RECURSIVE LINEAR PROGRAMS

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ABSTRACT

This paper summarizes the theory relating to recursive linear programming models, a branch of applied mathematical programming that is not well known outside the discipline of economics. The applications of recursive linear programming models are discussed, in addition to the existence and character of their solutions. The paper also attempts to extend their domain of application to more traditional areas of mathematical programming by demonstrating their implicit use in certain decomposition and non-linear programming algorithms.

CONTENTS

I. INTRODUCTION

A recursive linear programming (RLP) model belongs to a mathematical system whose members consist of a sequence of linear programming problems in which one or more of the coefficients of each problem depend on the solution vectors of preceding problems in the sequence. Their explicit use has been confined to the construction of various economic theories and to the development of empirical models of economic behavior. This paper reviews and summarizes the theory concerning the existence and character of solutions for RLP models, and attempts to extend their domain of application by demonstrating their implicit use in certain decomposition and non-linear programming algorithms.

II. THE GENERAL MODEL

Let $x, z \in E^m$, $c \in E^k$ and let $B \in E^{k \times m}$ be a $k \times m$ matrix, where E^m is a real m-dimensional Euclidean vector space. Define $w = (z, B, c) \in E^{m+k \times m+k}$. Consider a set of dual linear programming problems parametric on a set $W \subset E^{m+k \times m+k}$

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$$(1) \qquad \pi^* (w) = \max_x [z'x \mid Bx \leqq c, x \geqq 0]$$

$$(2) \qquad \rho^* (w) = \min_y [c'y \mid B'y \geqq z, y \geqq 0]$$

for $w \in W$.

The set of feasible decision vectors satisfying the constraints of (1) and (2), the primal and dual feasible regions respectively, are denoted

$$\Gamma_x(B, c) = \{x \mid Bx \leqq c, x \geqq 0\}$$

and

$$\Gamma_y(B, z) = \{y \mid B'y \geqq z, y \geqq 0\} .$$

The set of primal and dual optimal solutions[1] to (1) and (2) vary parametrically with the coefficient vector $w = (z, B, c)$. In what follows interest is not focussed on parametric programming in general, but rather on dual programs defined for sequences of coefficients $w_t = (z_t, B_t, c_t)$, and hence with sequences $\Gamma_x(B_t, c_t)$, $\Gamma_y(B_t, z_t)$, and sequences of optimal solution vectors.[2]

A sequence of problems denoted by (1) and (2), with appropriate time subscripts, can represent the sequence of choice problems of a single decision-maker, a set of decision-makers, or an organization in which the vector x_t^* represents activity levels and the vector y_t^* imputed marginal values of resources based on the coefficients of w at period t. If B is block diagonal, then the solution vectors may represent the choices of a set of independent decision-makers each of whom corresponds to a particular block of B. (Day and Kennedy [6]). If in a collection of decision-makers each possesses the same B matrix and if the z and c vectors for each lie on the same rays in E^m and E^k respectively, then x and c can represent the sum of the individual x and c vectors. (Day [4]). If B is block triangular or possesses other special structures, (1)—(2) can be used to represent the choice problem of a complex organization with a hierarchy of interdependent decision-makers. Block triangularity of B can also be interpreted as a situation in which

[1] The solutions of (1) and (2) are in general not unique. In much of the theory that follows such uniqueness is necessary. T_0 achieve this uniqueness a selection operator is applied to the set of optimal solutions of each l.p. problem, selecting one from among the several solutions in the solution set. An example of such a selection operator is the "conservative" selection operator: choose from among the current solutions the one closest to the previous period's selected solution.

[2] Recursive and dynamic programming are in some respects similar but are certainly not identical. A dynamic programming problem, say, to maximize an objective function in time period six, can be rewritten as a recursive programming problem in which the decision vector is a schedule of present *and* future decisions and in which the objective function grants positive payoffs only to results anticipated in time period six. But in general RLP models are very myopic in nature (as are many economic agents), utilizing an objective function granting positive payoffs to only results anticiptated in the immediate time period. To use such a structure to generate a path to an optimal position in period six would be courting disaster. As should be obvious, the resulting time path may even be anti-optimal, let alone non-optimal. The value of recursive programming in this context rests on a theoretical or empirical rationalization that this type of model captures actual behaviour more adequately than does a dynamic programming structure.

