Title	ON THE OPTIMAL GROWTH OF THE TWO SECTOR ECONOMY
Sub Title	
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Publisher	Keio Economic Society, Keio University
Publication year	1975
Jtitle	Keio economic studies Vol.12, No.1 (1975.) ,p.1- 36
JaLC DOI	
Abstract	
Notes	
Genre	Journal Article
URL	https://koara.lib.keio.ac.jp/xoonips/modules/xoonips/detail.php?koara_id=AA00260492-19750001-0001

ON THE OPTIMAL GROWTH OF THE TWO SECTOR ECONOMY*

JOHN Z. DRABICKI AND AKIRA TAKAYAMA

A. INTRODUCTION

The purpose of this paper is to investigate the optimal growth program for a two-sector economy under a non-linear objective function. As in the usual two-sector model, there are two industries—capital good (X) and consumption good (Y)— in the economy, each of which uses two factors of production, labor (L) and capital (K). It is assumed, as in the usual two-sector model, that the capital good is completely malleable and shiftable between the industries.

The optimal growth problem for such a two-sector economy is discussed in the literature quite extensively.[1] Srinivasan [20] and Uzawa [24][2] considered this problem under a linear objective function: the economy is to maximize the discounted stream of per capita consumption.[3] While Srinivasan solved the problem for the case when the consumption good industry is always more capital intensive than the capital good industry, Uzawa dispensed with this assumption while assuming a lower bound on per capita consumption.[4] Unfortunately, however, Uzawa's study contains a serious mistake which was pointed out by Haque [12]: Uzawa misspecified the direction of motion of the demand and supply price of capital which, subsequently, lead to an incorrect construction of the optimal path. In his paper, Haque first solved the problem considered by Srinivasan without assuming any capital intensity condition, and then, likewise, solved the problem considered by Uzawa. Thus, Haque

* This paper is a reproduction of paper No. 383 under the same title in the *Krannert Institute Papers*, Purdue University, January 1973. Prior to that, it was circulated privately in the fall of 1972 to Professors Ken-ichi Inada, Hukukane Nikaido, and Murray C. Kemp. The writing of the paper was motivated, in part, by Drabicki's discovery of some serious mistakes in Hadley and Kemp [11] in the spring of 1972 which was followed by our correspondences with Professor Kemp. He and Sheng Cheng Hu provided us with some useful comments.

[1] Several authors have also considered the optimal growth problem for a two-sector economy in which capital is not shiftable between the two sectors. See, for example, Kurz [16], Bose [4], Chakravarty [8], Dasgupta [9], Johansen [15], Ryder [18], and Hadley and Kemp [11].

[2] For an exposition, see, Shell [19] and Intriligator [14].

[3] In this paper, the economy will maximize the discounted stream of the utility of per capita consumption.

[4] That is, Uzawa [24] separately considered the two cases $k_x(\omega) > k_y(\omega)$ and $k_y(\omega) > k_x(\omega)$ for all $\omega > 0$, where k_i is the capital: labor ratio in sector i, $i = X, Y$, and ω is the wage: rental ratio, and then inferred, without demonstrating it, that his analysis for these two polar cases could be applied to deal with cases of factor intensity reversals.

solved the problem in full generality for the case of the linear objective function.

The two-sector optimal growth problem under a non-linear objective function was considered by Cass [6].[5] He, as Srinivasan, focused his attention to the case in which the consumption good industry is always more capital intensitive than the capital good industry.[6] But this seems to be a quite "unintuitive" assumption, and neither Cass nor Srinivasan give any economic justification for assuming it. On the contrary, the opposite assumption may be more palatable to some. In any case, a more comprehensive study which considers both of the two extreme cases was attempted by Hadley and Kemp [11], but it unfortunately contains some serious mistakes.[7]

Thus the problem[8] for the case in which the capital good industry is more capital intensive has yet to be solved. In addition, the assumption of no factor intensity reversals, which no one has yet confronted in the non-linear case, has to be dispensed with. It seems as if the assumption of no factor intensity reversals is rather restrictive and unrealistic as such reversals can and do occur.[9] Hence, the purpose of this paper is to solve the two-sector optimal growth problem without any capital intensity condition. Thus, we consider not only extreme cases where one industry is always more capital intensive than the other, but will allow for any number of factor intensity reversals. What is more, we do not confine ourselves to the usual factor intensity reversals which entail identical capital intensities in both industries for some isolated value of the wage : rental ratio; we allow for identical capital intensities for intervals of the wage : rental ratio. We may consider the latter to be a degenerate version of capital intensity reversals. We thus allow for all factor intensity situations that can possibly exist.

The plan of the paper is as follows. First, after a brief description of the model, the problem will be solved simultaneously for the two polar cases. This not only leads to the solution of the above-mentioned unsolved case, but also provides us with a framework to view these polar cases in a proper perspective, leading to the more general case of factor intensity reversals. In solving the problem a simple geometric construction is utilized and, as it

[5] Uzawa [25] considered the case in which utility was derived from consuming both a "pure" consumption good as well as another good which could be used for either consumption or investment purposes.

[6] An outline of Cass [6] can be seen in Foley, Shell, and Sidrauski [10].

[7] These will be pointed out from time to time in the course of the paper.

[8] Under a non-linear objective function.

[9] For example, as is well known, if both industries have production functions of the CES type with constant but different elasticities of factor substitution then a unique factor intensity reversal can always occur. See, Takayama [21], page 83.

may be of use elsewhere, it is described in the Appendix to this paper.[10] Then, with a few simple modifications, the analysis of these two cases will be utilized to study the behavior of two-sector economies which experience any number of factor intensity reversals in production. It will be shown that the intensity assumption is not relevant in terms of producing different solution paths. We will demonstrate that the optimal paths of economies which do not experience factor intensity reversals and the optimal paths of economies which do, are all of the same general nature. That is, capital intensity does not matter.

B. THE MODEL AND PRELIMINARY CONDITIONS[11]

If $K_x(t)$ and $L_x(t)$, respectively, are allowed to represent the amounts of capital and labor employed in the capital good sector at time t, the rate at which the capital good is produced is determined by the production relationship $X(t) = F[K_x(t), L_x(t)]$; similarly, $Y(t) = G[K_y(t), L_y(t)]$. Letting[12] $x \equiv X/L$, $y \equiv Y/L$, $k_i \equiv K_i/L_i$ and $l_i \equiv L_i/L$ for $i = X, Y$, if both production functions exhibit constant returns to scale with positive but diminishing marginal products for both factor inputs, then

$$\begin{aligned} x &= l_x f(k_x)\,, \quad \text{where } f(k_x) \equiv F(k_x, 1) \\ y &= l_y g(k_y)\,, \quad \text{where } g(k_y) \equiv G(k_y, 1) \end{aligned}$$

with

$$(1) \qquad \begin{matrix} f(k_x) > 0\,, & f'(k_x) > 0\,, & f''(k_x) < 0 & \text{for all } k_x > 0 \\ g(k_y) > 0\,, & g'(k_y) > 0\,, & g''(k_y) < 0 & \text{for all } k_y > 0\,. \end{matrix}$$

The following well-known conditions are also imposed,

$$(2) \qquad \begin{matrix} \lim\limits_{k_x \to 0} f(k_x) = 0\,, & \lim\limits_{k_x \to \infty} f(k_x) = \infty\,, & \lim\limits_{k_x \to 0} f'(k_x) = \infty\,, & \lim\limits_{k_x \to \infty} f'(k_x) = 0 \\ \lim\limits_{k_y \to 0} g(k_y) = 0\,, & \lim\limits_{k_y \to \infty} g(k_y) = \infty\,, & \lim\limits_{k_y \to 0} g'(k_y) = \infty\,, & \lim\limits_{k_y \to \infty} g'(k_y) = 0\,. \end{matrix}$$

For each of the inputs, the combined employment in the two sectors cannot exceed the amount supplied. Thus

$$\begin{aligned} k_x l_x + k_y l_y &\leqq k \\ l_x + l_y &\leqq 1 \end{aligned}$$

[10] Also, some of the details of our analysis are carried out in the Appendix so that readers interested solely in the major results will not have to be troubled with them in the paper itself.

[11] Needless to say, some of the information in this section is well known. However, it is included to keep the paper sufficiently self-contained.

[12] For convenience, the dependence of the time dependent variables on time will no longer be noted.

where $k \equiv K/L$ is the aggregate capital : labor ratio.[13] The supply of labor is given exogenously and grows at a positive and constant exponential rate n. If the rate of change in the capital stock is given by the amount of capital being produced less depreciation, then[14]

$$\dot{k} = x - \lambda k$$

where

$$\lambda \equiv n + \mu > 0 \tag{3}$$

and μ is a constant and positive depreciation factor.

At any point in time, social welfare is given by the utility of per capita consumption $U(y)$. The utility function exhibits positive but diminishing marginal utility,

$$U'(y) > 0\,, \qquad U''(y) < 0 \quad \text{for all} \quad y > 0 \tag{4}$$

as well as[15]

$$\lim_{y \to 0} U'(y) = \infty \quad \text{(or alternatively, } \lim_{y \to 0} U(y) = -\infty)\,. \tag{5}$$

Given the above technological and resource constraints, the government wants to choose the time path of resource allocations that will maximize the discounted stream of the utility of per capita consumption over an infinite planning horizon.

Thus, the problem can be stated as follows: choose the time path of l_x, l_y, k_x, k_y, x, y so as to maximize[16]

$$J \equiv \int_0^\infty U(y) e^{-\delta t}\, dt \tag{6}$$

subject of the constraints

$$\dot{k} = x - \lambda k\,, \qquad k(0) = k_0 \tag{7}$$

$$x = l_x f(k_x) \tag{8}$$

$$y = l_y g(k_y) \tag{9}$$

$$k_x l_x + k_y l_y \leqq k \tag{10}$$

$$l_x + l_y \leqq 1 \tag{11}$$

$$l_x \geqq 0\,, \quad l_y \geqq 0\,, \quad k_x \geqq 0\,, \quad k_y \geqq 0\,, \quad x \geqq 0\,, \quad y \geqq 0\,. \tag{12}$$

[13] These conditions are obtained by dividing the resource constraints $K_x + K_y \leqq K$ and $L_x + L_y \leqq L$ and noting the previous definitions.

[14] Logarithmic differentiation of $k = K/L$ yields $\dot{k}/k = \dot{K}/K - \dot{L}/L$ or $\dot{k} = \dot{K}/L - (\dot{L}/L)k$. Substituting $\dot{K} = X - \mu K$ and $\dot{L} = nL$ into the preceding expression gives $\dot{k} = x - \lambda k$.

[15] This means that the marginal utility of zero consumption if infinite, or, alternatively, the utility derived from consuming nothing is negative infinity. These assumptions will serve the same purpose of guaranteeing that the consumption good will always be produced.

[16] Here δ denotes the rate of discount, which is assumed to be a positive constant.

Introducing the multipliers $\hat{q}(t)$, $\hat{p}_x(t)$, $\hat{p}_y(t)$, $\hat{r}(t)$, and $\hat{w}(t)$, define the Lagrangian function L as

$$(13)\quad \begin{aligned} L \equiv U(y)e^{-\delta t} &+ \hat{q}[x - \lambda k] + \hat{p}_x[l_x f(k_x) - x] + \hat{p}_y[l_y g(k_y) - y] \\ &+ \hat{r}[k - k_x l_x - k_y l_y] + \hat{w}[1 - l_x - l_y]\,. \end{aligned}$$

Then, the solution to the above problem together with the corresponding bounded multipliers must satisfy the following Euler-Lagrange-Hamiltonian conditions where $q \equiv \hat{q}e^{\delta t}$, $p_x \equiv \hat{p}_x e^{\delta t}$, and so on

$$(14)\qquad \dot{k} = x - \lambda k$$

$$(15)\qquad \dot{q} = (\lambda + \delta)q - r$$

and

$$(16)\qquad q - p_x \leqq 0\,, \qquad x[q - p_x] = 0$$

$$(17)\qquad U'(y) - p_y \leqq 0\,, \qquad y[U'(y) - p_y] = 0$$

$$(18)\qquad p_x l_x f'(k_x) - r l_x \leqq 0\,, \qquad k_x[p_x l_x f'(k_x) - r l_x] = 0$$

$$(19)\qquad p_y l_y g'(k_y) - r l_y \leqq 0\,, \qquad k_y[p_y l_y g'(k_y) - r l_y] = 0$$

$$(20)\qquad p_x f(k_x) - r k_x - w \leqq 0\,, \qquad l_x[p_x f(k_x) - r k_x - w] = 0$$

$$(21)\qquad p_y f(k_y) - r k_y - w \leqq 0\,, \qquad l_y[p_y f(k_y) - r k_y - w] = 0$$

$$(22)\qquad r \geqq 0\,, \qquad r[k - k_x l_x - k_y l_y] = 0$$

$$(23)\qquad w \geqq 0\,, \qquad w[1 - l_x - l_y] = 0$$

as well as the constraints (8)–(12) plus the initial condition $k(0) = k_0$.

