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Title	POWER-POSITIVE MATRICES AND GLOBAL STABILITY OF COMPETITIVE EQUILIBRIUM
Sub Title	
Author	OKUGUCHI, KOJI
Publisher	Keio Economic Society, Keio University
Publication year	1975
Jtitle	Keio economic studies Vol.12, No.1 (1975.) ,p.37- 40
JaLC DOI	
Abstract	
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19750001-0037

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POWER-POSITIVE MATRICES AND GLOBAL STABILITY OF COMPETITIVE EQUILIBRIUM*

KOJI OKUGUCHI

I. INTRODUCTION

Except for pioneering works of Morishima [6], Arrow and Hurwicz [1] McKenzie [4] and Nikaido [10], serious studies of stability of competitive equilibrium in an economy containing both gross substitutes and gross complements have only recently been begun. We can cite Morishima [7], Quirk and Saposnik [12], Mukherji [8, 9], Ohyama [11] and Sato [13, 14] in this field.

Let there be $n + 1$ goods, good 0 being the numeraire and define:

P_i, price of the i-th non-numeraire good, $i = 1, 2, \ldots, n$,
$F_i(P_1, P_2, \ldots, P_n)$, excess demand function for the i-th non-numeraire good, $i = 1, 2, \ldots, n$,
$a_{ij} = \partial F_i / \partial P_j$, $i, j = 1, 2, \ldots, n$,
$A = [a_{ij}]$, the Jacobian matrix of excess demand functions,
$T = A + A'$, where "prime" denotes transposition.

According to Mukherji [8], a competitive economy is defined to be a *generalized gross substitute economy* (hereafter M-GGS for short) provided that there exists a square matrix $S \geqq 0$ or $S^{-1} \geqq 0$ such that

$$M = [m_{ij}] = STS^{-1}, \quad m_{ij} > 0, \quad m_{ii} < 0, \quad i, j = 1, 2, \ldots, n, \quad i \neq j.$$

G_2-Metzlerian due to Ohyama [11], on the other hand, is a competitive economy for which there exists a positive definite stochastic matrix G such that all off-diagonal elements of GT are positive and diagonal elements all negative. In other words, M and GT are Metzlerian. Earlier, Nikaido [10] introduced a concept of *generalized gross substitutability* (hereafter N-GGS) to allow for co-existence of gross substitutes and gross complements. Under N-GGS off-diagonal elements of $T = [t_{ij}]$ are all positive and diagonal elements all negative, that is, T is a Metzlerian. Thus N-GGS is a special case of M-GGS and of G_2-Metzlerian obtained by letting $S = I$ and $G = I$, respectively, where I is a unit matrix.

Noting that A, the Jacobian matrix of excess demand functions, can be rewritten as $A = B - \rho I$, $\rho > 0$, and ingeniously applying Brauer's theorem

* The author wishes to thank Professor Anjan Mukherji and Professor Ryuzo Sato for valuable comments. He is indebted also to Professor Michihiro Ohyama for prompting me to write this paper.

on power-positive matrices in [2], Sato [13, 14] calls A a *(power-transformed) gross substitute matrix of the* k-th *exponent* provided that a positive integer k can be found such that $B^k > 0$ (that is, B is power-positive[1]) and $C = B^k - \rho^k I$ becomes a gross substitute matrix, hence $C = [c_{ij}]$, $c_{ij} > 0$, $c_{ii} < 0$, $i \neq j$, $i, j = 1, 2, \ldots, n$. His principal finding is that the stability condition is *formally* identical with that of the original Metzler system provided that either (a) A is a gross substitute matrix of odd exponent, or (b) A is a gross substitute matrix of even exponent and at least one good is a weakly gross substitute of all other goods (or all other goods are weakly gross substitutes of at least one good).

Sato, however, is concerned with the case of linear excess demand functions, and both Mukherji and Ohyama deal only with *local* stability. Nikaido deals with global stability on the lines given by McKenzie [4, 5].