the decision vector x_t is composed of a sequence of subvectors each representing a set of choices for a specific anticipated time period. (Dorfman, Samuelson and Solow [8]). Becausethese various possibilities depend on the structure of B only they can be combined so that (1) — (2) can represent the choice problem of a collection of some independent and some interdependent decision-makers each of whom is scheduling an imminent and a sequence of anticipated future actions. Hence (1) — (2) can be used to represent the joint strategy of "persons" in a game setting that arises in modelling decentralized economic behavior. An alternative interpretation that focusses on numerical application instead of economic behavior is that each vector in the sequence $\{w_t\}$ represents the parameters of an l.p. problem that linearly approximates in the t^{th} iteration the constraints and payoff function of a *non-linear* programming problem. Both economic and numerical examples are discussed later in the paper.

There remains one major ingredient of RLP models that must be discussed, the feedback operator. The feedback operator, denoted by Ω, consists of $m + k \times m + k$ functions each determining the current value of an element of z, B, or c. Together, they determine the current w. They map the previous σ solution vectors of (1) and (2), along with a vector of exogenous variables, into new values of the parameters defining (1) and (2).
Thus

$$w_t = \Omega(x^*_{t-1} \ldots, x^*_{t-\sigma}, y^*_{t-1}, \ldots, y^*_{t-\sigma}, P_t)$$

where asterisks denote solution values (vectors) and P_t denotes a vector of exgeneous variables. The special case of primal (dual) feedback occurs if the Ω does not involve any of the $y^*_{t-i}(x^*_{t-i})$.

The feedback functions establish a recursive dependence of the parameters of a given problem in the sequence (1) — (2) and the primal and dual solution vector of σ immediately preceding dual problems in the sequence. In models of economic behavior such functions may define (i) the dependence of current capital stocks on past investment decisions (assuming a finite life for all capital stocks); (ii) behavioral constraints that represent frictional forces (learning new techniques) and decision rules about the maximum adjustment of capacity toward some forecasted desired capacity levels; (iii) forecasting equations for payoff coefficients and in some cases sales constraints; (iv) financial constraints that depend on past borrowing and debt repayment activities; (v) temporary equilibrium marketing conditions that connect past payoffs and available current supplies of inputs from "external" sectors to production and input utilization levels of the "internal" sector being modelled.

When (1) — (2) is a model of economic behavior then the vectors B_t, z_t, c_t are thought of as *states* or *information vectors* and the L.P. model leading to the sequence (1)—(2) augmented by a selection operator represents the decision strategy which associates with each information state (B_t, z_t, c_t) a particular choice x^*_t, y^*_t. The feedback operator then represents the way the environment (as defined by the *model builder*) and the forecasts assumed to be used by the decision-makers interact with the past choices of the decision unit or units. The feedback functions could be thought of as the strategy played by nature.

If (1) – (2) is defined by a non-linear programming (NLP) algorithm, then the feedback functions represent the current linear approximation to the constraint set or objective function of the non-linear problem as a function of the solution to the preceding approximates. The RLP model as a whole becomes a kind of Newton-like iterative process used to solve a static, nonlinear problem.

III. EXISTENCE OF SOLUTIONS

An RLP model of whatever type can be solved recursively, given the initial conditions of the previous σ solution vectors and the current vector of exogeneous variables. Using these initial conditions the current period's w, say w_1 for $t = 1$, can be found using the feedback function; w_1 defines the first period's l.p. problem. Solving this l.p. problem and then applying a selection operator generates unique solution vectors. This solution, plus new values for the exogenous variables, along with the previous $\sigma - 1$ solutions and $\sigma - 1$ values for the vector of exogeneous variables, allow w_2 to be computed using the feedback function. The second period's l.p. problem can then be solved. Continuing in this way, a sequence of solution vectors can be generated, such a sequence being called a solution to the RLP system.

Clearly a solution of an RLP system will exist only if the feedback mechanism t is such as to generate $w_t \in W$ for all t (i.e., so that Γ_x and Γ_y are non-empty). Such a system is said to be viable with respect to the initial conditions. Systems that are not viable can be represented by a sequence of solution vectors lasting only a finite number of periods. Such systems should not be considered to be of little interest. Although they are not of great interest from the mathematical point of view, they do have some relevance in the study of economic systems which possess structures that eventually lead to infeasibility. Frequently in history economies "break-down" and must be restructured by creating new economic institutions or modified "rules of the game". This is more often the case in microeconomic modelling, when the possibility of bankruptcy is common.