From assumption (5) and condition (17) we have

$$(24)\qquad y > 0\,.$$

That is, the consumption good will always be produced. Noting this, from (17) we obtain

$$(25)\qquad U'(y) = p_y$$

which, in turn, utilizing (4) yields

$$(26)\qquad p_y > 0\,.$$

Combining (24) with (8), (1), and (2), obtains

$$(27)\qquad l_y > 0\,, \qquad k_y > 0$$

which implies that (19) and (21) hold with equality. In particular, from (19)

$$(28)\qquad p_y g'(k_y) = r$$

which together with (1), (26), and (27) gives

$$(29)\qquad r > 0\,.$$

Utilizing (29), condition (22) implies

$$k_x l_x + k_y l_y = k \tag{30}$$

which states that capital is always fully employed. Interpreting the multiplier r as the rental rate on capital, the above conditions state that all of the capital available will be employed if the rental rate is positive. A similar result can be obtained for the labor market when the multiplier w is interpreted as the wage rate. That is, (27) with (21) and (28) yields

$$p_y[g(k_y) - k_y g'(k_y)] = w\,. \tag{31}$$

As is well known, $g(k_y) - k_y g'(k_y)$ is the marginal product of labor in the consumption good industry and is positive for all $k_y > 0$. Noting this, (26), (27), and (31) give

$$w > 0 \tag{32}$$

which, conbined with (23), yields[17]

$$l_x + l_y = 1 \tag{33}$$

which states that labor will be fully employed when its wage rate is positive.

It should be noted here that the full employment of both factors, etc., obtained above were derived from the fact that the consumption good is always produced. However, the same is not true for the capital good. Thus, two cases must be distinguished: (i) incomplete specialization, and (ii) complete specialization in the production of Y. Each case will be considered, in turn.

(i) Incomplete specialization: $y > 0$, $x > 0$

Production of the capital good implies

$$q = p_x \tag{34}$$

from (16). Also, from (8), (1) and (2)

$$l_x > 0\,, \qquad k_x > 0 \tag{35}$$

which makes (18) and (20) hold with equality. Recalling (28) and (31), from (18), (20), and (35) we obtain that under incomplete specialization

$$p_x f'(k_x) = p_y g'(k_y) = r \tag{36}$$

$$p_x[f(k_x) - k_x f'(k_x)] = p_y[g(k_y) - k_y g'(k_y)] = w\,. \tag{37}$$

But, these conditions are nothing more than the well-known marginal productivity rule that determines allocative efficiency between the two sectors

[17] As opposed to Cass [6] and Hadley and Kemp [11], who *a priori* assumed the full employment of both inputs, here the full employment conditions, (30) and (33), are derived from the maximization.

for inputs in a competitive economy. That is, interpreting the multiplies p_x and p_y, respectively, as the supply price of the capital good and the supply price of the consumption good, (36) states that the value of the marginal product of capital in both sectors must equal the rental rate. Similarly, condition (37) states that the value of the marginal product of labor in both sectors must equal the wage rate. Also, (36) implies $p_x > 0$.

Setting $\omega \equiv w/r$, which can be interpreted as the wage : rental ratio, division of (37) by (36) gives the well-known static allocation rule

$$\omega = \frac{f(k_x)}{f'(k_x)} - k_x = \frac{g(k_y)}{g'(k_y)} - k_y \,. \tag{38}$$

Since

$$\frac{d\omega}{dk_x} = -\frac{f(k_x)f''(k_x)}{[f'(k_x)]^2} > 0\,, \qquad \frac{d\omega}{dk_y} = -\frac{g(k_y)g''(k_y)}{[g'(k_y)]^2} > 0 \tag{39}$$

from (38) and (1), condition (38) can be inverted to yield k_x and k_y uniquely in terms of ω,

$$\begin{aligned} k_x &= k_x(\omega)\,, \qquad \frac{dk_x}{d\omega} > 0 \\ k_y &= k_y(\omega)\,, \qquad \frac{dk_y}{d\omega} > 0\,. \end{aligned} \tag{40}$$

Introducing $p \equiv p_x/p_y$, which can be interpreted as the price of capital in terms of the consumption good, from (36) and (40),[18]

$$p(\omega) = \frac{g'[k_y(\omega)]}{f'[k_x(\omega)]}\,. \tag{41}$$

Logarithmic differentiation of (41) yields

$$\frac{1}{p(\omega)}\frac{dp(\omega)}{d\omega} = \frac{1}{k_x(\omega) + \omega} - \frac{1}{k_y(\omega) + \omega} \gtreqless 0 \quad \text{as } k_y(\omega) \gtreqless k_x(\omega)\,. \tag{42}$$

Hence, regardless of what industry is more capital intensive, p and ω are uniquely related. Thus, from (40)–(42), it follows that k_x and k_y are uniquely determined by p,[19]

$$k_x = k_x(p)\,, \qquad k_y = k_y(p) \tag{43}$$

with

$$\frac{dk_x}{dp} \gtrless 0 \quad \text{and} \quad \frac{dk_y}{dp} \gtrless 0 \quad \text{as } k_y \gtrless k_x\,. \tag{44}$$

[18] Note that $p > 0$ for $\omega > 0$.

[19] Here we do not allow for factor intensity reversals for when they do occur condition (43) is no longer true. This can be seen from conditions (40)–(42).

Equations (30) and (33) can be solved to yield

$$l_x = \frac{k_y - k}{k_y - k_x}, \qquad l_y = \frac{k - k_x}{k_y - k_x}. \tag{45}$$

Combining (43) with (8), (9), and the above, we have

$$x = x(p, k) = \frac{k_y(p) - k}{k_y(p) - k_x(p)} f[k_x(p)] \tag{46}$$

$$y = y(p, k) = \frac{k - k_x(p)}{k_y(p) - k_x(p)} g[k_y(p)] \tag{47}$$

where[20]

$$\frac{\partial x}{\partial p} > 0, \qquad \frac{\partial y}{\partial p} < 0 \tag{48}$$

$$\frac{\partial x}{\partial k} \lesseqgtr 0 \quad \text{and} \quad \frac{\partial y}{\partial k} \gtreqless 0 \quad \text{as } k_y \gtreqless k_x. \tag{49}$$

From (46) and (47), x and y are uniquely determined in terms of p and k, while (48) states that if the relative price of capital rises, the production of capital will increase and the production of the consumption good will decrease. Condition (49), on the other hand, states that if the capital : labor endowment ratio increases, the output of the capital intensive good will increase while the output of the labor intensive good will decrease. The well-known Rybczynski's theorem is a special case of this as it assumes that one of the inputs is fixed.

In summary, p and k determine x and y uniquely. Given p, k_x and k_y are determined by (43). These values, together with the given k, when substituted into (46) and (47) yield x and y, respectively. Since p and ω are uniquely related, the determination of k_x and k_y can be illustrated by the well-known Samuelson-Harrod diagram[21] in Figures 1 and 2 for some $p = \bar{p}$.

Finally, it is noted that for a given $\bar{p}$, if $k \notin [\bar{k}_x, \bar{k}_y]$ when $k_y > k_x$ or $k \notin [\bar{k}_y, \bar{k}_x]$ when $k_x > k_y$, then from (45) the non-negativity of l_x or l_y will be

[20] To obtain (48) straightforward differentiation of (46) and (47) and rearrangement of terms yields

$$\frac{\partial x}{\partial p} = \frac{1}{(k_y - k_x)}\left((k_y - k)[(f - k_x f') + k_y f'] \frac{dk_x}{dp} + (k - k_x) f \frac{dk_y}{dp}\right)$$

$$\frac{\partial y}{\partial p} = \frac{1}{(k_y - k_x)}\left((k_x - k)[(g - k_y g') + k_x g'] \frac{dk_y}{dp} + (k - k_y) g \frac{dk_x}{dp}\right)$$

the sign of which is determined from (44). Similarly, we have

$$\frac{\partial x}{\partial k} = -\frac{f}{k_y - k_x}, \qquad \frac{\partial y}{\partial k} = \frac{g}{k_y - k_x}.$$

[21] See, for example, Takayama [21], especially footnote 11 on page 78. That both k_x and k_y pass through the origin follows from (2) and (38).

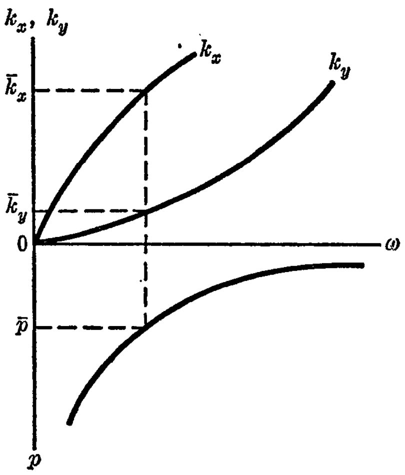

Fig. 1. Harrod-Samuelson Diagram: $k_x > k_y$

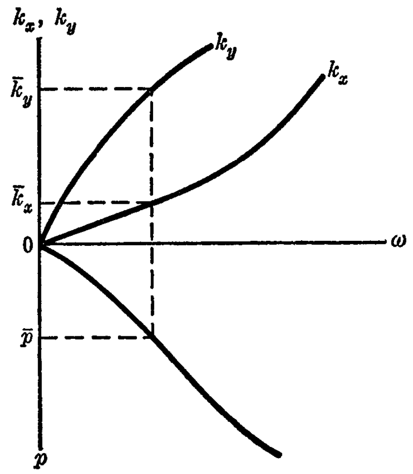

Fig. 2. Harrod-Samuelson Diagram: $k_y > k_x$

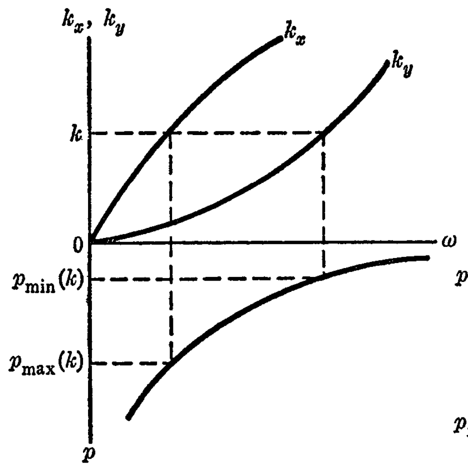

Fig. 3. p_{min} and p_{max}: $k_x > k_y$

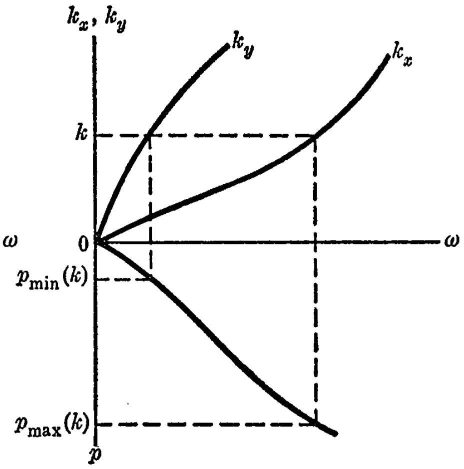

Fig. 4. p_{min} and p_{max}: $k_y > k_x$

violated. Thus, for the above relationships to hold, for a given k it must be that $p \in [p_{min}(k), p_{max}(k)]$ where the determination of $p_{min}(k)$ and $p_{max}(k)$ is illustrated in Figures 3 and 4. Note that from Figures 3 and 4, it follows easily that

$$\frac{dp_{min}(k)}{dk} \gtrless 0 \quad \text{as } k_y \gtrless k_x . \tag{50}$$

This will be of use later.

(ii) Complete specialization in Y: $y > 0$, $x = 0$

Since $x = l_x f(k_x)$, $x = 0$ can be attained when (a) $l_x = k_x = 0$, (b) $l_x = 0$, $k_x > 0$, or (c) $l_x > 0$, $k_x = 0$. However, we wish to show that (c) cannot hold in the present optimal growth problem. If $l_x > 0$, then from (9) and (33) $y = l_y g(k_y) < g(k_y)$, which means that the value of the integral (6) will be smaller than it would be if $l_x = 0$. Since $x = 0$, the lower level of consumption is not being compensated for by an increase in the capital stock which could lead to increased consumption at a future time.[22] Thus, we must have

$$l_x = 0 \tag{51}$$

which from (33) means that

$$l_y = 1 \,. \tag{52}$$

Utilizing the two previous equations, from (30)

$$k_y = k \,. \tag{53}$$

Lastly, from (9), (52), and (53)

$$y = g(k) \,. \tag{54}$$

It should be noted that conditions (52) and (53) merely state that when the capital good is not produced, the entire supplies of labor and capital are allocated to the production of the consumption good.

