MULTICOLLINEARITY AND THE EXACT DISTRIBUTION OF THE LEAST SQUARES ESTIMATORS IN THE ERROR MODELS

KAZUHIKO MATSUNO

I. INTRODUCTION

The multicollinearity problem is one of the classical problems of econometric methods. Multicollinearity effects on the Least Squares Estimator (LS), however, have hardly ever been discussed with a precise stochastic specification.

When the multicollinearity problem of the classical fixed variable regression model is discussed, we speak of a singularity of moment matrices. And stochastic characteristics of the moment matrices do not appear. For the multicollinearity problem of the error in variables model we speak of effects of observational errors which hide nuisance intercorrelations between explanatory variables. And it is warned that one might have nonsensical regression estimates.

This paper discusses a sampling distribution aspect of the LS in relation to the multicollinearity problem. In particular, the exact sampling distribution of the LS is obtained in the context of the error in variables models where multicollinearity is present. And some properties of the distribution are examined.

A breakthrough leading to the present analysis was established when Anderson and Girshick [1] presented probability density functions of the noncentral Wishart distribution and Tintner [9] noted significances of the distribution for econometric models. The noncentral Wishart distribution, however, became well known when it was utilized to obtain the sampling distributions of the econometric estimators in the simultaneous equations models.

The sampling distribution analysis of the simultaneous-equation models applies to the error in variables models. For a similarity exists between the stochastic natures of the simultaneous equations model and the error in variables models. Richardson and Wu [8] along this line present the sampling distributions of the LS and the Grouping Method Estimators of the error in variables model with two variables included. But there is no multicollinearity problem in the two variable model. The problem occures in models with three variables or more.

It is thought that sampling distribution problem becomes complicated when the error in variables model includes more than three variables. Presence of multicollinearity, however, simplifies the matter. For, in some cases, the sampling distribution problem can be reduced to that of the two variable model by the presence of multicollinearity, even if the model includes more than three variables.

Section II discusses specifications of the error-in-variables model and multicollinearity. And Section III presents our error models and the sampling distribution problems. Section IV provides a general statistical theory and our major interest is in its special case. In Section V, the results of the statistical theory are applied to the problems of Section III. Another application is given in Section VI. And this has a specific meaning in an econometric analysis.

II. ERROR IN VARIABLES MODELS AND MULTICOLLINEARITY

There have been two types of stochastic specification in the studies of the error in variables models. One type of specification is that the observations subject to economic interrelationships and measurement errors are distributed as normal with a constant mean value through observation periods. With a stochastic independence assumption this leads to the central Wishart distribution of the moment matrix of the observations.

The other one is that the observations are distributed as normal with varying mean values through observation periods. The moment matrix is then shown to be distributed as the noncentral Wishart distribution. This is a generalized form of the previous type in a statistical sense.

The moment matrix A of the noncentral Wishart distribution model consists of systematic variations A' and random variations A'' additively, i.e., $A = A' + A''$. Whereas the moment matrix of the central Wishart distribution model consists only of random variations A''. Therefore, when Farrar and Glauber [2] provide methods for detecting multicollinearity based on the central Wishart distribution, the multicollinearity refers to a singularity of A'' or to a singularity of a covariance matrix of measurement error vectors, a population counterpart of A''. But we understand that multicollinearity refers to presence of simultaneous relationships between systematic parts of economic variables. Therefore reference should be made to a singularity of the systematic variatition A' rather than a singularity of the random variation A'' for detecting multicollinearity, *see* Tintner [9].

In Marschack's terminology, [6], the central Wishart distribution model is self-contained and the noncentral Wishart distribution model is sectional Our model will be a sectional model with the A' of rank unity.

III. MODELS

Our error in variables models admit multicollinearity and shock disturbances, *see* Tintner [9] and Haavelmo [3].

The relationship between the true (latent) variables ξ_{kt},

$$(3.1) \qquad \xi_{1t} = -\beta_0 + \beta_2\xi_{2t} + \cdots + \beta_K\xi_{Kt} + u_t\,, \qquad t = 1, \cdots, T\,,$$

is estimated by the LS. Here the β_k are unknown parameters and T is a smaple size. The u_t are interpreted as shock disturbances.

The observations x_{kt} of the true variables are subject to the measurement errors ε_{kt},

$$x_{\cdot t} \equiv \begin{bmatrix} x_{1t} \\ \vdots \\ x_{Kt} \end{bmatrix} = \begin{bmatrix} \xi_{1t} \\ \vdots \\ \xi_{Kt} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{Kt} \end{bmatrix} \equiv \xi_{\cdot t} + \varepsilon_{\cdot t} . \tag{3.2}$$

The equations (3.1) and (3.2) constitute a simple shock and error model.

Let $l = [1 \cdots 1] = [1 \times T]$, $M = I - l(l'l)^{-1}l'$, $X' = [x_{\cdot 1} \cdots x_{\cdot T}]$ and

$$A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{c:c} 1 \times 1 & 1 \times \overline{T-1} \\ \hdashline \overline{T-1} \times 1 & \overline{T-1} \times \overline{T-1} \end{array}\right] \tag{3.3}$$

$$= X'MX = \sum_{t=1}^{T} \left(x_{\cdot t} - \frac{1}{T} \sum x_{\cdot t}\right)\left(x_{\cdot t} - \frac{1}{T} \sum x_{\cdot t}\right)' ,$$

then we have, for the LS $b' = [b_2 \cdots b_K]$ of $\beta' = [\beta_2 \cdots \beta_K]$,

$$b = A_{22}^{-1} A_{21} . \tag{3.4}$$

It is assumed that R "subsidiary" relationships exist between the ξ_{kt},

$$\begin{bmatrix} \alpha_{10} \\ \vdots \\ \alpha_{R0} \end{bmatrix} + \begin{bmatrix} \alpha_{12} & \cdots & \alpha_{1K} \\ \cdot & \cdots & \cdot \\ \alpha_{R2} & \cdots & \alpha_{RK} \end{bmatrix} \begin{bmatrix} \xi_{2t} \\ \vdots \\ \xi_{Kt} \end{bmatrix} = \begin{bmatrix} w_{1t} \\ \vdots \\ w_{Rt} \end{bmatrix} , \tag{3.5}$$

where the w_{rt} are shock disturbances. And the ξ_{kt} are said to be multicollinear. The occurence of multicollinearity might be accidental in the sense that the equations (3.5) hold only during the observation periods. Otherwise, the equations (3.5) describe inherent ecnomic relationships ruling the explanatory variables. And they are possibly unknown to an economist whose interest is in (3.1)[1].

