ON THE HICKSIAN LAWS OF COMPARATIVE STATICS AND THE CORRESPONDENCE PRINCIPLE

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I. INTRODUCTION

Samuelson, in deriving, "fruitful theorems in comparative statics" [11, page 258], had laid down the correspondence principle which said that there was an "intimate connection" between theorems of comparative statics and the fact that equilibrium is stable. While the stability of equilibrium is necessary for comparative statics information to be meaningful, whether stability is sufficient for comparative statics information to be available, is quite another matter. Our aim is to study the content of the correspondence principle in this note. More precisely, we investigate the direction of change of the equilibrium prices, when there is a shift in tastes from the numeraire to some good, assuming only that the equilibrium is stable. We show below, that such a frontal attack on this problem produces the following meagre results. If there is a shift in demand from the numeraire to good j, then a "weighted" sum of price changes is positive (Proposition I). Since the "weights" could be negative, it is not even clear whether any price increases. This is guaranteed only when some additional restriction is satisfied (Proposition II). And, there will always exist a good k which satisfies the first Hicksian law, i.e., if demand shifts from the numeraire to good k, then the price of k rises (Proposition III). Since the direct attack fails, we turn our attention in another direction.

It is now well known that to have local stability, excess demand functions have to be restricted in one of a number of alternate ways. Gross substitution (GS) is one such restriction; the Morishima case (MC) is another; the dominant diagonal hypothesis (DD) is a third possibility. The implications of GS, for comparative statics information are that all the Hicksian laws hold: if there is a shift in demand from the numeraire to good j, then

(a) the price of j rises (the first law)
(b) the price of all other goods increase (the second law) and
(c) the price of j increases relatively the most, (the third law).

The Morishima case, if stable, also admits of rich comparative statics information. Both GS and MC are *qualitative* restrictions on the system. In such a framework, the existence of comparative statics information has been

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exhaustively studied by Basset Habibagahi and Quirk [2], Basset Maybee and Quirk [3] and Quirk [9]. The DD hypothesis, is a *quantitative* restriction, and this may also be exploited to advantage as the rest of our principal propositions bear out.

We note that the first Hicksian law is always satisfied by the DD hypothesis (Proposition IV); also, there are positive weights such that the weighted sum of price changes is positive (Proposition V) and this holds even if demand is shifted to a group of commodities from the numeraire (Proposition VI): a sort of generalization of the second and third Hicksian laws. Finally, if a special form of the DD hypothesis holds, then the third Hicksian law holds (Proposition VII).

Thus, although the DD hypothesis "allows for extremely complex relationships" [1, page 251], some interesting results are still deduced. Since GS and the stable MC satisfy DD, these results may be interpreted as an attempt to get at conditions which are weaker than GS or MC but still imply the Hicksian laws. It should be pointed out that the implications of DD for comparative statics information, ought to be noted on its own merit. This is because, for comparative statics information to be meaningful, equilibrium must not only be stable but also *unique*. And one of the weakest restrictions which guarantees uniqueness is that DD holds at equilibrium [1, page 234].

II. THE SCOPE OF THE CORRESPONDENCE PRINCIPLE

Consider an economy with $n + 1$ goods, where the excess demand functions Z_j are continuously differentiable functions of the prices $p_1, \ldots, p_n$ and α. p_j denotes the price of good j relative to the numeraire good $n + 1$, and α is a shift parameter. $p^* = (p_1^*, \ldots, p_n^*, 1)$ is an equilibrium price relative to α^* if and only if $Z_j(p^*, \alpha^*) = 0$ for all $j = 1, 2, \ldots, n + 1$. Suppose now that α changes from α^*, reflecting a shift in tastes from the numeraire commodity $n + 1$ to the good 1. How do the prices p_j^*'s change?

To find the answer, differentiating the excess demand functions with respect to α, writing $a_{ij} = \partial Z_i(p^*, \alpha^*)/\partial p_j$, $p_{i\alpha} = dp_i/d\alpha$, $b' = (-Z_{1\alpha}, 0, \ldots, 0)$ where $'$ denotes transposition, $Z_{1\alpha} = \partial Z_1(p^*, \alpha^*)/\partial \alpha$ and the matrix $A = (a_{ij})$, $i, j = 1, \ldots, n$, we have

$$A[p_{i\alpha}] = b \,. \tag{1}$$

In fact, if A is non-singular, A^{-1} exists and denoting A^{-1} by (c_{ij}), we have

$$p_{i\alpha} = -c_{i1} Z_{1\alpha} \tag{2}$$

which is the answer to the question posed above. It should be pointed out that the change in α, we investigate, affects *only* the excess demand for good 1 and that of the numeraire; also $Z_{1\alpha}$ is positive. Thus from (2), $p_{i\alpha}$ has the same sign as $-c_{i1}$. Thus, before we can say whether the i-th price moves up or down, we would have to obtain the sign of the i-th entry in the first column of A^{-1}.

