).

We established a simple algorithm to obtain the solved form. The solved form of concerned endogenous variables is a linear combination of its lag terms, the current and the lag terms of both exogenous variables and disturbances. Therefore, given an econometric model, we continue to add a lagged equation to the model unitl we can solve the concerned variables from the system which contains more equation than the original model. After all the procedure results in solving a large-scale linear simultaneous equations. Finally, there remains the general problem of non-linearity. Whenever there are price variables in the macro-model, non-linearity of product or quotient appears. Furthermore, when we intend to reflect faithfully the theoretical developments in specifying some behavioral equations, we are apt to obtain non-linear specifications. Thus, in the general situation where non-linearity are popular, the analytical methods including this analysis to investigate the dynamic properties must make models linear as a first step, which would be a serious restriction.

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