The ξ_{kt} are *near* multicollinear since the equations (3.1) and (3.5) are stochastic. The ξ_{kt} are *exact* multicollinear when the shocks are empty.

A small value of R corresponds to large variations, in terms of degrees of freedom, of the ξ_{kt}. The analysis is confined to the cases of $R = K - 2$. Therefore, the model is sectional but close to self contained cases. One "exogenous" variable, say ξ_{Kt}, and the equations system (3.1) and (3.5) determine the systematic variations of the observations and the moment matrix.

The sampling distribution problems are based on the stochastic assumptions:

ξ_{kt} are fixed in repeated samples,

$$\begin{bmatrix} u_t \\ w_{\cdot t} \\ \varepsilon_{\cdot t} \end{bmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \begin{bmatrix} \sigma_{uu} & 0 & 0 \\ 0 & \Sigma_{ww} & 0 \\ 0 & 0 & \Sigma_{\varepsilon\varepsilon} \end{bmatrix}\right) , \tag{3.6}$$

$$\begin{bmatrix} u_t \\ w_{\cdot t} \\ \varepsilon_{\cdot t} \end{bmatrix} \text{ and } \begin{bmatrix} u_s \\ w_{\cdot s} \\ \varepsilon_{\cdot s} \end{bmatrix} \text{ are independent,}$$

[1] If one thinks that an objective of his investigation is the equations system (3.5) as well as (3.1), then methods of treating (3.1) and (3.5) simultaneously must be devised. We consider (3.5) as relationships reflecting different dimensions of an economy from that of (3.1).

where $w'_{\cdot t} = [w_{1t} \cdots w_{K-2t}]$.

Simple calculations give the recuced form

$$(3.7) \qquad x_{\cdot t} = \begin{bmatrix} \beta' \\ I \end{bmatrix} \gamma \xi_{Kt} - \begin{bmatrix} 1 & \Gamma \\ 0 & \end{bmatrix} \begin{bmatrix} \beta_0 \\ \alpha_{\cdot 0} \end{bmatrix} + \begin{bmatrix} 1 & \Gamma & I \\ 0 & & \end{bmatrix} \begin{bmatrix} u_t \\ w_{\cdot t} \\ \varepsilon_{\cdot t} \end{bmatrix}$$

$$= \pi \xi_{Kt} + \pi_0 + v_{\cdot t},$$

where

$$\pi = \begin{bmatrix} \beta' \\ I \end{bmatrix} \gamma, \quad \pi_0 = - \begin{bmatrix} 1 & \Gamma \\ 0 & \end{bmatrix} \begin{bmatrix} \beta_0 \\ \alpha_{\cdot 0} \end{bmatrix}, \quad v_{\cdot t} = \begin{bmatrix} 1 & \Gamma & I \\ 0 & & \end{bmatrix} \begin{bmatrix} u_t \\ w_{\cdot t} \\ \varepsilon_{\cdot t} \end{bmatrix},$$

$$(3.8) \quad \gamma = \begin{bmatrix} -D^{-1}\alpha_{\cdot K} \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} \alpha_{12} & \cdots & \alpha_{1K-1} \\ \cdot & \cdots & \cdot \\ \alpha_{K-2K} & \cdots & \alpha_{K-2K-1} \end{bmatrix}, \quad \alpha_{\cdot K} = \begin{bmatrix} \alpha_{1K} \\ \vdots \\ \alpha_{K-2K} \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} \beta' \begin{pmatrix} D^{-1} \\ 0 \end{pmatrix} \\ \begin{pmatrix} D^{-1} \\ 0 \end{pmatrix} \end{bmatrix}.$$

In view of (3.6), the distributions of the $x_{\cdot t}$ are

$$(3.9) \quad \begin{aligned} & x_{\cdot t} \sim N(\pi \xi_{Kt} + \pi_0, \Omega), \qquad t = 1, \cdots, T, \\ & x_{\cdot t} \text{ and } x_{\cdot s} \text{ are independent,} \end{aligned}$$

where

$$(3.10) \quad \Omega = \begin{bmatrix} \sigma_{uu} + \beta' \begin{pmatrix} D^{-1} \\ 0 \end{pmatrix} \Sigma_{ww} (D'^{-1} 0) \beta & \beta' \begin{pmatrix} D^{-1} \\ 0 \end{pmatrix} \Sigma_{ww} (D'^{-1} 0) \\ \begin{pmatrix} D^{-1} \\ 0 \end{pmatrix} \Sigma_{ww} (D'^{-1} 0) \beta & \begin{pmatrix} D^{-1} \\ 0 \end{pmatrix} \Sigma_{ww} (D'^{-1} 0) \end{bmatrix} + \Sigma_{\varepsilon\varepsilon}.$$ [2]

It is shown that the distribution of the moment matrix A is the non-central linear Wishart distribution with the *sigma matrix* Ω, the *mean sigma matrix* $\pi\xi' M \xi \pi'$, $\xi' = [\xi_{K1} \cdots \xi_{KT}]$, and $T-1$ degrees of freedom. This is written as

$$(3.11) \qquad A \sim W(A, \Omega, \pi\xi' M \xi \pi', K, T-1, \rho(\pi\xi' M \xi \pi')),$$

where K denotes dimensions of the random matrix A and $\rho(\pi\xi' M \xi \pi') = 1$.

The noncentral Wishart distributions give probability laws of moment matrices of independent normal vectors which have a constant covariance matrix and varying mean vectors during observation periods. In particular, the noncentral linear Wishart distributions are suitable when the varying mean vectors lie on a one dimen-

[2] This is due to the block diagonality of the covariance matrix of the u_t, $w_{\cdot t}$ and $\varepsilon_{\cdot t}$. The block diagonality is due partly to simplicity of the representation and partly to understanding that the u_t, $w_{\cdot t}$ and $\varepsilon_{\cdot t}$ are of different nature.

sional hyperplane of the mean vectors space. And this is our case by presence of multicollinearity.

Thus the problem is to derive the distributions of the random matrix $A_{22}^{-1}A_{21}$ provided that the distribution of $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is given by (3.11).