To investigate the scope of the correspondence principle, let us assume further, that the equilibrium is locally stable under an adjustment of the form

$$\begin{aligned} &(3) \qquad & \dot{p}_j &= Z_j(p, \alpha^*)\,, \quad j = 1, \ldots, n \\ && p_{n+1} &= 1\,. \end{aligned}$$

Then the matrix A defined in (1) is a stable matrix; i.e., each characteristic root of A has its real part negative. A^{-1} also exists and must be a stable matrix too; this follows immediately, if one recalls that t is a characteristic root of A if and only if $1/t$ is a characteristic root of A^{-1}. For stable matrices, it is known that the following holds:

Lyapunov theorem. A real $n \times n$ matrix C is stable if and only if there is a symmetric matrix B which is positive definite such that $BC + C'B$ is negative definite. [C is positive (negative) definite if and only if $x'Cx > (<)\, 0$ for all $x \neq 0$.]

Thus there must be a positive definite matrix $B = (b_{ij})$ such that $BA^{-1} + (A^{-1})'B$ is negative definite, since A^{-1} is stable. Then rewriting (1) as

$$[p_{i\alpha}] = A^{-1}b$$

we have

$$b'B[p_{i\alpha}] = b'(BA^{-1})b < 0 \qquad \text{or} \qquad Z_{1\alpha} \sum_{i=1}^{n} b_{1i}\, p_{i\alpha} > 0\,.$$

Hence the preliminary investigation undertaken above, allows us to conclude:

Proposition I. A is stable implies that there is a positive definite matrix B, such that if there is a shift in demand from the numeraire to good j, then

$$b^j[p_{i\alpha}] > 0$$

where b^j is the j-th row of B.

This, of course, is somewhat disappointing; note that we cannot, in general, even conclude that some prices rise; only, one may say that a "weighted" sum of price changes is positive where the "weights" b_{jk} can be negative if $j \neq k$. However, if in addition to the stability of A, we have some information like: The numeraire is a gross substitute for any other commodity (NGS). Then, since $\sum_{i=1}^{n} p_i Z_i + Z_{n+1} = 0$ must hold (Walras law), we have, differentiating with respect to p_j and evaluating at equilibrium,

$$p^{*\prime}A < 0\,.$$

Also

$$A\begin{pmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence it must be the case that $c_{j1} < 0$ for at least one j. This follows, because we know that for any matrix A, exactly one of the following holds [4]:

$$\begin{array}{lll} \text{Either} & Ax < 0 & \text{has a nonnegative solution} \\ \text{or} & Ay \geqq 0 & \text{has a semipositive solution}. \end{array}$$

Now if $c_{i1} < 0$, then from (2), we may conclude that $p_{i\alpha} > 0$. Thus:

Proposition II. If A is stable and (NGS) holds, then a shift in demand from the numeraire to commodity j, leads to some price rises.

But which prices rise? Does the price of the commodity to which demand has shifted go up? These questions go unanswered, even though we have an additional assumption in the form of (NGS). Returning to the fact that the stability of A implies that of A^{-1}, and that this surely implies that some diagonal element of A^{-1} is negative, one may note:

Proposition III. If A is stable, there would be at least one good j, such that if there is a shift in demand from the numeraire to commodity j, then there is a rise in the price of j.

Propositions I–III seem to exhaust the content of the correspondence principle. It should be pointed out that the last result has been noted before [10, page 209].

III. GS, MC AND THE HICKSIAN LAWS

Usually comparative statics information is sought when along with stability, some other conditions are satisfied. The most well known case is that of GS; see [10], for example. The GS assumption amounts to assuming $a_{ij} > 0$ for $i \neq j$ where $A = (a_{ij})$ is as defined in (1). Under GS, there is an unique $p^* > 0$ for eadh value of α; by redefining units of measurements, $p_j^* = 1$ for all j.[1] Also, under GS, $A^{-1} < 0$, see [6], for example. Hence from (2), one may conclude

(i) $p_{1\alpha} > 0$: the first Hicksian law,
(ii) $p_{j\alpha} > 0$: the second Hicksian law,
(iii) $p_{1\alpha} > p_{j\alpha}$: the third Hicksian law, since all the equilibrium prices have been chosen to be unity.