Two special kinds of solutions to RLP systems are of particular interest, stationary states and compact orbits. A stationary state is a solution in which the same decision is made in each time period. A compact orbit solution is one in which the decisions made in every time period belong to a specific subset of decisions (a zone of behaviour which, once entered, will not be left). A stationary state is thus a special case of a compact orbit in which the subset of decisions contains only one decision.

Existence of a stationary state solution for an RLP system can be shown by means of the general theory developed in Day and Kennedy [6][3]. An RLP system

[3] For a specific discussion of the proof in the context of RLP systems see Kennedy [13].

can be structured as a special case of their RDS (recursive decision system) and an existence proof can be derived in a similar fashion, involving fewer assumptions than those needed for the more general RDS case. In particular, for the existence of a stationary state, an RLP system must satisfy the following assumptions[4].

a) the system must be viable;
b) solution vectors must be bounded[5];
c) the feedback operator must be continuous, with no exogeneous feedback
d) the selection operator must be "conservative" (see fn. 1).

The existence of compact orbits requires the same assumptions; in more general recursive decision systems the assumptions required for existence of compact orbits are slightly weaker than those required for existence of stationary states.

IV. PHASE THEORY

As might seem obvious, RLP's are of primary use in the analysis of dynamic (as opposed to stationary state) problems. In empirical work using this technique (for example Day [3], Day et al. [5], Day and Nelson [7], and Heidehues [11]), the main objective has been to capture empirically the dynamic behaviour of the decision-maker (for example, a firm) over time. The great power of this approach in explaining dynamic behaviour lies in the extraordinarily rich variety of dynamic time paths capable of being generated by an RLP model. This may be seen more clearly by examining the phenomenon of phases.

Utilizing the familiar theoretical results of Goldman and Tucker [10] concerning equated constraints and linear programming[6], it can easily be shown that a solution to an RLP system must satisfy a sequence of equated constraints. If each system of equated constraints in this sequence is non-singular, the solution is said to be extreme (i.e., not simply a linear combination of two or more distinct solutions). The system of equated constraints at time t consists of those constraints that are "tight" (or equated) when solving the t^{th} period's l.p. problem. The particular constraints that are "tight" in period t clearly need not be the same as the constraints that were "tight" in the preceding period. If over time the set of equated constraints does not change (the parameter values of the constraints may change, however), the system is said to be "in phase". The system is said to "switch phases" whenever the set of equated constraints changes. The number of possible phases is thus the number of possible different sets of equated constraints. This is the number of possible partitionings of B into square non-singular sub-matrices.

A system in phase with an extreme solution can be represented by a familar

[4] The implications of these assumptions are discussed in Day and Kennedy [6].

[5] Conditions under which assumptions a) and b) are satisfied in RLP systems are discussed at some length in Kennedy [13]. These assumptions are the most important assumptions in the sense that they are not typical in analyses of this genre.

[6] This theory of equated constraints was first used by Shapely and Snow [19] in connection with game theory.

system of simultaneous σ-order difference equations:

$$\begin{aligned} x_1^*(t) &= [B_{11}(\sigma)]^{-1}c_1(\sigma) \\ x_2^*(t) &= 0 \\ y_1^*(t) &= [B'_{11}(\sigma)]^{-1}z_1(\sigma) \\ y_2^*(t) &= 0 \end{aligned}$$

where $B_{11}(\sigma)$ is the square non-singular sub-matrix of B corresponding to the equated constraints. The σ argument represents the fact that each element of B_{11} is a function of the preceding σ values of x^* and y^*. $c_1(\sigma)$ is the part of the c vector corresponding to the equated constraints, each element of which is also a frunction of the preceding σ values of x^* and y^*. $z(\sigma)$ is defined in a similar fashion.

The different types of dynamic paths capable of being generated by such a system have been extensively examined in the difference equation literature. The source of the rich variety of possible dyanmic paths characterizing an RLP system lies in the possibility of switching phases at any time. Switching phases means that the underlying base of equated constraints changes, i.e., we move to a new "corner" of the feasible region. This obviously permits the system to jump at will to completely different types of dynamic paths, a flexibility which allows it to capture, as empirical studies have shown, sudden changes in the behaviour of a firm, say a decision to invest heavily in a certain type of equipment.