IV. STATISTICAL THEORY[3]

The distribution of $\Theta^* = A_{22}^{-1}A_{21}$ is obtained under the assumption that

$$A \sim W(A, \Omega, \eta^*\eta^{*\prime}, p, n, 1)\,, \tag{4.1}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} r\times r & \vdots & r\times s \\ \cdots\cdots & \vdots & \cdots\cdots \\ s\times r & \vdots & s\times s \end{bmatrix}, \qquad p = r+s\,,$$

$$\eta^{*\prime} = [\eta_1^{*\prime} \vdots \eta_2^{*\prime}] = [1\times r \vdots 1 \times s]\,.$$

Consider an upper triangular matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ O & S_{22} \end{bmatrix} = \begin{bmatrix} r\times r & \vdots & r\times s \\ \cdots\cdots & \vdots & \cdots\cdots \\ s\times r & \vdots & s\times s \end{bmatrix}$ such that

$$S'S = \Omega^{-1}\,. \tag{4.2}$$

Partitioning as $\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} = \begin{bmatrix} r\times r & \vdots & r\times s \\ \cdots\cdots & \vdots & \cdots\cdots \\ s\times r & \vdots & s\times s \end{bmatrix}$, we have

$$\begin{aligned} S_{11}'S_{11} &= \Omega_{11\cdot 2}^{-1}\,, \\ S_{11}'S_{11}'^{-1} &= -\,\Omega_{22}^{-1}\Omega_{21}\,, \\ S_{22}'S_{22} &= \Omega_{22}^{-1} \end{aligned} \tag{4.3}$$

where $\Omega_{11\cdot 2} = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}$. By the transformation

$$A = S^{-1}BS'^{-1}\,, \tag{4.4}$$

we have

$$B \sim W(B, I, \eta\eta', p, n, 1)\,, \tag{4.5}$$

where

$$\eta' = [\eta_1' \vdots \eta_2'] = \eta^{*\prime}S' = [\eta_1^{*\prime}S_{11}' + \eta_2^{*\prime}S_{12} \vdots \eta_2^{*\prime}S_{22}]\,. \tag{4.6}$$

And the probability density function (pdf) of the B is, from Anderson and Girshick [1],

$$W(B, I, \eta\eta', p, n, 1) = \frac{e^{(-1/2)\eta'\eta}e^{(-1/2)\mathrm{tr}B}\,|B|^{(1/2)(n-p-1)}}{2^{(1/2)pn}\pi^{(1/4)p(p-1)}\prod_{i=1}^{p-1}\Gamma\left(\frac{n-i}{2}\right)}\sum_{j=0}^{\infty}\frac{(\eta'B\eta)^j}{2^{2j}j!\,\Gamma\left(\frac{n}{2}+j\right)}\,, \tag{4.7}$$

$$n \geq p\,.$$

[3] Except for a few minor alterations this part is a reproduction of a section in the author's paper [7], where the sampling distribution of the simultaneous equations estimators is studied.

Let

$$\Theta = B_{22}^{-1}B_{21}\,, \tag{4.8}$$

where $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{22} & B_{22} \end{bmatrix} = \begin{bmatrix} r\times r & \vdots & r\times s \\ \cdots\cdots & \vdots & \cdots\cdots \\ s\times r & \vdots & s\times s \end{bmatrix}$, then we have

$$\Theta = S_{22}'^{-1}(\Theta^* + S_{12}'S_{11}'^{-1})S_{11}'\,. \tag{4.9}$$

The pdf $f_1(\Theta)$ of Θ is first obtained, then it is transformed into the pdf $f_2(\Theta^*)$ of Θ^* by the relation

$$f_2(\Theta^*) = J(\Theta\colon \Theta^*)f_1(S_{22}'^{-1}[\Theta^* + S_{12}'S_{11}'^{-1}]S_{11}')\,. \tag{4.10}$$

It is shown that the Jacobian of the transformation (4.9) is

$$J(\Theta\colon \Theta^*) = |\Omega_{22}|^{r/2}|\Omega_{11\cdot 2}|^{-s/2}\,. \tag{4.11}$$

The B is transformed into Θ and Q,

$$\begin{aligned} B_{11} &= Q + \Theta' B_{22}\Theta\,, \\ B_{21} &= B_{22}\Theta\,, \\ J(B_{11}, B_{21}\colon \Theta, Q) &= |B_{22}|^r\,. \end{aligned} \tag{4.12}$$

Then we have the pdf of (Θ, Q, B_{22}),

$$\begin{aligned} f_3(\Theta, Q, B_{22}) = {} & c|Q|^{(n-p-1)/2}e^{(-1/2)\operatorname{tr}Q}|B_{22}|^{(n-p-1/2)+r}e^{(-1/2)\operatorname{tr}FB_{22}F'} \\ & \times \sum_{j=0}^{\infty}\frac{(\eta_1'Q\eta_1 + \eta' FB_{22}F'\eta)^j}{2^{2j}j!\Gamma\left(\frac{n}{2}+j\right)}\,, \end{aligned} \tag{4.13}$$

where

$$F = [\Theta \vdots I] \quad \text{and} \quad c = e^{(-1/2)\eta'\eta}\Big/ 2^{(1/2)pn}\pi^{(1/4)p(p-1)}\prod_{i=1}^{p-1}\Gamma\left(\frac{n-i}{2}\right).$$

After the binomial expanison, the pdf of (Θ, Q) may be written as

$$\begin{aligned} f_4(\Theta, Q) = {} & c|Q|^{(n-p-1)/2}e^{(-1/2)\operatorname{tr}Q}\sum_{j=0}^{\infty}\frac{1}{2^{2j}\Gamma\left(\frac{n}{2}+j\right)}\sum_{j_1+j_2=j}\frac{(\eta_1'Q\eta_1)^{j_1}}{j_1!j_2!} \\ & \times \int_{B_{22}>0} e^{(-1/2)\operatorname{tr}FB_{22}F'}|B_{22}|^{(n-p-1/2)+r}(\eta' FB_{22}F'\eta)^{j_2}dB_{22}\,. \end{aligned} \tag{4.14}$$

And it can be shown that the integral equals to

$$\begin{aligned} & (\eta' F(F'F)^{-1}F'\eta)^{j_2}|F'F|^{-(n+r)/2}\left\{2^{(n+r/2)+j_2}\Gamma\left(\frac{n+r}{2}+j_2\right)\right\} \\ & \times \left\{2^{(n+r)/2}\Gamma\left(\frac{n+r}{2}\right)\right\}^{s+1}\left\{\prod_{i=1}^{s-1}\frac{\Gamma\left(\frac{n+r-i}{2}\right)}{\Gamma\left(\frac{n+r}{2}\right)}\right\}. \end{aligned} \tag{4.15}$$