[1] We shall subsequently assume that $p_j^* = 1$ for all j holds, for the rest of the paper.

In fact, under GS, the same conclusions may be drawn when large parameter shifts are considered, see [8], for example.

A somewhat weaker restriction is that of MC. Non-numeraire commodities $1, 2, \ldots, n$ are assumed to satisfy

(a) $a_{ii} < 0$ for all $i = 1, 2, \ldots, n$,
(b) $a_{ij} \cdot a_{ji} > 0$ for all i, j,
(c) sign $a_{ij} =$ sign $(a_{ik} \cdot a_{kj})$ for all $i \neq j \neq k$.

For such a sign pattern, if the equilibrium is stable, then it is well known [7]:

(i′) $p_{1\alpha} > 0$
(ii′) $p_{j\alpha} >(<) 0$ if $a_{1j} >(<) 0$.

The third Hicksian law[2] may not hold here as we shall indicate below.

IV. THE DD HYPOTHESIS

GS and MC, if stable, satisfy the DD hypothesis. In this section, we work out the implications of the dominant diagonal hypothesis (DD) for comparative statics results. Usually, the matrix $A = (a_{ij})$ is said to have a negative dominant diagonal, if $a_{jj} < 0$ for all j and there are positive numbers $d_1, \ldots, d_n$ such that

$$d_j|a_{jj}| > \sum_{i \neq j} d_i|a_{ij}| , \quad j = 1, \ldots, n . \tag{4}$$

This is equivalent to saying that there are positive numbers $w_1, \ldots, w_n$ such that

$$w_j|a_{jj}| > \sum_{i \neq j} w_i|a_{ji}| , \quad j = 1, \ldots, n . \tag{5}$$

Usually (4) refers to the case of column dominant diagonal, whereas (5) refers to the case of row dominant diagonal. It is a fact that the existence of a column dominant diagonal for any matrix is equivalent to the existence of a row dominant diagonal for the same matrix; the number d's and w's in (4) and (5) are, in general, different. A proof of this is provided in an Appendix, since we could not find a reference for this result.

Thus, when A has DD, both (4) and (5) hold. For such a matrix A, A is Hicksian, i.e., a principal minor of A order r has the sign of $(-1)^r$. Consequently, diagnal elements of A^{-1} are negative. Thus, from (2), it follows that:

[2] With some price changes negative, the third Hicksian law, would be taken to imply, in the light of the previous footnote, that

$$|p_{1\alpha}| \geqq |p_{j\alpha}| \quad \text{for all } j$$

when there is a shift in demand from the numeraire to good 1.

Proposition IV. If A has a negative dominant diagonal, then a shift in demand from the numeraire to some commodity j, raises the price of j.

But how does a shift in demand form the numeraire to good j affect the prices of the other commodities? A rather interesting answer is possible in this connection. Just as Proposition IV was a strengthened form of Proposition III, due to the DD assumption, we can now give a more meaningful version of Proposition I. To see this, let $d' = (d_1, \ldots, d_n)$ where d_j's are as in (4). Then

$$\begin{aligned} & A'd = h < 0 \quad \text{or} \\ (6) \qquad & [p_{i\alpha}]'A'd = b'd \quad \text{from (1)} \quad \text{or} \\ & \sum_i (-h_i)p_{i\alpha} > 0\,; \quad \text{we have therefore.} \end{aligned}$$

Proposition V. If A has DD, then there are positive weights $t_1, \ldots, t_n$, such that if there is a shift in demand from the numeraire to any other good, then $\sum t_i p_{i\alpha} > 0$ where $t_i = -\sum d_j a_{ji}$ where d_j's satisfy (4).