From the comments above it is clear that there is a finite number of phases. Thus if the RLP system does not settle down into a single phase, it will move from phase to phase, at times returning to phases it had been in earlier. In fact it is easy to see that phase "cycles" could be generated, involving a periodic movement amongst several phases. For a more complete description of these and other interesting dynamic patterns that can be generated by RLP's, as well as the development of several economic illustrations, the reader is referred to Tinney [20]. The theory of these dynamic systems as far as stability is concerned is still a relatively open field. Tinney [20] has generated stability argments for some special cases, and the last sections of this paper show how some stability arguments in the literature can be re-interpreted as RLP stability arguments for special cases, but beyond that very few conclusions have been drawn concerning general RLP stability. Certainly if a system is in phase, the stability problem can be solved by a straightforward application of difference equation theory. It is the phase switching which creates problems for the stability analysis—perhaps the price paid for gaining such a rich source of dynamic time paths.

V. NONLINEAR PROGRAMMING ALGORITHMS

RLP structures can be used to simulate an algorithm for the solution of certain types of static non-linear programming problems. The methods of feasible directions and of set approximation yield examples of such applications of RLP's. Two examples are presented below.

a) *Conditional Gradient Method*

A special case of the conditional gradient method, based on Frank and Wolfe [9], is illustrated first. Consider the problem of maximizing a concave function $f(x)$ where x, an m-dimensional vector, is constrained by the linear relations $Bx \leqq c$, $x \geqq 0$.

To begin the algorithm an initial basic feasible vector, x_1^*, say, must be found. Let q_1 be an alternative feasible vector where for the purposes of initiating the algorithm $q_1 = x_1^*$. The LP problem

$$\max\,[z_2' x_2] \qquad \text{s.t.} \qquad Bx_2 \leqq c,\, x_2 \geqq 0$$

is now solved, where $z_2' = \partial f_{q_1}$, the vector partial evaluated at q_1. This, in essence, maximizes a linearization of f (i.e., its gradient) at the point q_1. The solution to this LP is x_2^*.

The feeback mechanism is structured as follows. Let S be arbitrarily chosen such that

$$-\,(x_t^* - q_{t-1})' \partial^2 f_r (x_t^* - q_{t-1}) \leqq S \geqq 0$$

for all r on the segment $\overline{q_{t-1} x_t^*}$. Compute

$$\mu = \min\,\{\partial f_{q_{t-1}} (x_t^* - q_{t-1})/S,\ 1\}$$

and set $q_t = q_{t-1} + \mu(x_t^* - q_{t-1})$.
Thus $z_{t+1} = \partial f_{q_t}$ and the conditional gradient method can be represented by the sequence of programming problems

$$\max\,[z_t' x_t] \qquad \text{s.t.} \qquad Bx_t \leqq c,\, x_t \geqq 0\,.$$

The technical details of this alogorithm are explained in Frank and Wolfe [9], as are convergence proofs demonstrating that in fact this RLP system converges on the solution of the non-linear programming problem. In the quadratic case, an algorithm is constructed for which it is demonstrated that the RLP system actually reaches the desired solution.[7]

b) *The Cutting Plane Method*

The cutting plane method, based on Kelley [12], is an example of the more general category of set approximation methods. Here the problem is to maximize a linear function $z'\,x$ over a compact convex set $R = \{x \mid G(x) \leqq 0\}$ of dimension m, where $G(x)$ is a continuous convex function defined in the m-dimensional compact polyhedral convex set $S = \{x \mid Bx \leqq c\}$ Thus $R \subset S$, and we assume R to be non-empty.

The RLP problem may be set up as

$$\max\,[z' x_t] \qquad \text{s.t.} \qquad B_t x_t \leqq c_t,\, x_t \geqq 0$$

with a feedback of adding a new (additional) constraint to the present constraints of the form $\partial G_{x_t^*} x_{t+1} \leqq \partial G_{x_t^*} x_t^* - G(x_t^*)$.

[7] Proof of convergence of an algorithm implies the existence of a stationary state, as well as the existence of a compact orbit.

This is simply an extreme support to the graph of $G(x)$ at x_t^* (the tangent hyperplane to G at x_t^* if $G(x)$ is "smooth").

The essence of this is that a polyhedral set S is constructed to contain R and then the objective is maximized over S. The feedback then "cuts down" the size of S to more closely approximate R and the process is repeated. Although one might argue that this system is not an RLP since *additional* constraints are being added in each period (rather than simply changing the parameters of old constraints), they are in fact theoretically equivalent since the original problem could have been constructed with extra inoperative constraints which were altered one by one for each iteration.