Substituting (4.15) into the integral of (4.14), the pdf of Θ may be written as

$$(4.16)\quad f_1(\Theta) = c2^{((n+r)/2)s}\pi^{(1/4)s(s-1)}\prod_{i=1}^{s-1}\Gamma\left(\frac{n+r-i}{2}\right)|F'F|^{-(n+r)/2}$$

$$\times\sum_{j=0}^{\infty}\frac{1}{2^{2j}\Gamma\left(\frac{n}{2}+j\right)}\sum_{j_1+j_2=j}\frac{\Gamma\left(\frac{n+r}{2}+j_2\right)2^{j_2}}{j_1!j_2!}(\eta'F(F'F)^{-1}F'\eta)^{j_2}$$

$$\times\int_{Q>0}e^{-(1/2)\mathrm{tr}Q}|Q|^{(n-p-1)/2}(\eta_1'Q\eta_1)^{j_1}dQ\,.$$

Evaluating the integral of the same type as the previous one and changing the method of summing the series, we have

$$(4.17)\quad f_1(\Theta) = \frac{e^{(-1/2)\eta'\eta}\prod_{i=1}^{s-1}\Gamma\left(\frac{n+r-i}{2}\right)\prod_{i=1}^{r-1}\Gamma\left(\frac{n-s-i}{2}\right)}{\pi^{(1/2)rs}\prod_{i=1}^{p-1}\Gamma\left(\frac{n-i}{2}\right)}|I+\Theta\Theta'|^{-(n+r)/2}$$

$$\times\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{n-s}{2}+i\right)\left(\frac{\eta_1'\eta_1}{2}\right)^i\Gamma\left(\frac{n+r}{2}+j\right)}{i!\,\Gamma\left(\frac{n}{2}+i+j\right)j!}$$

$$\times\left[\frac{1}{2}(\eta_1'\Theta'+\eta_2')(I+\Theta\Theta')^{-1}(\Theta\eta_1+\eta_2)\right]^j\,.$$

If the mean sigma matrix satisfies the condition $\eta_1 = 0$ then

$$(4.18)\qquad f_1(\Theta\,|\,\eta_1=0) = f_1(-\Theta\,|\,\eta_1=0)\,,$$

that is, the pdf f_1 is symmetric about the origin.

Consider the pdf (4.17) with $r=1$, therefore $\Theta=\theta=[\theta_2\cdots\theta_p]'$ and η_1 is a scalar. After some calculations by making use of relations

$$(4.19)\qquad \begin{aligned}|I+\theta\theta'| &= 1+\theta'\theta\,,\\ (I+\theta\theta')^{-1} &= I-\frac{\theta\theta'}{1+\theta'\theta}\,,\end{aligned}$$

the pdf of θ is written as

$$(4.20)\quad f_6(\theta) = c'\sum_{i=0}^{\infty}\varphi(i)\sum_{j=0}^{\infty}\phi(i,j)\left[\frac{\eta'\eta}{2}-\frac{(\eta_1-\eta_2'\theta)^2}{2(1+\theta'\theta)}\right]^j(1+\theta'\theta)^{-(n+1)/2}\,,$$

where c', $\varphi(i)$ and $\phi(i,j)$ may be correctly identified with reference to (4.17). By iterations of the binomial expansion we have

$$(4.21)\quad f_6 = c'\sum\varphi(i)\sum\sum\phi(i,j+k)\binom{j+k}{j}\left(\frac{\eta'\eta}{2}\right)^k\left(-\frac{1}{2}\right)^j$$

$$\times\sum_{l_1+\cdots+l_p=2j}\frac{(2j)!(-\eta_1)^{l_1}\eta_2^{l_2}\cdots\eta_p^{l_p}\theta_2^{l_2}\cdots\theta_p^{l_p}}{l_1!\cdots l_p!\left(1+\sum_2^p\theta_g^2\right)^{(n+1/2)+j}}\,.$$

To obtain the moment $E(\theta_2^{h_2}\cdots\theta_p^{h_p})$ of order $h_2+\cdots+h_p$ we must calculate

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\theta_2^{h_2+l_2}\cdots\theta_p^{h_p+l_2}}{(1+\sum\theta_g^2)^{(n+1/2)+j}}d\theta_2\cdots d\theta_p\,, \tag{4.22}$$

$$l_2,\cdots,l_p=0,1,\cdots,2j$$
$$l_1\equiv 2j-(l_2+\cdots+l_p)=0,1,\cdots,2j$$
$$j=0,1,2,\cdots.$$

The transformation to polar coordinates shows that the integral (4.22) equals to

$$\int_0^{\infty}\frac{x^{l_2+\cdots+h_p+l_2+\cdots+l_p+p-2}}{(1+x^2)^{(n+1/2)+j}}dx \tag{4.23}$$

multiplied by the beta functions. The integrand in (4.23) is at most of order $1/x$ raised to $(n-p+3)-(h_2+\cdots+h_p)+l_1$ power, $l_1\geq 0$. And the integral (4.23) exists if and only if $(n-p+3)-(h_2+\cdots+h_p)+l_1>1$.

The integral (4.23) for all $l_1\geq 0$ must exist for the moment $E(\theta_2^{h_2}\cdots\theta_p^{h_p})$ to exist. Thus we have

$$E(\theta_2^{h_2}\cdots\theta_p^{h_p})<\infty\,,\quad\text{if and only if}\quad n-p+2>h_2+\cdots+h_p\,, \tag{4.24}$$

that is, the moments exist up to order $n-p+1$.