Suppose now that there is a shift in demand from the numeraire not only to good 1, but also to goods 2, ..., k as well. This is, of course, more natural; when the demand for a good k shifts due to a change in tastes, say, it is only natual to expect that there is a shift in demand for goods complementary to k. A shift in demand for tea should be acompanied by shifts in demands for sugar and milk, for example. Now, in (1), $b' = (-Z_{1\alpha}, -Z_{2\alpha}, \ldots, -Z_{k\alpha}, 0, \ldots, 0)$, where $Z_{i\alpha}$ is positive for $i = 1, \ldots, k$. Note that Proposition IV does not necessarily hold anymore. But from (6),

$$[p_{i\alpha}]'A'd = b'd = -\sum_{i=1}^{k} d_i Z_{i\alpha} < 0$$

so that

$$\sum_{i=1}^{n} t_i p_{i\alpha} > 0$$

as before. Thus:

Proposition VI. Under DD, if there is a shift in demand from the numeraire to a group of goods 1, ..., k then $\sum_{i=1}^{n} t_i p_{i\alpha} > 0$ where $t_i > 0$ are defined in Proposition IV.

Thus, if there is a shift in demand from the numeraire to commodity j, then price of j rises; further, a positive weighted sum of price changes is positive; the last holds even if there is a shift in demand to a group of commodities. Thus the DD hypothesis has already provided us with more information regarding the validity of the Hicksian laws; we do not expect, of course, the clear-cut conclusions that could be drawn in the GS and MC cases. But what about the third Hicksian law?

Consider a special case, when A has DD and in (5),

$$w_1 = w_2 = \cdots = w_n = 1 . \tag{7}$$

Writing $A^{-1} = (c_{ij})$, as before, we can now assert that $|c_{11}| \geqq |c_{j1}|$ for all j. For, suppose not; then there is $k \neq 1$, such that $|c_{k1}| \geqq |c_{j1}|$ for all j. For this particular k, consider $\sum_j a_{kj} c_{j1} = 0$, since $AA^{-1} = I$ or $|a_{kk} c_{k1}| = |\sum_{j \neq k} a_{kj} c_{j1}| \leqq \sum_{j \neq k} |a_{kj}||c_{j1}| \leqq |c_{k1}| \sum_{j \neq k} |a_{kj}|$ or $|a_{kk}| \leqq \sum_{j \neq k} |a_{kj}|$ which contradicts (5) if $w_j = 1$ for all j. Thus no such k exists. As asserted, $|c_{11}| \geqq |c_{j1}|$ holds. Therefore, $|p_{1\alpha}| = |c_{11} Z_{1\alpha}| \geqq |c_{j1} Z_{1\alpha}| = |p_{j\alpha}|$. Thus:

Proposition VII. If A has a negative dominant diagonal and (7) holds, then a shift in demand from the numeraire to good j causes the price of good j to rise and this rise can not be less than the absolute value of the change in the price of good k, $k \neq j$.

Thus the special case of the DD hypothesis leads to the third Hicksian law of comparative statics. A word about the condition (7) becomes necessary; this condition is not wildly impossible—it is satisfied by the GS case. Since the equilibrium prices have been redefined so that they are all unity, (7) may therefore, be interpreted as the case when the matrix A has a row DD with the equilibrium prices as weights. In general, (7) need not hold and neither need Proposition VII. For example, let

$$A = \begin{pmatrix} -4 & -2 \\ -5 & -3 \end{pmatrix} \quad \text{then,} \quad A^{-1} = \begin{pmatrix} -3/2 & 1 \\ 5/2 & -2 \end{pmatrix}.$$

A represents a three goods economy with the Morishima sign pattern for the non-numeraire commodities. $d_1 = 11$, $d_2 = 6$ and $w_1 = 11$, $w_2 = 20$ yield DD for A. However, if there is a shift in demand from the numeraire to good 1, then $|p_{1\alpha}| < |p_{2\alpha}|$. But note that if there is a shift in demand from the numeraire to good 2, then $|p_{2\alpha}| > |p_{1\alpha}|$. Unfortunately, it has not been possible to prove this in general, i.e., there will always be one commodity satisfying the third Hicksian law, under DD. This is true if there are two non-numeraire commodities but an extension to an arbitrary number of commodities has not been possible.

V. CONCLUDING REMARKS

If there is a shift in demand from the numeraire to some good, or a group of goods, under DD, prices will generally rise; in fact, a weighted sum of price changes would be positive. This is in contrast to the rigid GS assumption, under which all prices rise. Consider the following passage [5, pages 75–76]: "Taking these things into account, it does appear that an increase in demand for a particular good (or a group of goods) is most likely to have an upward effect on prices in general. Of course, the good or goods for which demand increases must be of considerable importance if this upward tendency is to be at all widespread. And it is always probable that there will be a few particular goods, directly or indirectly complementary with the first, whose prices will actually fall." As is apparent, this conclusion obtains more easily from the DD hypothesis than from say, the GS assumption.