C. THE REGIONS OF INCOMPLETE AND COMPLETE SPECIALIZATION

In the subsequent sections, we will study the dynamic behavior of the optimal path by using the phase diagram technique to examine the behavior of the multiplier q and the aggregate capital : labor ratio k on the q–k plane, (q will be plotted on the vertical axis and k on the horizontal axis). Although, in the present model, the consumption good will always be produced, it is possible for the output of the capital good to be zero. Therefore, we have to distinguish two regions, the region in which both commodities are produced (incomplete specialization) and the region in which only the consumption good is produced (complete specialization). In this section, we determine these two regions as well as the locus which separates them (such a locus will be termed the *boundary curve*).

In the region of incomplete specalization, from (25) and (34), we have

$$q = U'[y(p, k)]p$$

which combined with Figures 3 and 4 and condition (54) implies that the boundary curve separating the regions of incomplete and complete specialization is defined by

$$q = U'[g(k)]p_{\min}(k) \equiv q(k) \,. \tag{55}$$

[22] If $y = l_y g(k_y) < g(k_y)$ and $x = 0$, then the economy is foregoing some of the consumption that it could have while not accumulating any capital which could incease consumption in the future. Clearly, such a policy cannot be optimal.

Along the boundary curve, production is specialized in Y. But for a fixed k, increasing p above $p_{\min}(k)$ will move us into the area of incomplete specialization since by (48) it will induce the production of the capital good. Since in the region of incomplete specialization $\partial q/\partial p > 0$,[23] this region must lie above the boundary curve.

Similarly, for a fixed k, decreasing p below $p_{\min}(k)$ will result in the continued specialization in Y. But here, from (16), (25), and (54) we have

$$q \leqq U'[g(k)]p$$

which implies that decreases in p will eventually decrease q. Thus the region of complete specialization lies below the boundary curve.

To determine the shape of the boundary curve, which will be labeled B, we calculate its slope

$$\left.\frac{dq}{dk}\right|_{\mathrm{B}} = U''[g(k)]g'(k)p_{\min}(k) + U'[g(k)]\frac{dp_{\min}(k)}{dk}\,. \tag{56}$$

From (1), (4), (50), and (56), it follows that

$$\begin{aligned} &\left.\frac{dq}{dk}\right|_{\mathrm{B}} < 0 && \text{if } k_x > k_y \\ &\left.\frac{dq}{dk}\right|_{\mathrm{B}} \text{ is of indeterminate sign} && \text{if } k_y > k_x\,. \end{aligned} \tag{57}$$

Moreover, (1), (4), (55) and Figures 3 and 4 imply that along the boundary curve

$$q(k) > 0 \quad \text{for } k > 0\,. \tag{58}$$

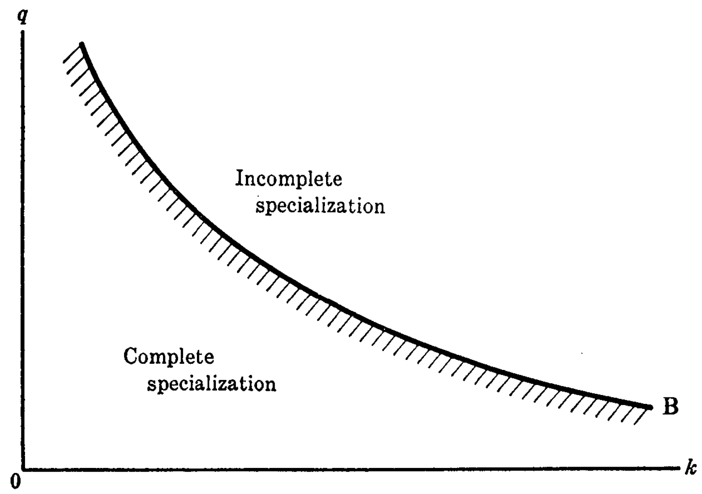

Fig. 5. Regions and Boundary Curve B: $k_x > k_y$

[23] This is obtained from $q = U'[y(p, k)]p$. See, equation (65) and footnote 27.

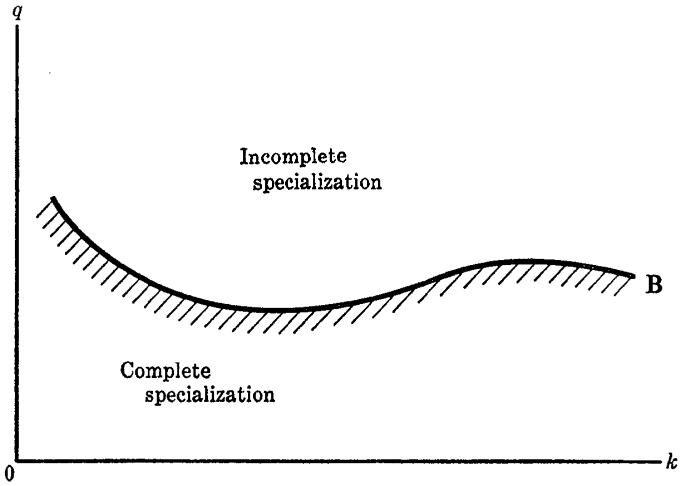

Fig. 6. Regions and Boundary Curve B: $k_y > k_x$

Lastly

(59) $$\lim_{k \to 0} q(k) = \infty \quad \text{if } k_x > k_y$$

from (2), (5), and (55). The above results may be seen in Figures 5 and 6.

D. THE $\dot{q} = 0$ CURVE

(i) Incomplete specialization

Under incomplete specialization, from (15), (34), and (36)

(60) $$\dot{q} = q[\lambda + \delta - f'(k_x)] .$$

Hence $\dot{q} = 0$ if and only if

(61) $$f'(k_x) = \lambda + \delta$$

which is satisfied by some unique k_x^*,[24] which, in turn, is satisfied by a unique p^* by (43) and (44).[25] Therefore, $\dot{q} = 0$ if and only if $p = p^*$. But from (25), (34), and (47)

(62) $$q = U'[y(p, k)]p \equiv q(p, k)$$

from which it follows that when $\dot{q} = 0$

(63) $$\tilde{q} = U'[y(p^*, k)]p^* .$$

Defferentiating the above and utilizing (4) and (49), the slope of the $\dot{q} = 0$

[24] The existence and uniqueness of k_x^* are guaranteed by assumption (2).

[25] Recall that since the consumption good is always produced, $p < p_{\max}$ always. Needless to say, $p^* < p_{\max}$.

curve can be calculated as[26]

$$(64)\qquad \left.\frac{dq}{dk}\right|_{\dot{q}=0} = U'' \frac{\partial y}{\partial k} p^* \lesseqgtr 0 \quad \text{as } k_y \gtreqless k_x .$$

That is, the slope of the $\dot{q} = 0$ curve is negative (positive) if $k_y > k_x$ ($k_x > k_y$). Now from (62), conditions (4), (48), and (49) imply[27]

$$(65)\qquad \frac{\partial q}{\partial p} > 0 , \qquad \frac{\partial q}{\partial k} \lesseqgtr 0 \quad \text{as } k_y \gtreqless k_x$$

which means that (62) can be inverted to give

$$(66)\qquad p = p(q, k) , \qquad \frac{\partial p}{\partial q} > 0 , \qquad \frac{\partial p}{\partial k} \gtreqless 0 \quad \text{as } k_y \gtreqless k_x .$$

Thus, from (66) and (43), equation (60) can be rewritten as

$$(67)\qquad \dot{q} = q\{(\lambda + \delta) - f'[k_x(p(q, k))]\} \equiv \dot{q}(q, k)$$

which utilizing (1), (49), and (66) yields

$$(68)\qquad \frac{\partial \dot{q}}{\partial k} = -qf'' \frac{dk_x}{dp} \frac{\partial p}{\partial q} > 0 .$$

This means that $\dot{q} > 0$ ($\dot{q} < 0$) to the right (left) of the $\dot{q} = 0$ curve.

(ii) Complete specialization in Y

When production is specialized in the consumption good, from (15), (25), (28), and (54) we have

$$(69)\qquad \dot{q} = q(\lambda + \delta) - U'[g(k)]g'(k) \equiv \dot{q}(q, k)$$

from which it follows that $\dot{q} = 0$ if and only if

$$(70)\qquad q = \frac{U'[g(k)]g'(k)}{\lambda + \delta} .$$

Utilizing the above expression as well as (1) and (4), we calculate slope of the $\dot{q} = 0$ curve to be

$$(71)\qquad \left.\frac{dp}{dk}\right|_{\dot{q}=0} = \frac{U''(g')^2 + U'g''}{\lambda + \delta} < 0 .$$

[26] Hadley and Kemp [11] set the sign of this expression as negative for both cases which is incorrect. Also they failed to investigate the behavior of $\dot{q}$ in the area of complete specialization in Y. Cass [6], of course, dealt only with the case $k_y > k_x$.

[27] Differentiation of (62) gives

$$\frac{\partial q}{\partial p} = pU'' \frac{\partial y}{\partial k} + U' , \qquad \frac{\partial q}{\partial k} = pU'' \frac{\partial y}{\partial k}$$

the sign of which is determined from (4), (48), and (49).

That is, the slope of the $\dot{q} = 0$ curve is negative in the region of complete specialization. What is more, by comparing (56) and (71), it can be shown that at the point of intersection of the boundary curve B with the $\dot{q} = 0$ curve, the slope of the boundary curve B is greater than the slope of the $\dot{q} = 0$ curve. We also obtain the following limiting result from (2), (4), (5), and (70): along the $\dot{q} = 0$ curve

$$\lim_{k \to \infty} q = 0 . \tag{72}$$

Finally from (69)

$$\frac{\partial \dot{q}}{\partial q} = \lambda + \delta > 0 \tag{73}$$

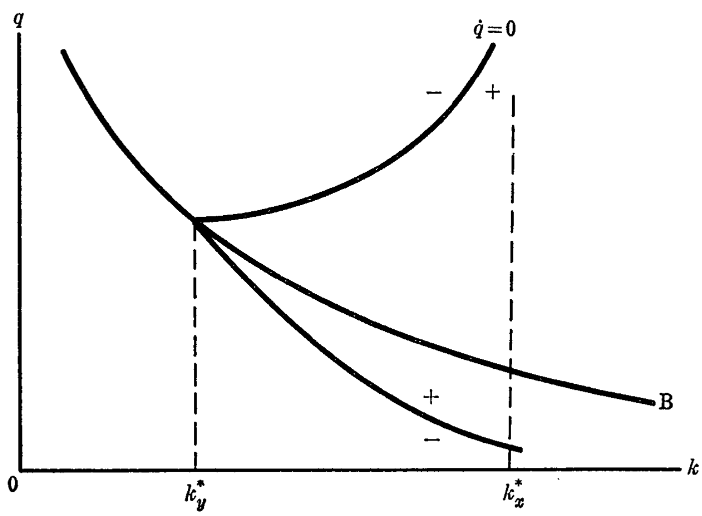

Fig. 7. $\dot{q} = 0$ Curve and Boundary Curve B: $k_x > k_y$

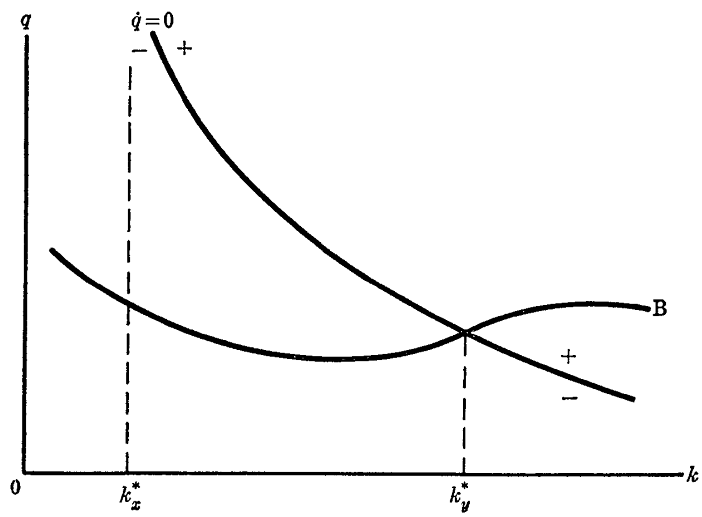

Fig. 8. $\dot{q} = 0$ Curve and Boundary Curve B: $k_y > k_x$.

which implies that $\dot{q} > 0$ ($\dot{q} < 0$) above (below) the $\dot{q} = 0$ curve. Utilizing the results of this section of the paper, the behavior of q can be summarized in Figures 7 and 8.[28]

E. THE $\dot{k} = 0$ CURVE

The behavior of the dynamic equation $\dot{k} = x - \lambda k$ is investigated in this section of the paper. Clearly, $\dot{k} = 0$ if and only if

$$x = \lambda k \,. \tag{74}$$

Hence $x > 0$ as long as $k > 0$. Since $y > 0$, the $\dot{k} = 0$ curve lies in the region of incomplete specialization.[29] Therefore, utilizing (46), $\dot{k} = 0$ if

$$x(p, k) = \lambda k \,. \tag{75}$$

Since we want to determine the shape of the $\dot{k} = 0$ curve in the $q - k$ plane, from (66) and (75) we have

$$x[p(q, k), k] = \lambda k \tag{76}$$

which upon being differentiated yields[30]

$$\left.\frac{dq}{dk}\right|_{\dot{k}=0} = pU'' \frac{\partial y}{\partial k} - \frac{\partial x/\partial k - \lambda}{\partial x/\partial p}\left[pU'' \frac{\partial y}{\partial p} + U'\right]. \tag{77}$$

The sign of (77) is generally indeterminate as can be seen from (4), (48), (49), and (A7). However, some information can be obtained. Noting that the intersection of the $\dot{k} = 0$ and $\dot{q} = 0$ curves must occur in the region of incomplete specialization, at their point of intersection we have[31]

$$\left.\frac{dq}{dk}\right|_{\dot{k}=0} \gtrless \left.\frac{dq}{dk}\right|_{\dot{q}=0} \quad \text{as } k_y \gtrless k_x \,. \tag{78}$$

Condition (78) simply states that when $k_y > k_x$, at their point of intersection the slope of the $\dot{k} = 0$ curve is greater than the slope of the $\dot{q} = 0$ curve.