In view of (4.18), if $r=1$, $n\geq p$ and $\eta_1=0$ then the pdf $f_1(\theta)$ is symmetric about the origin and the first order moments exist. And we have

$$E\begin{bmatrix}\theta_2\\ \vdots\\ \theta_p\end{bmatrix}=0\,,\quad\text{if}\quad n\geq p,\ \eta_1=0\,. \tag{4.25}$$

The transformation (4.10) is a matter of matrix calculations. To write out $f_2(\Theta^*)$ we need some definitions:

$$\begin{aligned} q'(\eta^*,\Theta^*)&=[\eta_2^{*\prime}\Omega_{22}^{-1}+(\eta_1^{*\prime}-\eta_2^{*\prime}\Omega_{22}^{-1}\Omega_{21})\Omega_{11\cdot2}^{-1}(\Theta^*-\Omega_{22}^{-1}\Omega_{21})']\,,\\ p(\eta^*,\eta^*)&=(\eta_1^{*\prime}-\eta_2^{*\prime}\Omega_{22}^{-1}\Omega_{21})\Omega_{11\cdot2}^{-1}(\eta_1^*-\eta_2^*\Omega_{22}^{-1}\Omega_{21})\,,\\ Q(\Theta^*,\Theta^*)&=[\Omega_{22}^{-1}+(\Theta^*-\Omega_{22}^{-1}\Omega_{21})\Omega_{11\cdot2}^{-1}(\Theta^*-\Omega_{22}^{-2}\Omega_{21})']\,.\end{aligned} \tag{4.26}$$

After some rearrangements by noting (4.3), we obtain the pdf of Θ^*,

$$\begin{aligned} f_2(\Theta^*)=&\frac{e^{(-1/2)\eta^{*\prime}\Sigma^{-1}\eta^*}\prod_{i=1}^{s-1}\Gamma\left(\frac{n+r-i}{2}\right)\prod_{i=1}^{r-1}\Gamma\left(\frac{n-s-i}{2}\right)}{\pi^{(1/2)rs}\prod_{i=1}^{p-1}\Gamma\left(\frac{n-i}{2}\right)}\\ &\times|\Omega_{22}|^{-h/2}|\Omega_{11\cdot2}|^{-s/2}|Q(\Theta^*,\Theta^*)|^{-(n+r)/2}\\ &\times\sum_i\sum_j\frac{\Gamma\left(\frac{n-s}{2}+i\right)\left(\frac{1}{2}p(\eta^*,\eta^*)\right)^i\Gamma\left(\frac{n+r}{2}+j\right)}{j!\,\Gamma\left(\frac{n}{2}+i+j\right)j!}\\ &\times\left[\frac{1}{2}q'(\eta^*,\Theta^*)Q^{-1}(\Theta^*,\Theta^*)q(\eta^*,\Theta^*)\right]^j.\end{aligned} \tag{4.27}$$

From observations of (4.27) or from (4.18), (4.6) and (4.9), it follows that if $\eta_1^{*\prime} = \eta_2^{*\prime}\Omega_{22}^{-1}\Omega_{21}$ then the pdf $f_2(\Theta^*)$ is symmetric about the coordinates $\Theta^* = \Omega_{22}^{-1}\Omega_{21}$,

$$(4.28)\quad f_2(\Omega_{22}^{-1}\Omega_{21} + \Theta^* \,|\, \eta_1^{*\prime} = \eta_2^{*\prime}\Omega_{22}^{-1}\Omega_{21}) = f_2(\Omega_{22}^{-1}\Omega_{21} - \Theta^* \,|\, \eta_1^{*\prime} = \eta_2^{*}\Omega_{22}^{-1}\Omega_{21}) .$$

Consider cases with $r = 1$, $\Theta^* = \theta^* = [\theta_2^* \cdots \theta_p^*]'$ and a scalar S_{11}. Then (4.9) becomes

$$(4.29)\qquad \theta^* = \Phi\theta + \phi ,$$

where

$$(4.30)\qquad \begin{aligned} \Phi &= S_{11}^{-1}S_{22}' \equiv \begin{bmatrix} \phi_{22} & & 0 \\ \vdots & \ddots & \\ \phi_{p2} & \cdots & \phi_{pp} \end{bmatrix}, \\ \phi &= -\, S_{12}'S_{11}'^{-1} = \Omega_{22}^{-1}\Omega_{21} \equiv \begin{bmatrix} \phi_2 \\ \vdots \\ \phi_p \end{bmatrix}. \end{aligned}$$

Therefore, if (4.25) holds or if $n \geq p$ and $\eta_1 = 0$ (or $\eta_1^* = \eta_2^{*1}\Omega_{22}^{-1}\Omega_{21}$), then the pdf $f_2(\theta^*)$ is symmetric, (4.28), and the first order moments of θ^* exist. Consequently,

$$(4.31)\qquad E\begin{bmatrix} \theta_2^* \\ \vdots \\ \theta_p^* \end{bmatrix} = \Omega_{22}^{-1}\Omega_{21} , \quad \text{if} \quad n \geq p \quad \text{and} \quad \eta_1^* = \eta_2^{*\prime}\Omega_{22}^{-1}\Omega_{21} .$$

The moment of the marginal distribution of, say, θ_2^* is the next problem. From (4.29), the h th moment of θ_2^* is written as

$$(4.32)\qquad E(\theta_2^{*h}) = E(\phi_{22}\theta_2 - \phi_2)^h .$$

Therefore the existence condition of $E(\theta_2^{*h})$ is equivalent to that of $E(\theta_2^h)$. Since the reasoning applies interchangeably to every θ_g^*, we conclude that

$$(4.33)\qquad E(\theta_g^{*h}) < \infty , \quad \text{if and only if} \quad n - p + 2 > h ,$$

i.e., the marginal distribution of θ_g^* has finite moments up to order $n - p + 1$.

The pdf $f_2(\Theta^*)$ is a noncentral (linear) generalization of the multivariate distribution of regression coefficients, *see* Kshirsagar [5], and a multivariate generalization of Richardson-Wu distribution [8].

V. SAMPLING DISTRIBUTIONS OF THE LEAST SQUARES ESTIMATORS

The results of Section IV are applied to the sampling distribution problem of the LS under influences of measurement errors and multicollinearity.

When the covariance matrix Ω, (3.10), of the observations is an identity matrix the pdf of the LS b is given by (4.17) with parameters $r = 1$, $p = K$, $n = T - 1$, $(T \geq K + 1)$[4] and

[4] The analysis is confined to cases with $T \geq K+1$. When $T < K$, the LS does not exist with probability one. It exists when $T = K$ and another method of analysis is needed for such a case.

$$\eta' = [\eta_1' \quad \eta_2'] = \xi' M \xi [\gamma'\beta \quad \gamma'] . \tag{5.1}$$