Our purpose in this paper is two-fold. First, to extend Mukherji's results on stability by applying Sato's method based on Brauer's theorem on power-positive matrices. Second, to derive a sufficient condition ensuring global stability of competitive equilibrium independently of choice of numeraire.

II. GLOBAL STABILITY OF COMPETITIVE EQUILIBRIUM

In this section we are concerned with global stability of a normalized adjustment process

$$dP_i/dt = F_i(P_1, P_2, \ldots, P_n)\,, \quad i = 1, 2, \ldots, n\,.$$

Under the assumption of M-GGS, $m_{ij} > 0$, $m_{ii} < 0$, $i \neq j$ and M can be rewritten as

$$M = N - \rho I\,,$$

where all elements of $N = [n_{ij}]$ are positive and ρ is a positive scalar[2]. By Frobenius theorem on non-negative matrices, N has the absolute greatest characteristic root λ^*. Thus a necessary and sufficient condition for M to be an invariably stable matrix—that is, all of M's characteristic roots have negative real parts always—is $\rho > \lambda^*$. T being symmetric, its characteristic roots are all real. Further, T is similar to M. It follows therefore that under the assumption of $\rho > \lambda^*$, T's characteristic roots are all real

[1] According to Brauer [2], a real square matrix H is power-positive provided that a positive integer k can be found such that H^k is positive. If k is odd, H is power-positive of odd exponent, otherwise power-positive of even exponent. H may be power-positive of odd and even exponent at the same time.

[2] Note, however, that since a_{ij}'s are all functions of P_i's, hence of time, M, N and ρ all depend on time. This is in sharp contrasts to the case of linear excess demand functions or that of local stability where a_{ij}'s are constant.

and A negative quasi-definite. Hence global stability of the normalized adjustment process.[3] In short, a necessary and sufficient condition for global stability in M-GGS is $\rho > \lambda^*$.

In order to weaken the concept of M-GGS, let us now assume that N is power-positive, so that there exists a positive integer k satisfying

$$H = [h_{ij}] = N^k - \rho^k I ,$$

where all elements of N^k are positive and $\rho > 0$. In the case of odd k, the absolute greatest root, say λ^{**}, of N is positive, while if k is even and at least all elements of a column or a row of N are non-negative, the same can be said.[4] Assume $\rho > \lambda^{**}$. Then M becomes a stable matrix. Hence follows negative quasi-definiteness of A which is sufficient for global stability. These results are stated as the following

Theorem A. A necessary and sufficient condition for global stability in an economy which is invariably power-transformed M-GGS is formally identical with that for the original M-GGS provided that (a) either M is M-GGS of odd exponent, or (b) M is M-GGS of even exponent and all elements of at least one column or row of N are non-negative.

Before proceeding to consider the stability of competitive economy with regards to choice of numeraire, let us note that the following theorem is derivable by application of McKenzie's Theorem 2′ [5, p. 58].[5]

Theorem B. A necessary and sufficient conition for global stability in M-GGS economy is that M is invariably a matrix with dominant negative diagonals.

III. CHOICE OF NUMERAIRE AND STABILITY

Mukherji [9] has shown by using an example that stability of competitive equilibrium is in general not independent of choice of numeraire, and derived a sufficient condition which will ensure *local* stability regarless of choice of numeraire. Adapting his results [9, Theorem 2, p. 431 and Theorem 4, p. 432] we may prove that the condition in our Theorem B above is sufficient for *global* stability independently of choice of numeraire.[6]

We have derived our theorems without recource to Mukherji's crucial

[3] For the fact that negative quasi-definiteness implies global stability, see Arrow and Hurwicz [1], Hartman and Olech [3] and Quirk and Saposnik [12].

[4] See Brauer [2].

[5] Notice that T and M have the same characteristic roots, and take into consideration of negative quasi-definiteness of A.

[6] Mukherji's concern in [9] as in [8] is restricted to local stability.

assumption that $S \geqq 0$ or $S^{-1} \geqq 0$. What really matters is non-singularity of S only, as a closer look at our arguments reveals.

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