[28] We note here that the depicted k_x^* and k_y^* will be later shown to correspond, respectively, to the balanced growth values of k_x and k_y given by $\dot{q} = \dot{k} = 0$. The interested reader is referred to Section B′ of the Appendix for a detailed discussion of the range of the $\dot{q} = 0$ curve.

[29] Thus the $\dot{k} = 0$ curve will lie above the boundary curve B. See Section C′ of the Appendix for additional results.

[30] Hadley and Kemp [11] calculated this expression incorrectly; this led to further errors.

[31] This result can be seen as follows. From (64) and (77) at $\dot{q} = \dot{k} = 0$ we have

$$\left.\frac{dq}{dk}\right|_{\dot{k}=0} - \left.\frac{dq}{dk}\right|_{\dot{q}=0} = \frac{\partial x/\partial k - \lambda}{\partial x/\partial p}\left[pU'' \frac{\partial y}{\partial p} + U'\right]$$

from which condition (78) is obtained by noting (4), (48), and (A7).

And, when $k_x > k_y$, the slope of the $\dot{q} = 0$ curve is greater than the slope of the $\dot{k} = 0$ curve at their intersection point. In both cases, this guarantees the uniqueness of the point determined by $\dot{q} = \dot{k} = 0$.

The above result, (78), can be strengthened, in fact, when $k_x > k_y$. It can be shown that (77) can be written as

$$\left.\frac{dq}{dk}\right|_{\dot{k}=0} = p^2U'(f - \lambda) - \frac{U'}{\partial x/\partial p}\left(\frac{\partial x}{\partial k} - \lambda\right). \tag{79}$$

From (61) we obtain that if $\dot{q} = 0$, then

$$f' - \lambda = \delta > 0 .$$

Noting this as well as (4), (48), and (A7) of Section C′ of the Appendix, equation (79) states that at $\dot{q} = \dot{k} = 0$

$$\left.\frac{dq}{dk}\right|_{\dot{k}=0} < 0 \quad \text{if } k_x > k_y . \tag{80}$$

That is, if the capital good industry is more capital intensive than the consumption good industry, the slope of the $\dot{k} = 0$ curve at its point of intersection with the $\dot{q} = 0$ curve is negative.

Defining the maximum sustainable capital : labor ratio $\bar{k}$ by

$$f(\bar{k}) = \lambda\bar{k} \tag{81}$$

from (5), (62), and (75) and Figures A.1, A.2, A.7, and A.8 we can obtain the following limiting results[32]

$$\lim_{k\to\bar{k}} q|_{\dot{k}=0} = \infty \tag{82}$$

$$\lim_{k\to 0} q|_{\dot{k}=0} = \infty \quad \text{if } k_x > k_y . \tag{83}$$

We also note that since $\bar{k}$ is defined by $f(\bar{k}) = \lambda\bar{k}$ and k_x^* by $f'(k_k^*) = \lambda + \delta$,[33] it follows from (1) that

$$k_x^* < \bar{k} \tag{84}$$

always. Thus, it is obvious that

$$k_y^* < \bar{k} \quad \text{when } k_x > k_y . \tag{85}$$

However, when $k_y > k_x$, while the relation $k_y^* > k_x^*$ must necessarily hold, in this case no *a priori* relationship between k_y^* and $\bar{k}$ can be determined.

[32] That is, $\bar{k}$ is that value of k which would produce $\dot{k} = 0$ if all of the inputs were allocated to the production of capital. Needless to say, by (24) this will never occur.

[33] Recall equations (61) and (81).

That is[34]

(86) $$k_y^* \gtreqless \tilde{k} \quad \text{when } k_y > k_x \,.$$

Lastly, from (46) and (66), equation (7) can be written

(87) $$\dot{k} = x[p(q, k), k] - \lambda k \equiv \dot{k}(q, k) \,.$$

Thus from (48), (66), and (87)

(88) $$\frac{\partial \dot{k}}{\partial q} = \frac{\partial x}{\partial p} \frac{\partial p}{\partial q} > 0 \,.$$

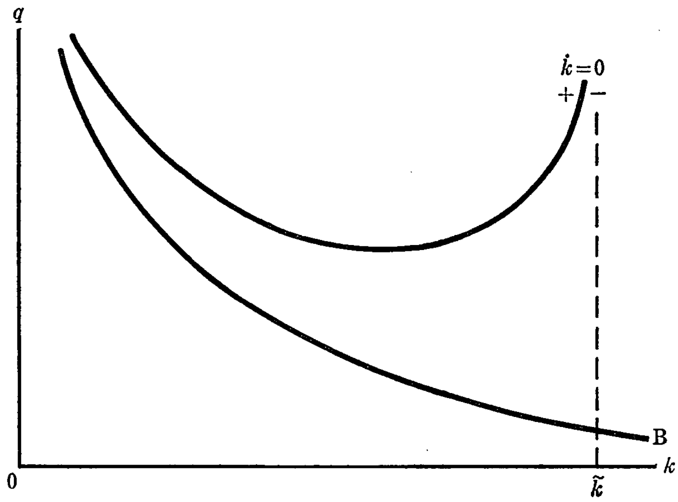

Fig. 9. $\dot{k} = 0$ Curve and Boundary Curve B: $k_x > k_y$

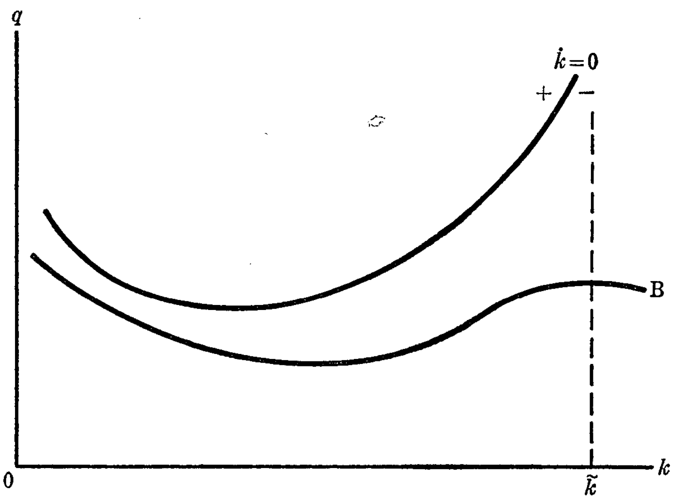

Fig. 10. $\dot{k} = 0$ Curve and Boundary Curve B: $k_y > k_x$

[34] This possibility was not pointed out by either Cass [6] or Hadley and Kemp [11].

Hence, in the area of incomplete specialization,[35] the area above the boundary curve B, $\dot{k} < 0$ below the $\dot{k} = 0$ curve and $\dot{k} > 0$ above the $\dot{k} = 0$ curve. However, when production is specialized in Y, the area on and below the boundary curve B, from (7) we have $\dot{k} = -\lambda k < 0$. Hence $\dot{k} < 0$ for all points lying below the $\dot{k} = 0$ curve. The above results can be seen in Figures 9 and 10.

F. THE DYNAMIC PATH

Utilizing the information of the previous sections the dynamic behavior of the economy can be described by the phase diagrams of Figures 11 and 12. Observe that there are three types of feasible paths:

$$\begin{aligned} &\text{Type } \alpha: \quad k(t) \to 0 \\ &\text{Type } \beta: \quad k(t) \to \tilde{k} \\ &\text{Type } \gamma: \quad k(t) \to k^* . \end{aligned}$$

Observe that the type γ path is the only path which satisfies the following right-end point condition[36]

$$\lim_{t \to \infty} e^{-\delta t} q(t) = 0 \tag{89}$$

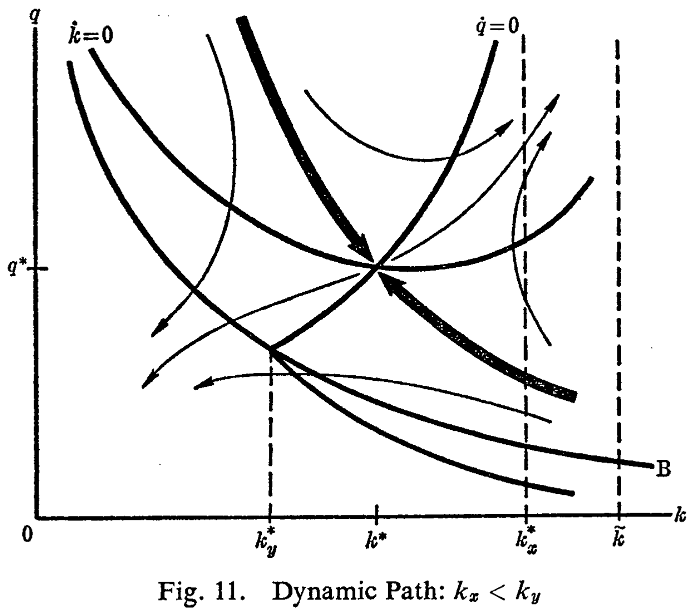

Fig. 11. Dynamic Path: $k_x < k_y$

[35] Recall that the $\dot{k} = 0$ curve lies in the region of incomplete specialization.

[36] That such conditions are necessary for optimality is not established for a general class of optimal growth problems. (See, for example, Arrow and Kurz [2], page 46.) However, it is known that such a condition is necessary for the present class of problems. See, Cass [6], Haque [12], and also Arrow and Kurz [2], for example.

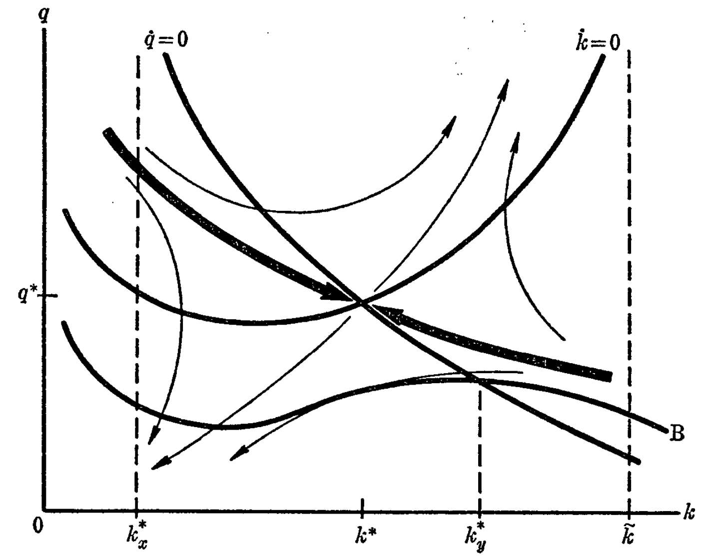

Fig. 12. Dynamic Path: $k_y > k_x$

or by explicitly requiring $\lim_{t\to\infty} k(t) \geqq 0$,

(89)′ $$\lim_{t\to\infty} e^{-\delta t}q(t) \geqq 0 \quad \text{and} \quad \lim_{t\to\infty} e^{-\delta t}q(t)k(t) = 0 .$$

We now show that the type γ path is indeed optimal. Let x, y, and so on, be the values of the variables and the multipliers along the type γ path and let x^0, y^0, and so on, be the value of the corresponding variables along any feasible path starting from the same initial value of $k(0)$. Then consider the following utility difference[37]

(90) $$D \equiv \int_0^\infty U(y)e^{-\delta t}\,dt - \int_0^\infty U(y^0)e^{-\delta t}\,dt .$$

In order to show that $D \geqq 0$, which establishes the optimality of the type γ path, first obtain the following inequality using (7)–(11), (30), and (33),[38]

(91) $$\begin{aligned} D \geqq \int_0^\infty e^{-\delta t}\{U(y) - U(y^0) &+ [x - \lambda k - \dot{k} - (x^0 - \lambda k^0 - \dot{k}^0)] \\ &+ p_x[l_x f(k_x) - x - (l_x^0 f(k_x^0) - x^0)] \\ &+ p_y[l_y g(k_y) - y - (l_y^0 g(k_y^0) - y^0)] \end{aligned}$$

[37] Obviously this statement is meaningless unless the integrals in (90) converge. Since it can be easily seen that the type γ path converges, we ignore all feasible paths whose discounted utility sums J diverge to $-\infty$. That is, we focus our attention to those feasible paths whose value of J is bounded from below. Hence it remains to be shown that the value of J is bounded from above along any feasible path. As can easily be seen from Figures 11 and 12, if $k(0) \leqq \tilde{k}$, then $k(t) \leqq \tilde{k}$ for all $t \geqq 0$ so that $y \leqq f(\tilde{k})$. Therefore $U(y) \leqq U[f(\tilde{k})]$ and J is bounded from above. Similarly, if $\tilde{k} < k(0) < \infty$ then $k(t) \leqq k(0)$ for all $t \geqq 0$ so that $y \leqq f(k(0))$. Thus $U(y) \leqq U[f(k(0))]$ and J is bounded from above.