The pdf of b with $\Omega = I$ is written as

$$f_\eta(b \mid \Omega = I) = \frac{\Gamma\left(\frac{T-1}{2}\right) \exp\left(-\frac{1}{2}\xi' M \xi \gamma'(I + \beta\beta')\gamma\right)}{\pi^{1/2(K-1)} \Gamma\left(\frac{T-K+1}{2}\right) \Gamma\left(\frac{T-K}{2}\right)(1 + b'b)^{T/2}} \tag{5.2}$$
$$\times \sum_i \sum_j \frac{\Gamma\left(\frac{T-K}{2} + i\right)\left(\frac{\beta'\gamma\xi' M \xi \gamma'\beta}{2}\right)^i \Gamma\left(\frac{T}{2} + j\right)}{i!\, \Gamma\left(\frac{T-1}{2} + i + j\right) j!}$$
$$\times \left[\frac{\xi' M \xi}{2} \gamma'(I + \beta b')(I + bb')^{-1}(b\beta' + I)\gamma\right]^j .$$

The result (4.24) shows that the LS has finite moments up to order $T - K$ provided $\Omega = I$,

$$E(b_2^{h_2} \cdots b_K^{h_K} \mid \Omega = I) < \infty , \quad \text{if and only if} \quad T - K + 1 > h_2 + \cdots + h_K . \tag{5.3}$$

Particularly, it is seen that the first order moments exist

$$E(b_k \mid \Omega = I) < \infty , \tag{5.4}$$

when $T \geq K + 1$.

Furthermore, from (4.25), it is seen that

$$E\left(\begin{bmatrix} b_2 \\ \vdots \\ b_K \end{bmatrix} \middle| \Omega = I\right) = 0 , \quad \text{if} \quad \xi' M \xi \gamma'\beta = 0 \quad \text{and} \quad T \geq K + 1 . \tag{5.5}$$

The condition $\xi' M \xi \beta'\gamma = 0$ is satisfied when one of the following conditions is satisfied,

$$\begin{aligned} &\xi' M \xi = 0 \\ &\beta = 0 \\ &\beta'\gamma = \beta_K - [\beta_2 \cdots \beta_{K-1}] D^{-1} \alpha_{\cdot K} = 0 \end{aligned} \tag{5.6}$$

The first one is said to be a "central condition" since this reduces our noncentral Wishart distribution to the central one. The second one is said to be a "null condition" since it specifies a null hypothesis. It is noted that $\xi' M \xi \gamma'\beta$ is a systematic variation part of the dependent variable x_{1t}.

We turn to non-identity Ω cases. The pdf of the LS for general cases is given by (4.27). With definitions;

$$\begin{aligned} \Psi(x, y) &= (x - \Omega_{22}^{-1}\Omega_{21})\Omega_{11\cdot 2}^{-1}(y - \Omega_{22}^{-1}\Omega_{21})' , \\ \Phi(x, y) &= (\Omega_{22}^{-1} + \Psi(x, y)) , \end{aligned} \tag{5.7}$$

it is written as

$$(5.8)\quad f_8(b) = \frac{|\Omega_{22}|^{-(T-1)/2}|\Omega_{11\cdot 2}|^{-(K-1)/2}\Gamma\left(\frac{T-1}{2}\right)}{\pi^{(1/2)(K-1)}\Gamma\left(\frac{T-K+1}{2}\right)\Gamma\left(\frac{T-K}{2}\right)}\cdot|\Phi(b,b)|^{-T/2}$$
$$\times \exp - \frac{\xi' M\xi}{2}\gamma'\Phi(\beta,\beta)\gamma$$
$$\times \sum_i \sum_j \frac{\Gamma\left(\frac{T-K}{2}+i\right)\left(\frac{1}{2}\xi' M\xi\gamma'\Psi(\beta,\beta)\gamma\right)^i \Gamma\left(\frac{T}{2}+j\right)}{i!\,\Gamma\left(\frac{T-1}{2}+i+j\right)j!}$$
$$\times\left[\frac{\xi' M\xi}{2}\gamma'\Phi(\beta,b)\Phi^{-1}(b,b)\Phi(b,\beta)\gamma\right]^j,$$

where the parameters γ and Ω are given by (3.8) and (3.10).

It is shown from (4.33) that the marginal distribution of the b_k has finite moments up to order $T-K$,

$$(5.9)\qquad E(b_k^h) < \infty\,,\quad \text{if and only if}\quad T-K+1 > h\,.$$

Finally, it follows from (4.31) and (3.10) that

$$(5.10)\qquad E(b) = (G+\Sigma_{\mathrm{II\,II}})^{-1}(G\beta+\Sigma_{\mathrm{II\,I}})\,,$$
$$\text{if}\quad \xi' M\xi\gamma'\beta = \xi' M\xi\gamma'(G+\Sigma_{\mathrm{II\,II}})^{-1}(G\beta+\Sigma_{\mathrm{II\,I}})\,,$$

where

$$G = \begin{bmatrix} D^{-1}\Sigma_{ww}D'^{-1} & 0 \\ 0 & 0\end{bmatrix}\quad\text{and}$$

$$\Sigma_{\varepsilon\varepsilon} = \begin{bmatrix}\Sigma_{\mathrm{II}} & \Sigma_{\mathrm{I\,II}}\\ \Sigma_{\mathrm{II\,I}} & \Sigma_{\mathrm{II\,II}}\end{bmatrix} = \left[\begin{array}{c:c} 1\times 1 & 1\times\overline{K-1} \\ \hdashline \overline{K-1}\times 1 & \overline{K-1}\times\overline{K-1}\end{array}\right].$$

Therefore, if $\xi' M\xi = 0$ (the central Wishart distribution case) then the LS is an unbiased estimator of $(G+\Sigma_{\mathrm{II\,II}})^{-1}(G\beta+\Sigma_{\mathrm{II\,I}})$. Or it follows that

$$(5.11)\qquad E(b) = \beta\,,\quad \text{if}\quad \beta = \Sigma_{\mathrm{II\,II}}^{-1}\Sigma_{\mathrm{II\,I}}\,.$$

But this unbiasedness is exceptional. Even if the unbiasedness appears in unusual cases, we may as well understand that the LS is an estimate of $\Sigma_{\mathrm{II\,II}}^{-1}\Sigma_{\mathrm{II\,I}}$, not of β.

VI. ANOTHER EXAMPLE

It may be appropriate to consider a more specific example for application of the statistical theory in Section IV. The example is the LS estimation of the tangency condition equation which appears in the theory of consumers' behavior.