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REFERENCES

[1] Arrow, K. J. and F. H. Hahn, *General Competitive Analysis*, Holden-Day, 1971.

[2] Basset, L., H. Habibagahi and J. Quirk, "Qualitative Economics and Morishima Matrices," *Econometrica*, 1967.

[3] Basset, L., J. Maybee and J. Quirk, "Qualitative Economics and the Scope of the Correspondence Principle," *Econometrica*, 1968.

[4] Gale, D., *The Theory of Linear Economic Models*, McGraw-Hill, 1960.

[5] Hicks, Sir J. R., *Value and Capital* (Second Edition), Oxford University, 1946.

[6] McKenzie, L. W., "Matrices with Dominant Diagonals and Economic Theory," in *Mathematical Methods in the Social Sciences*, Arrow, *et al.*, eds., Stanford, 1959.

[7] Morishima, M., "On the Laws of Change in the Prsce System in an Economy with Complementary Commodities," *Osaka Economic Papers*, 1952.

[8] Morishima, M., *Equilibrium Stability and Growth*, Oxford, 1964.

[9] Quirk, J., "The Competitive Equilibrium: A Qualitative Analysis," in *Economic Models, Estimation and Risk Programming*: Essay in honor of G. Tintner, Beckmann, *et al.*, eds., Springer-Verlag, 1969.

[10] Quirk, J., and R. Saposnik, *Introduction to General Equilibrium Theory and Welfare Economics*, McGraw-Hill, 1969.

[11] Samuelson, P. A., *Foundations of Economic Analysis*, Harvard University, 1967.

APPENDIX: ROW DD $\Longleftrightarrow$ COLUMN DD

For easy reference, we have the following version of Gale [4, page 49, Theorem 2.10]: For any matrix A,

$$\begin{array}{lll} \text{either} & Ax < 0 & \text{has a non-negative solution} \\ \text{or} & yA \geqq 0 & \text{has a semipositive solution}. \end{array}$$

Suppose A has a column DD, i.e., (4) holds, i.e., there are positive numbers $d_1, \ldots, d_n$ such that $d_j|a_{jj}| > \sum_{i \neq j} d_i|a_{ij}|$ for all $j = 1, 2, \ldots, n$. Consider $A^* = (a^*_{ij})$, where $a^*_{ij} = |a_{ij}|$, $i \neq j$, $a^*_{jj} = -|a_{jj}|$. If A has to have a row DD, then there must be a non-negative solution to

$$\text{(i)} \qquad A^*w < 0$$

(any $w \geqq 0$ satisfying the above must satisfy $w > 0$). And suppose, that there is no non-negative solution to (i). Then from the Gale theorem, there is $y \geq 0$ such that $yA^* \geqq 0$.

Let $J = (i/y_i > 0) \neq \emptyset$. Thus, for each $j \in J$, we have:

(ii)
$$-y_j a^*_{jj} \leqq \sum_{\substack{i \in J \\ i \neq j}} y_i a^*_{ij} .$$

But since A has column DD, so has (a_{ij}), $i, j \in J$, being a principal minor of A. In other words,

(iii)
$$d_j |a_{jj}| > \sum_{\substack{i \in J \\ i \neq j}} d_i |a_{ij}| , \quad j \in J .$$

From (ii) and (iii), it follows that

$$\sum_{\substack{i \in J \\ i \neq j}} (d_i/d_j - y_i/y_j)|a_{ij}| < 0 \quad \text{for } j \in J$$

or

$$\sum_{\substack{i \in J \\ i \neq j}} y_i/d_j(d_i/y_i - d_j/y_j)|a_{ij}| < 0 \quad \text{for } j \in J$$

but this cannot hold for j such that $d_j/y_j = \min_{i \in J} d_i/y_i$ which is a contradiction.* Hence there cannot exist $y \geq 0$ and $yA^* \geqq 0$. Thus there is $w \geqq 0$ and $A^*w < 0$. In fact, $w > 0$ and (5) holds so that A has a row DD.

For the converse, if A has a row DD, then A' has a column DD, whence by the above, A' has a row DD or A has a column DD. This completes the proof.

* The argument in the last step is borrowed from [2], where it was shown that if A has, in our terminology, a column DD, then for any other set of positive constants c_j's,

$$c_j |a_{jj}| > \sum_{i \neq j} c_i |a_{ij}|$$

must hold for at least one j. We reproduce, the argument, for the sake of completeness.