[38] We may note that this inequality becomes an equality if $l_x^0 + l_y^0 = 1$ and $l_x^0 k_x^0 + l_y^0 k_y^0 = k^0$.

$$+ r[k - l_x \dot{k}_x - l_y k_y - (k^0 - l_x^0 k_x^0 - l_y^0 k_y^0)]$$
$$+ w[1 - l_x - l_y - (1 - l_x^0 - l_y^0)]\} \, dt .$$

Substituting (18)–(21), (28), and (31) into (91) yields

$$(92)\quad D \geqq \int_0^\infty e^{-\delta t}\{U(y) - U(y^0) - U'(y)(y - y^0) + (q\lambda - r)(k^0 - k) + q(\dot{k}^0 - \dot{k})\} \, dt .$$

By the concavity of U

$$U(y) - U(y^0) - U'(y)(y - y^0) \geqq 0$$

so that, utilizing (15), the RHS of (92) is greater than or equal to

$$(93)\quad \int_0^\infty e^{-\delta t}\{(q - \delta q)(k^0 - k) + q(\dot{k}^0 - \dot{k})\} \, dt = e^{-\delta t}p(t)[k^0(t) - k(t)]|_0^\infty .$$

But along the type γ path

$$\lim_{t\to\infty} e^{-\delta t}q(t) = 0$$

so that the value of expressinon (93) is zero. Thus $D \geqq 0$ so that the type γ path is clearly an optimal path.

The saddle-like nature of the solution can be confirmed by linearizing the two dynamic equations (67) and (87) around the point (k^*, q^*) and calculating the eigenvalues of the coefficient matrix

$$\begin{bmatrix} \dfrac{\partial \dot{k}}{\partial k} & \dfrac{\partial \dot{k}}{\partial q} \\ \dfrac{\partial \dot{q}}{\partial k} & \dfrac{\partial \dot{q}}{\partial q} \end{bmatrix}$$

which are equal to

$$(94)\quad \lambda_1, \lambda_2 = \frac{a \pm (a^2 - 4b)^{1/2}}{2}$$

where

$$(95)\quad a \equiv \frac{\partial \dot{k}}{\partial k} + \frac{\partial \dot{q}}{\partial q}$$

$$(96)\quad b \equiv \frac{\partial \dot{k}}{\partial k}\frac{\partial \dot{q}}{\partial q} - \frac{\partial \dot{k}}{\partial q}\frac{\partial \dot{q}}{\partial k}$$

all evaluated at (k^*, q^*), From (67) and (87)

$$\frac{\partial \dot{k}}{\partial k} = \frac{\partial x}{\partial p}\frac{\partial p}{\partial k} + \left(\frac{\partial x}{\partial k} - \lambda\right)$$

$$\frac{\partial \dot{q}}{\partial q} = -qf''\frac{dk_x}{dp}\frac{\partial p}{\partial q}$$

which substituted with (68) and (88) into (96) yields

$$b = -qf'' \frac{dk_x}{dp} \frac{\partial p}{\partial q} \left(\frac{\partial x}{\partial k} - \lambda \right). \tag{97}$$

From (1), (44), (66), (97), and (A7), we have

$$b < 0 \tag{98}$$

regardless of the capital intensity condition. Thus from (94) and (98) the eigenvalues λ_1 and λ_2 are real and of opposite signs, and hence (k^*, q^*) is indeed a saddle point.

We can thus assert the following:

Given an arbitrary initial value $k(0)$, *there exists a unique* $q(0)$ *through which passes a unique optimal path which approaches the balanced growth path* (k^*, q^*) *monotonically, regardless of whether* $k_x(\omega) > k_y(\omega)$ *or* $k_y(\omega) > k_x(\omega)$ *for all* $\omega > 0$.[39]

Finally, as a point of interest, it should be pointed out that from Figures 11 and 12 it can be seen that in both cases blocked intervals can occur along the optimal path. By blocked intervals we mean segments of the k-axis for which investment, that is, production of the capital good, is equal to zero.[40] Therefore incomplete specialization is not ruled out along the optimal path in either of the two cases, as blocked intervals occur when $k(0)$ is sufficiently large.

G. THE CASE OF FACTOR INTENSITY REVERSALS

Our analysis for the two polar cases, together with the results for the case of $k_x(\omega) = k_y(\omega)$, can be adapted to obtain the optimal growth path for two-sector economies which experience any number or kinds of factor intensity reversals. Here we explain briefly how such a general solution is obtained and state our final conclusions while illustrating some possible cases in Figures 13 thru 16. The details of our analysis as well as an exposition of the case $k_x(\omega) = k_y(\omega)$ can be found in Section D′ of the Appendix.

The solution of the general case is obtained by using the results obtained for the polar cases for those regions of k where they apply. Even when factor intensity reversals occur, for a given value of k, only one intensity condition holds. For example, for the case illustrated in Figure 13, $k_y(\omega) > k_x(\omega)$ for

[39] Since at (k^*, q^*) we have $p = p^*$, it is clear that the k_x^* and k_y^* of Figures 7 and 8 are indeed those values, respectively, of k_x and k_y which exist when the economy is at (k^*, q^*). See, the discussion in the Appendix.

[40] See, Arrow and Kurz [2].

all values of k in the interval $(0, k_r)$. Similarly, $k_y(\omega) = k_x(\omega)$ for all $k \in [k_r, k_s]$ and $k_x(\omega) > k_y(\omega)$ for all $k > k_s$. Thus, the k-axis is first partitioned on the basis of whether $k_y \gtreqless k_x$, and then the corresponding polar result is "applied" to each such interval. Again using Figure 13 as an example, since $k_y(\omega) > k_x(\omega)$ in the interval $(0, k_r)$, the shape of the boundary curve B is indeterminate in that region [recall condition (57)]. For $k \in [k_r, k_s]$ we have $k_x(\omega) = k_y(\omega)$ so that the slope of the boundary curve B becomes negative, by condition (A9) of the Appendix, in that interval. That is why abrupt changes in the shape of the boundary curve B are seen at $k = k_r$ in Figures 13 thru 16. Finally, for $k > k_s$ the condition $k_x(\omega) > k_y(\omega)$ holds so that the

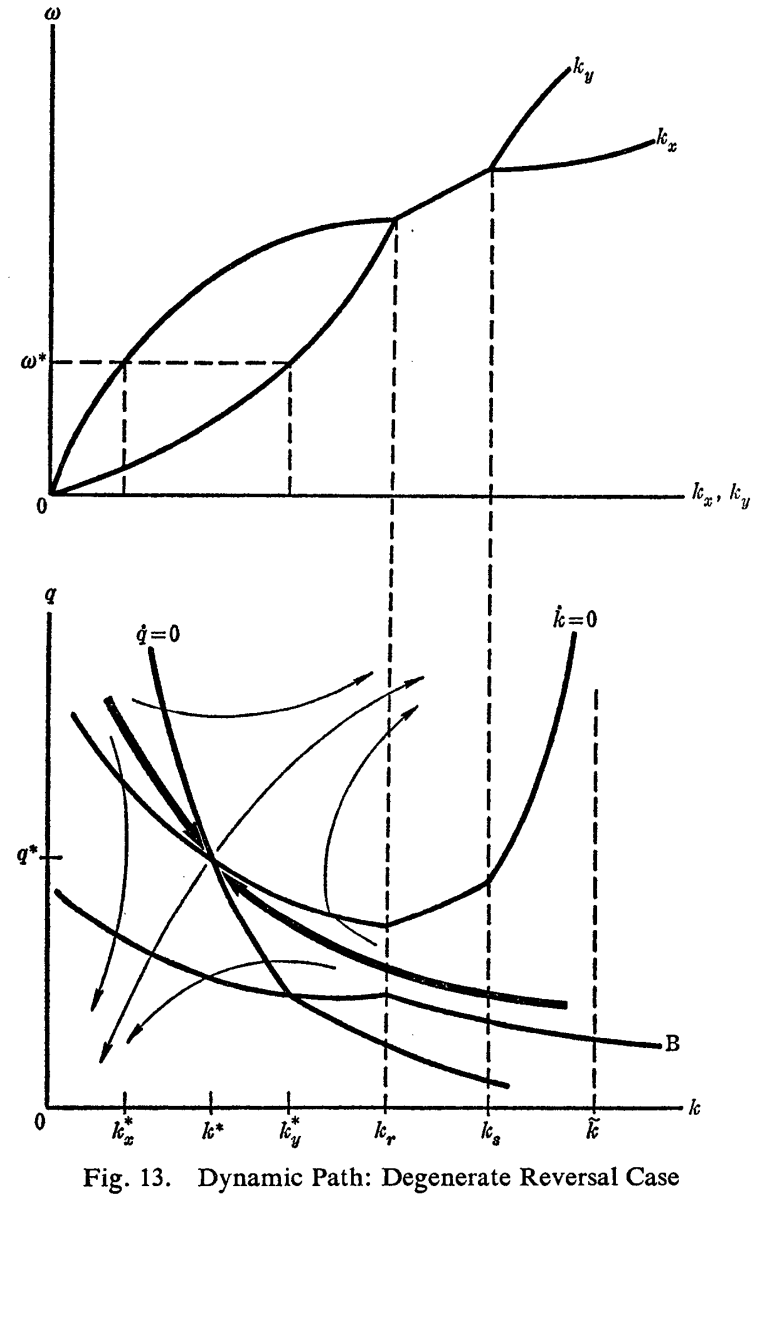

Fig. 13. Dynamic Path: Degenerate Reversal Case

boundary curve continues to have a negative slope by condition (57). The shape of the $\dot{k} = 0$ curve is determined analogously, as is the shape of the $\dot{q} = 0$ curve. However, in the case of the $\dot{q} = 0$ curve, an additional factor complicates the analysis: the shape of the curve depends on whether $k_y \gtreqless k_x$ for the value of k at which $f'(k) = \lambda + \delta$. This complication as well as other considerations are discussed in the Appendix.

As is suggested by Figures 13 thru 16, regardless of the number of factor intensity reversals that may occur, the optimal path for the two-sector economy is of the same general nature. As can be seen, factor intensity reversals produce kinks in and affect the slopes of the boundary curve B, and so on.