The model employes a utility indicator,

$$(6.1)\qquad I = (\alpha_1+q_1)^{\beta_1}(\alpha_2+q_2)^{\beta_2}\,,$$

where q_1 and q_2 denote quantities of two goods consumed, say, food and non-food consumptions. The α_1, α_2, β_1 and β_2 are parameters. We obtain the structural equations system, the balance equation and the tangency condition,

$$\begin{bmatrix} 1 & 1 \\ 1 & -\beta \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \gamma_1 & \gamma_2 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \tag{6.2}$$

where p_1, p_2 and y stand for the prices of the goods and the total expenditure, respectively, the e_1 and e_2 are expenditures on the goods and u is a shock. The new parameters are given by

$$\alpha' \equiv [\beta \quad \gamma'] = \left[\frac{\beta_1}{\beta_2}, \; -\alpha_1, \frac{\beta_1}{\beta_2}\alpha_2\right]. \tag{6.3}$$

The structural system (6.2) is a shock model with respect to the theoretical variables.

The reduced form of (6.2) is

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ y \end{bmatrix} + \begin{bmatrix} u/\beta + 1 \\ -u/\beta + 1 \end{bmatrix}, \tag{6.4}$$

where

$$\Pi \equiv \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \frac{1}{\beta + 1}\begin{bmatrix} \gamma' & \beta \\ -\gamma' & 1 \end{bmatrix}. \tag{6.5}$$

It is assumed that linear relationships exist between the exogenous variables,

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ y \end{bmatrix} \equiv [D : d]\begin{bmatrix} p_1 \\ p_2 \\ y \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{6.6}$$

or

$$\begin{bmatrix} p_1 \\ p_2 \\ y \end{bmatrix} = \begin{bmatrix} \delta \\ 1 \end{bmatrix} y + \begin{bmatrix} D^{-1} \\ 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{6.7}$$

where w_1 and w_2 are shocks and $\delta = -D^{-1}d$. The p_1, p_2, y are near multicollinear, or exact multicolli near with empty w_1, w_2. Substituting (6.7) into (6.4), we get

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \Pi\begin{bmatrix} \delta \\ 1 \end{bmatrix} y + \begin{bmatrix} 1/\beta + 1 \\ -1/\beta + 1 \end{bmatrix} u + \Pi\begin{bmatrix} D^{-1} \\ 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{6.8}$$

The time series budget data $E_{1t}, E_{2t}, P_{1t}, P_{2t}, Y_t$ of the theoretical variables $e_{1t}, e_{2t}, p_{1t}, p_{2t}, y_t$ are assumed to be subject to additive measurement errors,

$$\begin{bmatrix} E_{1t} \\ E_{2t} \\ P_{1t} \\ P_{2t} \\ Y_t \end{bmatrix} = \begin{bmatrix} e_{1t} \\ e_{2t} \\ p_{1t} \\ p_{2t} \\ y_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \\ \varepsilon_{4t} \\ \varepsilon_{5t} \end{bmatrix}, \qquad t = 1, \cdots, T. \tag{6.9}$$

Various sources of measurement error as well as error of index number or of aggregation are included in the additive measurement errors. So the specification (6.9) may be a great simplification.

The stochastic data generating mechanism is summarized as

$$\text{(6.10)}\quad \begin{bmatrix} E_{1t} \\ E_{2t} \\ P_{1t} \\ P_{2t} \\ Y_t \end{bmatrix} = \begin{bmatrix} \Pi \\ I_3 \end{bmatrix}\begin{bmatrix} \delta \\ 1 \end{bmatrix} y_t + \left[\begin{array}{c:c:c} \begin{pmatrix} 1/\beta + 1 \\ -1/\beta + 1 \end{pmatrix} & \Pi\begin{pmatrix} D^{-1} \\ 0 \end{pmatrix} & \\ 0 & \begin{pmatrix} D^{-1} \\ 0 \end{pmatrix} & I_5 \end{array}\right] \begin{bmatrix} u_t \\ w_{1t} \\ w_{2t} \\ \varepsilon_{1t} \\ \vdots \\ \varepsilon_{5t} \end{bmatrix}.$$

Under the stochastic assumptions,

y_t are fixed in repeated samples,

$$\text{(6.11)}\quad \begin{aligned} &\begin{bmatrix} u_t \\ w_{1t} \\ w_{2t} \\ \varepsilon_{1t} \\ \vdots \\ \varepsilon_{5t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \sigma_{uu} & 0 & 0 \\ 0 & \Sigma_{ww} & 0 \\ 0 & 0 & \Sigma_{\varepsilon\varepsilon} \end{bmatrix}\right), \\ &\begin{bmatrix} u_t \\ w_{1t} \\ w_{2t} \\ \varepsilon_{1t} \\ \vdots \\ \varepsilon_{5t} \end{bmatrix} \text{ and } \begin{bmatrix} u_s \\ w_{1s} \\ w_{2s} \\ \varepsilon_{1s} \\ \vdots \\ \varepsilon_{5s} \end{bmatrix} \text{ are independent,} \end{aligned}$$

it follows that

$$\text{(6.12)}\quad \begin{aligned} &\begin{bmatrix} E_{1t} \\ E_{2t} \\ P_{1t} \\ P_{2t} \end{bmatrix} \sim N\left(\begin{bmatrix} \Pi\begin{pmatrix} \delta \\ 1 \end{pmatrix} \\ \delta \end{bmatrix} y_t \quad \Omega\right), \\ &\begin{bmatrix} E_{1t} \\ E_{2t} \\ P_{1t} \\ P_{2t} \end{bmatrix} \text{ and } \begin{bmatrix} E_{1s} \\ E_{2s} \\ P_{1s} \\ P_{2s} \end{bmatrix} \text{ are independent}, \end{aligned}$$

where

$$\text{(6.13)}\quad \Omega = \begin{bmatrix} \dfrac{\sigma_{uu} + \gamma'\Phi\gamma}{(\beta+1)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \end{bmatrix}\dfrac{\gamma'\Phi}{(\beta+1)} \\ \dfrac{\Phi\gamma}{(\beta+1)}[1 \quad -1] & \Phi \end{bmatrix} + \begin{bmatrix} \Sigma_{\mathrm{II}} & \Sigma_{\mathrm{I\,II}} \\ \Sigma_{\mathrm{II\,I}} & \Sigma_{\mathrm{II\,II}} \end{bmatrix},$$

the last term being a covariance matrix of $\varepsilon_{1t}, \cdots, \varepsilon_{4t}$ with a scalar Σ_{II}, and $\Phi = D^{-1}\Sigma_{ww}D^{-1'}$.