Proofs for the convergence of this algorithm and technical details for the conditions under which it is operative are given in Kelley [12].

VI. DECOMPOSITION PROCESSES

An additional application of RLP's is to the solution of a large and cumbersome linear programming problem via some type of decomposition technique. In this approach the original problem is partitioned into a group of sub-problems which are then individually solved. The solution of these sub-problems will not necessarily yield the optimal over-all solution, so a feedback is instigated to adjust these sub-problems and the process is repeated. The basic theory and some examples of this type of approach are given in Dantzig and Wolfe [2], Rosen [16], Rosen [17], and Rosen and Ornea [18].

Because the feedback mechanisms in the aforementioned articles are somewhat artificial, we have chosen to use as an example of this type of technique the two-level ecnomic planning idea suggested by Kornai and Liptak [14]. Here a very large linear programming problem of central planning is solved by decomposing the overall problem into sub-problems to be solved by mutually independent sectors. These sectors are co-ordinated by the central planning bureau by having the latter allocate resources to the sectors and having the sectors report back their shadow prices.

Consider the overall central planning problem with its corresponding dual:

$$\max\,[z'x] \quad \text{s.t.} \quad Bx \leqq c,\ x \geqq 0$$

$$\min\,[c'y] \quad \text{s.t.} \quad B'y \geqq z,\ y \geqq 0$$

Now let this problem be partitioned into n sub-problems as follows:

$$\max\,[z_i'x_i] \quad \text{s.t.} \quad B_ix_i \leqq c_i.\ x_i \geqq 0$$

for $i = 1, \ldots, n$ with $\sum_{i=1}^n c_i = c$ and where

$$B = [B_1, \ldots, B_n],\ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \qquad z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

The corresponding n dual problems may be written as

$$\min\, [c_i' y_i] \qquad \text{s.t.} \qquad B_i' y_i \geqq z_1,\, y_i \geqq 0$$

where y_i is of the same dimension as y.

If we write $$C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix},$$

the central planning bureau can be regarded as attempting to allocate resources (choosing a C) such that the sectors when they independently solve their own problems will generate the overall optimal solution. At the same time the individual sectors, looking now at their dual problems, are interested in minimizing their cost of production (by choosing y_is and thus a Y), keeping in mind that feeding back to the central planner a y_i too low will cause a cutback in the resources made available to them, and a y_i too high could cause them to operate at a loss if the central planner decided to charge them that price for their resources.

It is shown in Kornai and Liptak [14] that such a "two-level" planning problem can be represented by a polyhedral game in which the overall optimum can be written as

$$\max_C \min_Y\, [C'Y]$$

and furthermore can be solved using the fictitious play method developed in Brown [1] and Robinson [15]. This fictitious play can be couched in an RLP format and formalized in one of two ways—either as a two-level RLP in which there are two distinct optimization operations and two feedbacks within a single iteration of the RLP, or as an artificially constructed "combined" RLP in which both the central planner and the sectors make their decisions simultaneously. In each case there are several (to be explicit, $n + 1$) decision makers, rather than a single decision maker.

To be more explicit, the system can be written as a two-level RLP as follows:

$$\max_{C(t)}\, [Y(t)'C(t)] \qquad \text{s.t.} \qquad C(t) \in C_0$$

where C_0 is the set of non-negative $C(t)$ that not only satisfy $\sum_{i=1}^{n} c_1(t) = c$ but permit feasible operation of each of the sectors (i.e., gives enough resources to each sector so that they are able to meet their production quotas, etc.). The first feedback is given by

$$C(t) = \frac{t-1}{t}\, C(t-1) + \frac{1}{t}\, C^*(t)\,.$$

Then at the second level we have

$$\min_{y_i}\, [c_i(t)' y_i(t)] \qquad \text{s.t.} \qquad B_i' y_i(t) \geqq z_i,\, y_i(t) \geqq 0$$

for all i. The second feedback is given by

$$Y(t+1) = \frac{t}{t+1}\, Y(t) + \frac{1}{t+1}\, Y^*(t)\,.$$

Thus each "player" of the game is operating by choosing a strategy on the assumption that his opponent's future actions will resemble the past—in fact he plays the strategy which is optimal in terms of his total experience with his opponent, assigning equal weights to his opponent's previous actions. For further comments on the economic applications and implications of such a structure in the context of central planning, see Kornai and Liptak [14]; for a proof that this "ficticious" game process converges to the optimal overall solution, see Robinson [15].

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