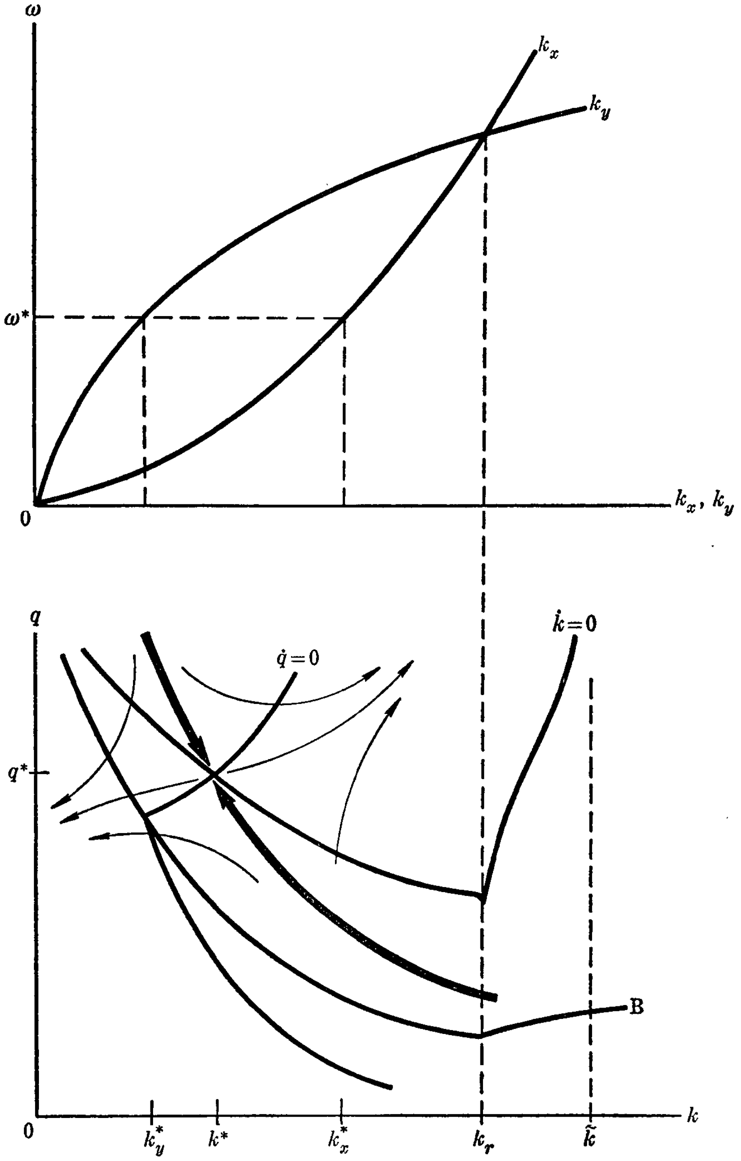

Fig. 14. Dynamic Path: Case of One Reversal

However, the general form of the optimal path is not affected. We thus conclude the following:

> *Regardless of the number or kinds of factor intensity reversals that may or may not occur, for each interval value $k(0)$ there exists a unique optimal path which approaches (k^*, q^*) monotonically.*

But this is the result that was obtained for the two previous polar cases.[41] That is to say that capital intensity reversals have no effect on the nature of

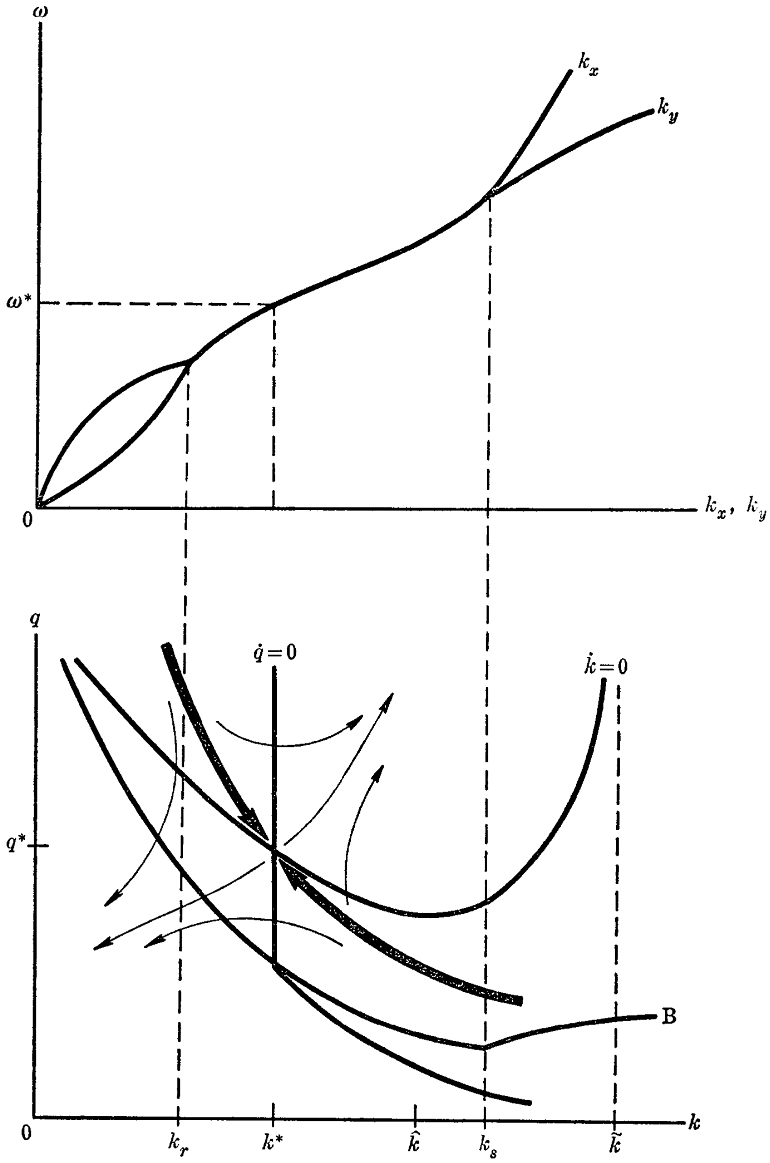

Fig. 15. Dynamic Path: Degenrate Reversal Case

[41] Note that this is also the result that is obtained in the usual one-sector optimal growth model. See, Cass [7], for example.

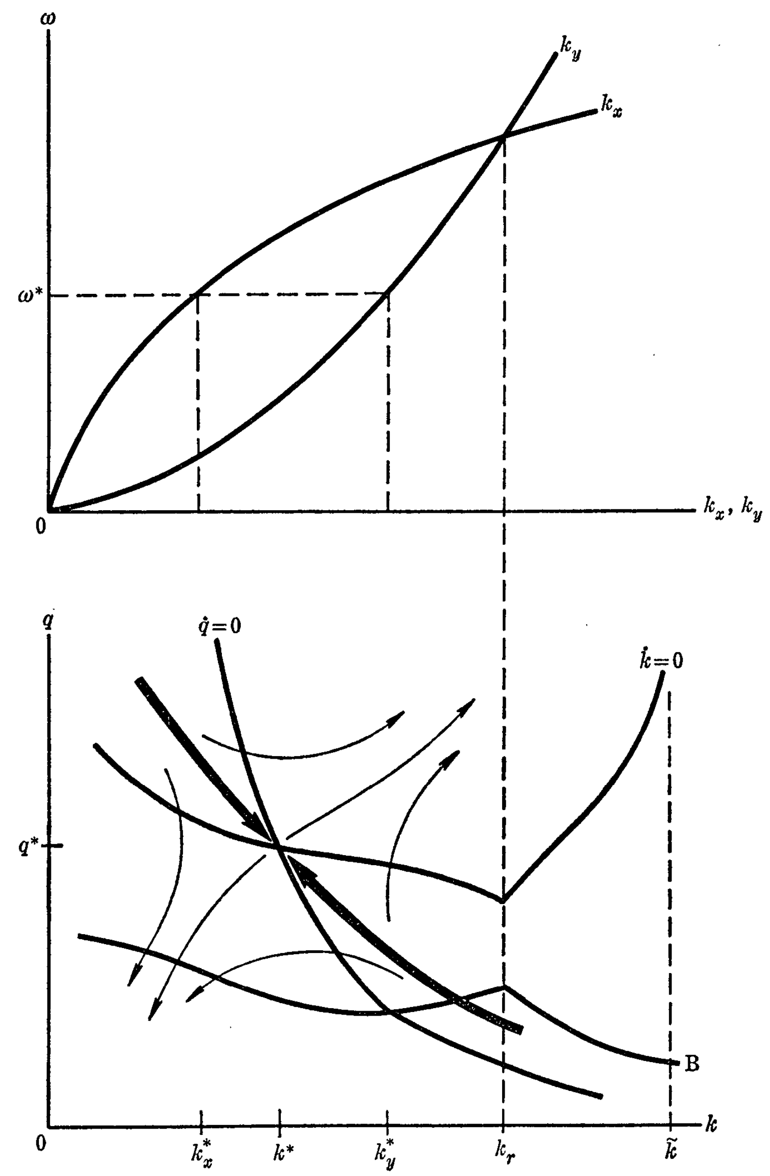

Fig. 16. Dynamic Path: Case of One Reversal

the optimal growth path. Hence we conclude that for the two-sector optimal growth problem, capital intensity does not matter.

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APPENDIX

A′. THE GEOMETRIC CONSTRUCTION

In this Appendix we shall derive certain important results used in the text by constructing a simple diagram. This diagram will facilitate the clear understanding of these results.[1] The construction of this diagram is based on well-known results for the two-sector economy.

As is well known, for a given k[2]

$$p \leqq p_{\min}(k) \Rightarrow x = 0\,, \quad y = g(k)$$

$$p_{\min}(k) \leqq p \leqq {}_{\max}(k) \Rightarrow \begin{cases} x = x(p, k) = \dfrac{k_y - k}{k_y - k_x} f(k_x) \\[2ex] y = y(p, k) = \dfrac{k - k_x}{k_y - k_x} g(k_y) \end{cases}$$

where $k_x = k_x(p)$, $k_y = k_y(p)$

$$p_{\max}(k) \leqq p \Rightarrow x = f(k)\,, \quad y = 0\,.$$

Thus, for a given p, say $p = p^*$, we have

$$\text{(A1)} \qquad p^* \leqq p_{\min}(k) \Rightarrow x = 0\,, \quad y = g(k)$$

$$\text{(A2)} \qquad p_{\min}(k) \leqq p^* \leqq p_{\max}(k) \Rightarrow \begin{cases} x = x(p^*, k) = \dfrac{k_y^* - k}{k_y^* - k_x^*} f(k_x^*) \\[2ex] y = y(p^*, k) = \dfrac{k - k_x^*}{k_y^* - k_x^*} g(k_y^*) \end{cases}$$

where $k_x^* = k_x(p^*)$, $k_y^* = k_y(p^*)$

$$\text{(A3)} \qquad p_{\max}(k) \leqq p^* \Rightarrow x = f(k)\,, \quad y = 0\,.$$

The above results can be summarized in Figures A.1 and A.2. We will only explain Figure A.1, the case of $k_x > k_y$, as the two cases are similar.

Observe that the top graph of Figure A.1 is noting more than the Samuelson-Harrod diagram. As illustrated, given $p = p^*$, k_x and k_y are uniquely determined as k_x^* and k_y^*, respectively. Clearly for all k such that $0 \leqq k \leqq k_y^*$, we have $p^* \leqq p_{\min}(k)$. Hence from (A1)

$$\text{(A4)} \qquad x(p^*, k) = 0\,, \qquad y(p^*, k) = g(k) \quad \text{for } 0 \leqq k \leqq k_y^*\,.$$

That is, for $p = p^*$, if $0 \leqq k \leqq k_y^*$, the capital good will not be produced, i.e., $x = 0$, and hence all resources will be utilized in the production of the

[1] It appears that some of the major errors in Hadley and Kemp [11] can be attributed to their incorrect version of this diagram. See, for example, their Figure 6.5 (b), page 346, which makes their subsequent figure and analysis incorrect.

[2] Here we assume away any factor intensity reversals.

consumption good, i.e., $y = g(k)$. Similarly, for all k such that $k_y^* \leqq k \leqq k_x^*$, we have $p_{\min}(k) \leqq p^* \leqq p_{\max}(k)$ so that (A2) applies. Thus

$$\text{(A5)} \qquad \left.\begin{aligned} x(p^*, k) &= \frac{k_y^* - k}{k_y^* - k_x^*} f(k_x^*) \\ y(p^*, k) &= \frac{k - k_x^*}{k_y^* - k_x^*} g(k_y^*) \end{aligned}\right\} \quad \text{for } k_y^* \leqq k \leqq k_x^* .$$

Observe that $x(p^*, k)$ and $y(p^*, k)$ are linear increasing and linear decreasing functions of k, respectively, for $k_y^* \leqq k \leqq k_x^*$.[3] Finally, if $k_x^* \leqq k$, $p_{\max}(k) \leqq$