Let A be a moment matrix of E_{1t}, E_{2t}, P_{1t}, P_{2t} about the origin,

$$(6.14)\qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 1\times 1 : 1\times 3 \\ \cdots\cdots\cdots \\ 3\times 1 : 3\times 3 \end{bmatrix} = \sum_{t=1}^{T} \begin{bmatrix} E_{1t} \\ E_{2t} \\ P_{1t} \\ P_{2t} \end{bmatrix} [E_{1t}E_{2t}P_{1t}P_{2t}] ,$$

then we have, for the LS $a' = [b, c_1, c_2]$ of $[\beta, \gamma_1, \gamma_2]$,

$$(6.15)\qquad a = A_{22}^{-1}A_{21} .$$

Furthermore, it is shown that

$$(6.16)\qquad A \sim W(A, \Omega, \eta\eta', 4, T, 1) ,$$

where

$$(6.17)\qquad \eta = \mu\begin{bmatrix} \Pi\begin{pmatrix} \delta \\ 1 \end{pmatrix} \\ \delta \end{bmatrix} = \mu\begin{bmatrix} \beta + \gamma'\delta/\beta + 1 \\ 1 - \gamma'\delta/\beta + 1 \\ \delta \end{bmatrix}, \qquad \mu^2 = \sum y_t^2 .$$

Therefore the results of Section IV apply to the sampling distribution of the LS a.

The pdf of a is given by (4.17), when $\Omega = I$ and $T \geq 4$, with parameters replaced by (6.17) and $r = 1$, $p = 4$, $n = T$, $\Theta = a$.

It is shown from (4.24) that the moments of a exist up to order $T - 3$,

$$(6.18)\qquad E(b^h \cdot c_1^i \cdot c_2^j \mid \Omega = I) < \infty , \quad \text{if and only if} \quad T - 2 > h + i + j ,$$

and from (4.25) that

$$(6.19)\qquad E\left(\begin{bmatrix} b \\ c_1 \\ c_2 \end{bmatrix} \middle| \Omega = I\right) = 0 , \quad \text{if} \quad \mu(\beta + \gamma'\delta) = 0 , \quad T \geq 4 .$$

For non-indentity Ω cases, the pdf of a is given by (4.27) with parameters Ω and η^* replaced by (6.13) and (6.17) respectively and $r = 1$, $p = 4$, $n = T$, $\Theta = a$.

It is seen from (4.33) that the moments of each marginal distribution of a exist up to order $T - 3$, for instance,

$$(6.20)\qquad E(b^h) < \infty , \quad \text{if and only if} \quad T - 2 > h .$$

It is also seen from (4.31) that

$$(6.21)\qquad E(a) = (G_1 + G_2 + \Sigma_{\mathrm{II\,II}})^{-1}(g_1 + g_2 + \Sigma_{\mathrm{II\,I}}) ,$$
$$\text{if} \quad T \geq 4 \quad \text{and}$$
$$\mu\left(\frac{\beta + \gamma'\delta}{\beta + 1}\right) = \mu\left[\frac{1 - \gamma'\delta}{\beta + 1}, \delta'\right](G_1 + G_2 + \Sigma_{\mathrm{II\,II}})^{-1}(g_1 + g_2 + \Sigma_{\mathrm{II\,I}}) ,$$

where

$$(6.22)\qquad \begin{aligned} G_1 &= \begin{bmatrix} \sigma_{uu}/(\beta+1)^2 & 0 \\ 0 & 0 \end{bmatrix}, \\ G_2 &= \begin{bmatrix} -\gamma'/\beta+1 \\ I \end{bmatrix} \Phi[-\gamma/\beta+1 \quad I], \\ g_1 &= \begin{bmatrix} -\sigma_{uu}/(\beta+1)^2 \\ 0 \end{bmatrix}, \\ g_2 &= \begin{bmatrix} -\gamma'/\beta+1 \\ I/\beta+1 \end{bmatrix} \Phi\gamma . \end{aligned}$$

The "if" condition of (6.21) is complicated but (6.21) is expressed in another form after some calculations,

$$(6.23)\qquad E(a) = \alpha\,, \quad \text{if} \quad \alpha = \Sigma_{\mathrm{II\,II}}^{-1}\left[\Sigma_{\mathrm{II\,I}} - \begin{pmatrix} \sigma_{uu}/\beta+1 \\ 0 \end{pmatrix}\right], \quad T \geq 4\,.$$

This tells us that the LS cannot, in general, be expected as an unbiased estimate.

VII. CONCLUDING REMARKS

In this article we obtained the exact sampling distribution of the LS in the error-in-variables models effected by multicollinearity. Some properties of the sampling distribution were presented. The function form of the distribution and the existence condition of moments for the two variable model appeared again in a multivariate fashion.

The model admits both of measurement errors and multicollinearity. It appears that the effects of the measurement errors dominate those of multicollinearity. And marginal effects of multicollinearity will be identified when the sampling distribution is obtained in non-multicollinearity situations.

Keio University

REFERENCES

[1] Anderson, T. W. and M. A. Girshick, "Some Extensions of the Wishart Distribution," *Annals of Mathematical Statistics*, Vol. 15 (1944) 345–357.

[2] Farrar, D. E. and R. R. Glauber, "Multicollinearity in Regression Analysis: The Problem Revisited," *Review of Economics and Statistics*, Vol. XLIX (1967) 92–107.

[3] Haavelmo, T., "Remarks on Frish's Confluence Analysis and Its Use in Econometrics," Chapter V in [4] 258–265.

[4] Koopmans, T. (ed.) *Statistical Inference in Dynamic Economic Models*, New York: John Wiley and Sons, Inc. (1950).

[5] Kshirsagar, G. M., "Some Extensions of the Multivariate t-Distribution and the Multivariate Generalization of the Distribution of the Regression Coefficient," *Proceedings of the Cambridge Philosophical Society*, Vol. 57 (1961) 80–85.

[6] Marschak, J., "Statistical Inference in Economics: An Introduction," Chapter I in [4] 1–50.

[7] Matsuno, K., "On the Exact Sampling Distributions of the Ordinary and Two Stage Least Squares Estimators," (in Japanese) *Mita Journal of Economics*, Vol. 67 (1974) 753–773.

[8] Richardson, D. H. and D. Wu, "Least Squares and Grouping Method Estimators in the Errors in Variables Model," *Journal of the American Statistical Association*, Vol. 65 (1970) 724–748.

[9] Tintner, G., "A Note on Rank, Multicollinearity and Multiple Regression," *Annals of Mathematical Statistics*, Vol. 16 (1945) 304–308.