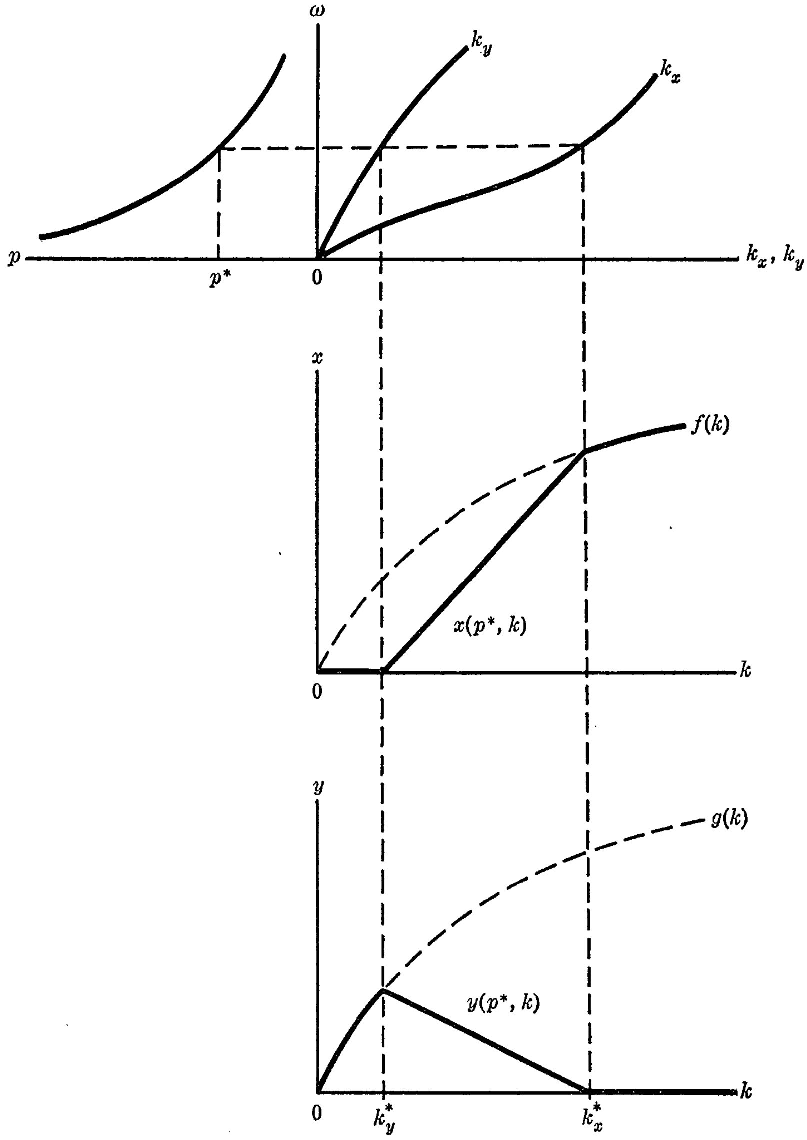

Fig. A.1. Determination of $x(p^*, k)$ and $y(p^*, k)$: $k_x > k_y$

[3] Recall condition (49) and footnote 20 of the text.

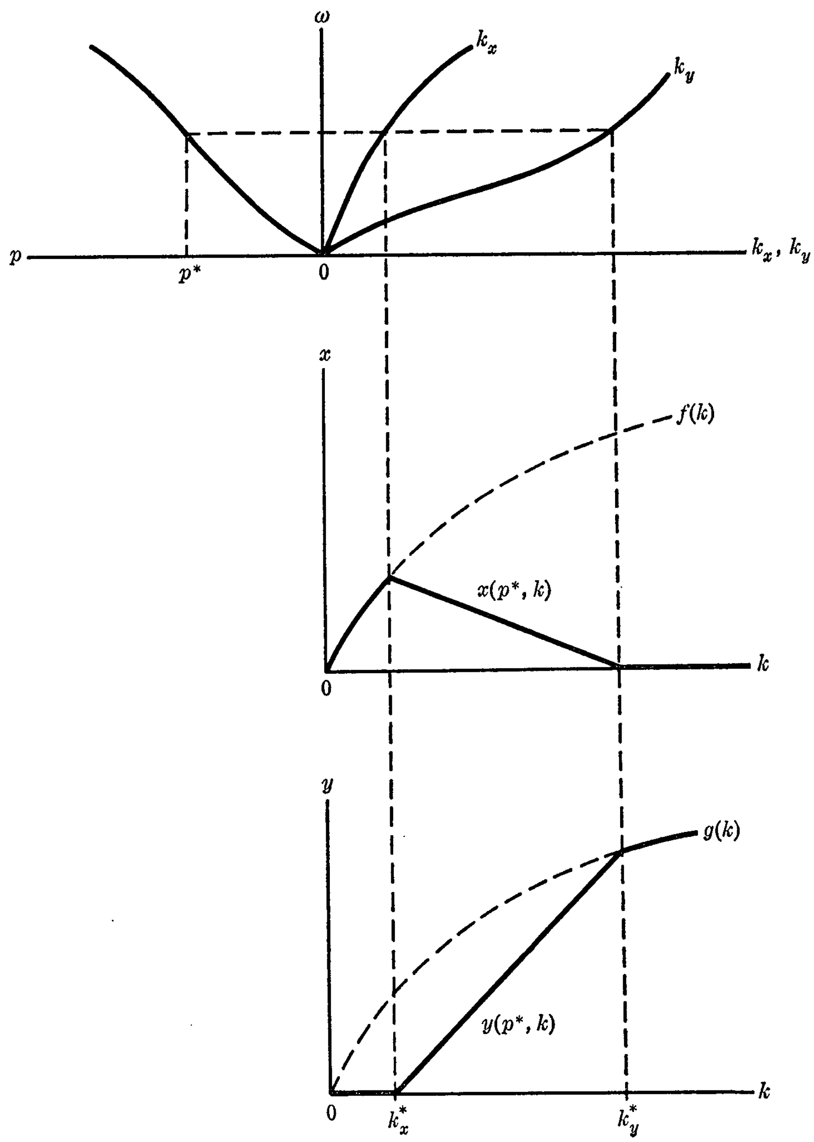

Fig. A.2. Determination of $x(p^*, k)$ and $y(p^*, k)$: $k_y > k_x$

p^*, so that from (A3)

$$\text{(A6)} \qquad x(p^*, k) = f(k)\,, \qquad y(p^*, k) = 0 \quad \text{for } k_x^* \leqq k\,.$$

The above results (A4)–(A6) are seen in the bottom graphs of Figures A.1 and A.2. The functions $x(p^*, k)$ and $y(p^*, k)$ are illustrated by the heavy lines, and the per-capita production functions f and g are illustrated by the thin broken lines in Figures A.1 and A.2.

Clearly, both $x(p^*, k)$ and $y(p^*, k)$ depend upon the chosen fixed value of p^*. If a different fixed value of p is chosen different curves will be obtained. In particular, if we choose some $\bar{p}$, such that $\bar{p} > p^*$, we will obtain the results depicted in Figures A.3 and A.4. We illustrate the results for the capital good

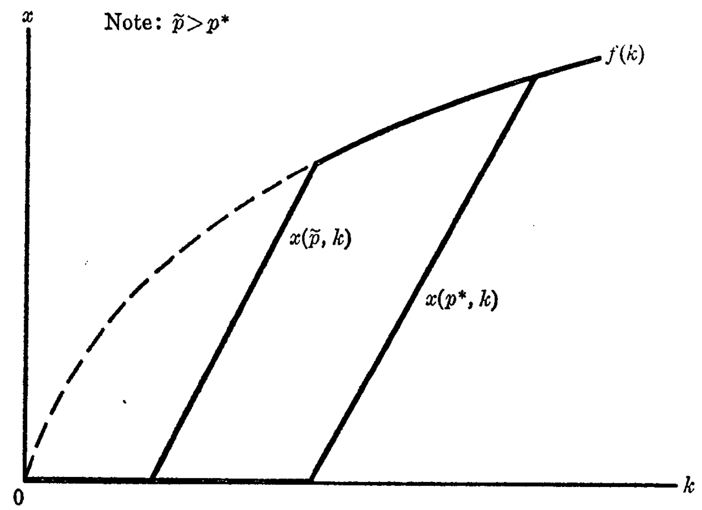

Fig. A.3. A Change in p: $k_x > k_y$

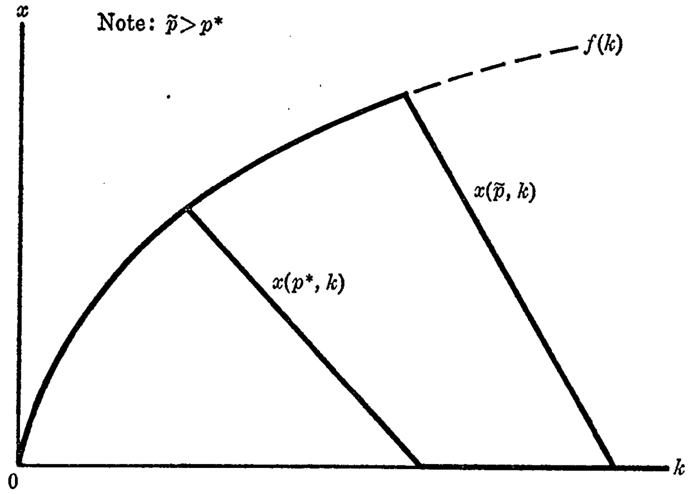

Fig. A.4. A Change in p: $k_y > k_x$

industry but the other case follows analogously. Observe that Figures A.3 and A.4 illustrate $\partial x/\partial p > 0$, that is, condition (48), for the case of incomplete specialization.

B′. LOCATION OF $\dot{q} = 0$ CURVE

Utilizing the preceding discussion, the location of the $\dot{q} = 0$ curve can be determined. Recall that under incomplete specialization $\dot{q} = 0$ if and only if $f'[k_x(p^*)] = \lambda + \delta$. The determination of this p^* is illustrated in Figures A.5 and A.6. Once p^* is determined, the values of k for which incomplete and complete specialization occurs are likewise determined. As can be seen from Figures A.5 and A.6, if $k_x > k_y$ then in the region of incomplete specialization the $\dot{q} = 0$ curve is defined only for values of k in the interval (k_y^*, k_x^*). For

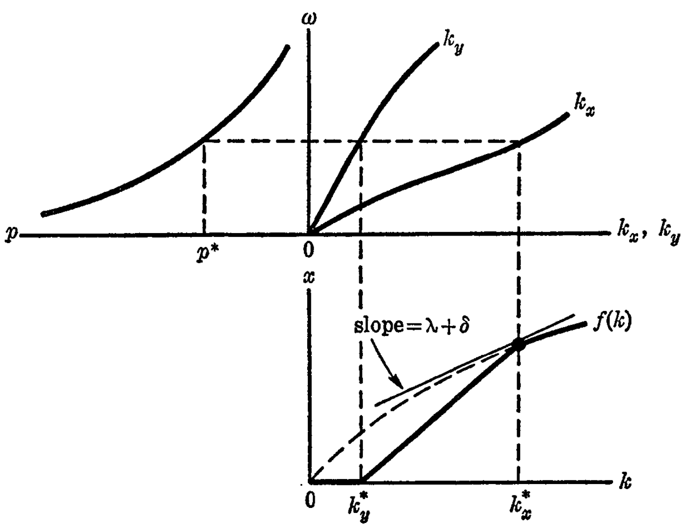

Fig. A.5. Determination of p^*, k_x^* and k_y^*: $k_x > k_y$

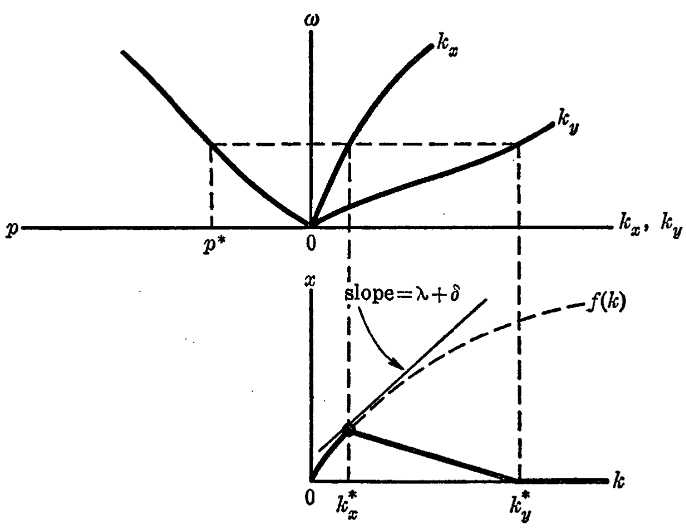

Fig. A.6. Determination of p^*, k_x^* and k_y^*: $k_y > k_x$

values of k greater than or equal to k_y^*, the $\dot{q} = 0$ curve lies in the region of complete specialization (see Section D). And for values of k less than k_y^*, the $\dot{q} = 0$ curve is not defined. Analogous results hold for the case $k_y > k_x$.

Lastly, from Figures A.5 and A.6 and condition (4), (5) and (63), we observe that along the $\dot{q} = 0$ curve[4]

$$\lim_{k \to k_x^*} q = \infty \, .$$

[4] This result was overlooked by both Cass [6] and Hadleya nd Kemp [11], and thus led Cass to specify the balanced growth value k_I^* (k_x^* in our notation) somewhat incorrectly. See, his Figure 1, page 34.

Utilizing the above information and that of Section D of the text, the behavior of q is illustrated in Figures 7 and 8 of the text.

C′. LOCATION OF $\dot{k} = 0$ CURVE

In this section, with the use of our geometric construction, we determine a few results which we use in the analysis of the $\dot{k} = 0$ curve in Section E of the text. Recall that $\dot{k} = 0$ if and only if $x = \lambda k$. The solution of this equation may be illustrated graphically as in Figures A.7 and A.8. Excluding the origin and noting that production will never be specialized in the production of the capital good, we can see from Figures A.7 and A.8 that there is

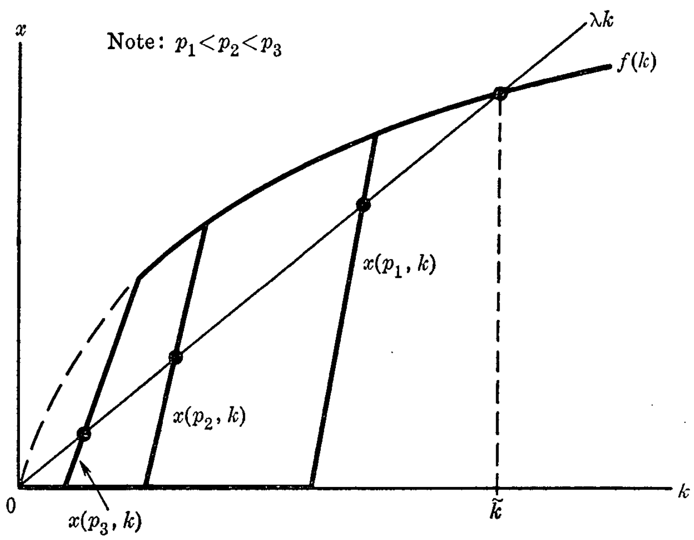

Fig. A.7. Graphical Solution of $x(p, k) = \lambda k$: $k_x > k_y$

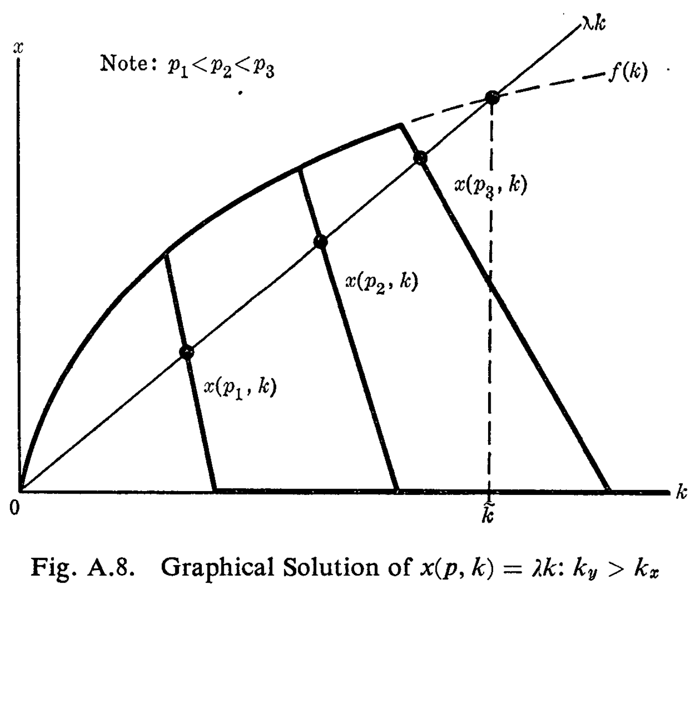

Fig. A.8. Graphical Solution of $x(p, k) = \lambda k$: $k_y > k_x$

always incomplete specialization along the $\dot{k} = 0$ curve.[5] Thus along the $\dot{k} = 0$ curve, $x(p, k) = \lambda k$. The determination of (p, k) pairs which satisfy this condition is illustrated in Figures A.7 and A.8 for several arbitrary fixed values of p. Thus we can easily see that

$$\text{(A7)} \qquad \frac{\partial x}{\partial k} - \lambda \lesseqgtr 0 \quad \text{as } k_y \gtreqless k_x$$

along the $\dot{k} = 0$ curve. Condition (A7) is utilized extensively in the text.

D′. FACTOR INTENSITY REVERSALS CASE: SOME RESULTS

The general solution, in which any factor intensity relationship is allowed, is obtained by utilizing the two polar results as well as the results for the polar case $k_x(\omega) = k_y(\omega)$. Therefore, before turning to the general solution, we will discuss the behavior of the boundary curve B, the $\dot{q} = 0$ curve, and the $\dot{k} = 0$ curve for the case $k_x(\omega) = k_y(\omega)$, whether for some isolated value of ω (the usual factor intensity reversal) or for some intervals of ω's (*the degenerate factor intensity reversal*).

When $k_x(\omega) = k_y(\omega)$, from (42) we know that $dp/d\omega = 0$ from which it can be deduced that

$$\text{(A8)} \qquad \frac{dp_{\min}(k)}{dk} = 0 \quad \text{when } k_x = k_y\,.$$

From the above, as well as (1), (4), and (56), we have

$$\text{(A9)} \qquad \left.\frac{dq}{dk}\right|_{B} = U''[g(k)]g'(k)p_{\min}(k) < 0 \quad \text{when } k_x = k_y\,.$$

That is, when $k_x = k_y$ the slope of the boundary curve B is negative.

The analysis of the $\dot{q} = 0$ curve must be modified only for the case of incomplete specialization when $k_x = k_y$. In particular, since $k_x = k_y = k$, from (61) $\dot{q} = 0$ if and only if

$$\text{(A10)} \qquad f'(k) = \lambda + \delta\,.$$

The above is solved by a unique positive value of k, say k^*, and it is clear that the $\dot{q} = 0$ curve is a vertical line with k-th coordinate equal to k^* in the region of incomplete specialization. With this exception, all of the other results concerning $\dot{q}$ obtained in Section D hold.

In the case of the $\dot{k} = 0$ curve, it can be shown that in the interior of the

[5] Thus the $\dot{k} = 0$ curve lies in the area of incomplete specialization, above the boundary curve B.

region in which $k_x(\omega) = k_y(\omega)$, we have

(A11) $$\left.\frac{dq}{dk}\right|_{\dot{k}=0} = p^2U''(f' - \lambda) .$$

Thus, in the interiors of intervals for which $k_x(\omega) = k_y(\omega)$,

(A12) $$\left.\frac{dq}{dk}\right|_{\dot{k}=0} \gtreqless 0 \quad \text{as } k \gtreqless \hat{k}$$

where $\hat{k}$ is defined by

(A13) $$f'(\hat{k}) = \lambda .$$

Thus, the slope of the $\dot{k} = 0$ curve is positive (respectively zero, negative) when k is greater than (respectively equal to, less than) the value $\hat{k}$. With this exception, the results concerning $\dot{k}$ derived in Section E hold. Finally, we note that the results that we have shown in this section for the case $k_x(\omega) = k_y(\omega)$ are identical to those obtained in the usual one-sector optimal growth model.[6] But this should not be surprising for when capital intensities are equal, the two industries are somewhat "identical." We now turn to the general solution.

When factor intensity reversals occur, an added complication enters our analysis due to the fact that p is no longer uniquely related to ω. Thus depending on the number of reversals that do occur, there will be several values of p which will solve $f'[k_x(p)] = \lambda + \delta$. However this relationship can be rewritten as $f'[k_x(\omega)] = \lambda + \delta$ which will always be solved by a unique value of ω, which will, in turn, give a unique p. Also, even though factor intensity reversals occur, for a given value of k, only one factor intensity relation holds and hence in that region the relationship between p and ω is unique. We note that the above is not ture when $k_x(\omega) = k_y(\omega)$ since then $dp/d\omega = 0$; however in such a case the lack of a unique relationship between p and ω causes no difficulties, as the shape of the boundary curve B, and so on, can be determined as these results do not depend upon the relationship between p and ω.

In any case, the optimal path for cases in which factor intensity reversals occur can be obtained by simply "piecing" together the results for the two polar cases together with results for the case $k_x = k_y$ for the corresponding values of k for which they do apply. The procedure can be seen in Figure A.9 where one possible case is illustrated. However it should be noted that similar results can be obtained for cases in which any number of factor intensity reversals occur.

[6] See, Cass [7].

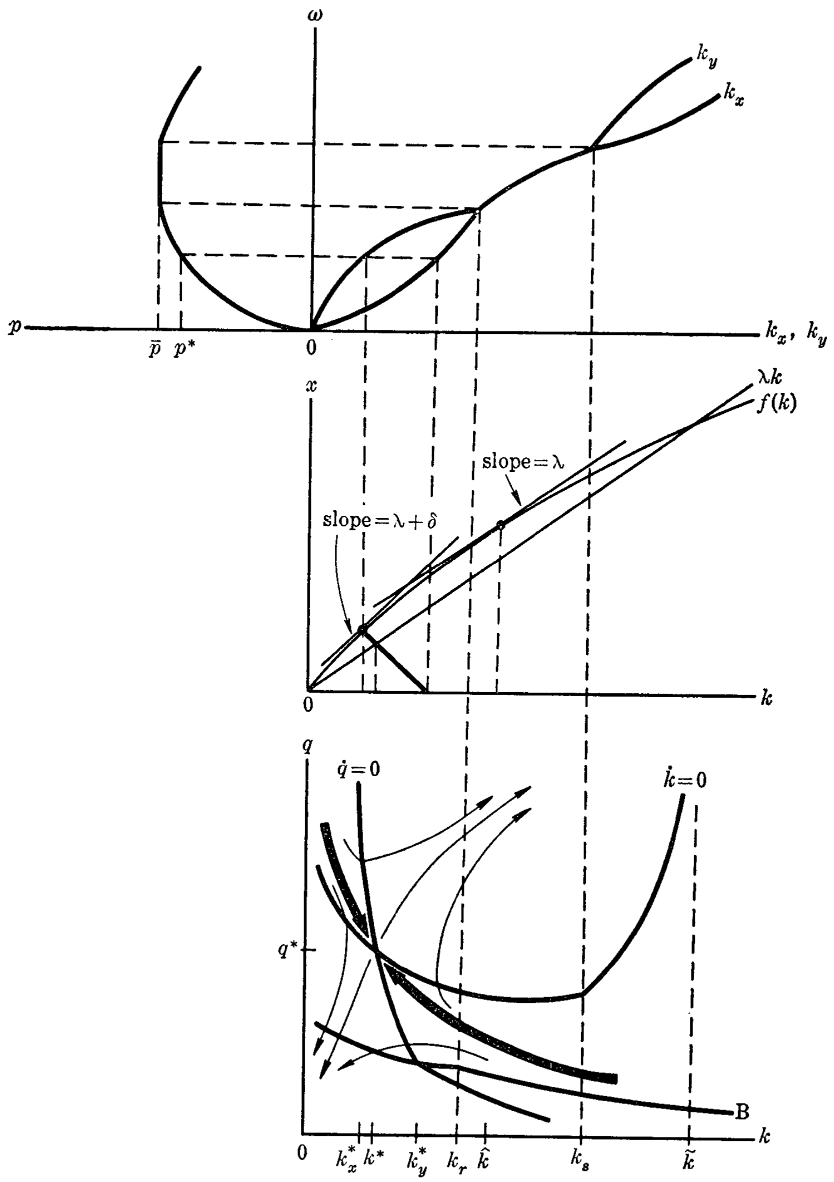

Fig. A.9. Determing Dynamic Path under Factor Intensity Reversal

After the k-axis is partitioned into intervals on the basis of whether $k_x \gtreqless k_y$, the corresponding polar result is applied to each such interval. For example, in Figure A.9 for $k \in (0, k_r)$ we have $k_y(\omega) > k_x(\omega)$, and hence in that interval the shape of the boundary curve **B** is indeterminate from condition (57). Similarly, for $k > k_r$, either $k_x(\omega) = k_y(\omega)$ or $k_x(\omega) > k_y(\omega)$ so that the boundary curve **B** has a negative slope. We do similarly to construct the $\dot{k} = 0$ curve. The general shape of the $\dot{q} = 0$ curve, however, depends on what the capital intensity condition is for the value of k at which $f'(k) = \lambda + \delta$. For example, if for that value of k, $k_y(\omega) > k_x(\omega)$, as is the case in Figure A.9, then the $\dot{q} = 0$ curve will have the general shape of the $\dot{q} = 0$ curve for the polar case $k_y > k_x$. That is, in such a case, in the area of incomplete specialization the $\dot{q} = 0$ curve is defined for values of k in the interval (k_x^*, k_y^*), and for values of k such that $k \geqq k_y^*$ in the area of complete specialization. Similar results hold when $k_x(\omega) > k_y(\omega)$ for the value of k which solves $f'(k) = \lambda + \delta$. Lastly we note that if $f'(k) = \lambda + \delta$ holds at a point of factor intensity reversal, the $\dot{q} = 0$ curve will be of the general shape of the $\dot{q} = 0$ curve in the usual one-sector optimal growth model.[7] The analysis of the optimal path in cases of factor intensity reversals is found in Section G of the text.

[7] See, Cass [7].