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OPTIMAL CAPITAL ACCUMULATION IN AN OPEN ECONOMY

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I. INTRODUCTION

The optimal pattern of international capital movements has been investigated by P. K. Bardhan [1] and K. Hamada [2, 3], while the optimal pattern of international trade associated with the international capital movements has not explicitly been characterized by them. On the other hand, the optimal pattern of capital accumulation with international trade has been investigated by H. E. Ryder [5] under the assumption of the absence of international capital movements. The purpose of the present paper is to characterize the optimal capital accumulation in an open economy which faces both an international capital market and an international commodity market. For this purpose, we shall consider a two-commodity two-factor model. In the model without international capital movements, the range of admissible trade policies is narrow since one commodity must be exported if the other commodity is to be imported; while, in our model, both commodities may be imported or exported simultaneously and hence the range of admissible trade policies is wider than in Ryder's model. Therefore the properties of optimal trade policy are of particular interest in the present paper.

The conditions will be obtained for the boundedness of feasible paths, the optimality of feasible paths, the uniqueness of optimal path, the existence of an optimal balanced growth path, and the convergence of the optimal path to an optimal balanced growth path.

II. THE MODEL

We shall be concerned with an economy which faces a perfectly competitive international commodity market; and hence, it will be assumed that the international terms of trade are exogenously given to the economy. This assumption may be justified if the economy considered is small relative to the international commodity market(1). For the sake of simplicity, we further assume that the terms of trade are kept constant over time. On the other hand, the economy will be assumed to face an imperfectly competitive capital market. This imperfection would not necessarily be incompatible with the relative smallness of the economy; for, because of risks involved in foreign investment, international capital markets

(1) The relationships between international capital movements and international terms of trade are important problems to be investigated. But we do not treat them in the present paper.

are more likely to be imperfect than commodity markets.

We suppose that there are two homogeneous commodities, the first good being a pure consumption good and the second available for both consumption and investment. Furthermore, both commodities are assumed to be produced with two homogeneous factors: capital and labour. We shall use the notation:

$K(t)$ = the capital stock in the economy at time t,
$K_i(t)$ = the capital employed in sector i at time t, $(i = 1, 2)$,
$L(t)$ = the labour force in the economy at time t,
$L_i(t)$ = the labour employed in sector i at time t, $(i = 1, 2)$,
$Y_i(t)$ = the production of sector i at time t, $(i = 1, 2)$,
$C_i(t)$ = the consumption of goods i at time t, $(i = 1, 2)$,
$Z(t)$ = gross investment at time t,
$A(t)$ = the capital invested abroad at time t, measured in terms of the second goods,
p = the international price of the first good in terms of the second goods, a constant,
$\rho(t)$ = the international rate of interest at time t,
n_1 = the rate of growth of the labour force, a constant,
n_2 = the rate of depreciation of capital stock, a constant.

The production function $F^i(K_i, L_i)$ is assumed to be homogeneous of degree one in K_i and L_i; hence, we can define the function $f_i(K_i/L_i) = F^i(K_i/L_i, 1)$ for each $i = 1, 2$. In the present paper, we shall consider the following system:

$$K_1(t) + K_2(t) = K(t),$$
$$L_1(t) + L_2(t) = L(t),$$
$$Y_i(t) = L_i(t)f_i(K_i(t)/L_i(t)), \quad (i = 1, 2),$$
$$\dot{L}(t) = n_1L(t),$$
$$\dot{K}(t) = Z(t) - n_2K(t),$$
$$\dot{A}(t) = \rho(t)A(t) + pY_1(t) + Y_2(t) - pC_1(t) - C_2(t) - Z(t),$$
$$\rho(t) = r(A(t)/L(t));^{(2)}$$

or in terms of per-capita quantitities,

$$k_1(t)l_1(t) + k_2(t)l_2(t) = k(t)\,, \tag{1}$$

$$l_1(t) + l_2(t) = 1\,, \tag{2}$$

$$\dot{k}(t) = z(t) - (n_1 + n_2)k(t)\,, \tag{3}$$

$$\dot{a}(t) = g(a(t)) - n_1a(t) + pl_1(t)f_1(k_1(t)) + l_2(t)f_2(k_2(t)) - pc_1(t) - c_2(t) - z(t)\,, \tag{4}$$

where

(2) The last equation represents the behaviour of the rest of the world, that is, its demand for (or supply of) capital. The assumption that the international rate of interest depends upon the per capita asset is borrowed from K. Hamada [3].

$$k_i(t) = K_i(t)/L_i(t),\ k(t) = K(t)/L(t),\ l_i(t) = L_i(t)/L(t),\ z(t) = Z(t)/L(t)\,,$$
$$a(t) = A(t)/L(t),\ c_i(t) = C_i(t)/L(t),\ g(a(t)) = a(t)r(a(t)),\ (i = 1, 2)\,.$$

Throughout the present paper, we make

Assumption 1. $f_i(0) = 0$, $f_i'(0) = \infty$, $f_i'(\infty) = 0$, and $f_i''(k_i) < 0$ for all $k_i > 0$, $(i = 1, 2)$;

Assumption 2. $g''(a) \leqq 0$ for all a.

The meanings of Assumption 1 and equations (1) through (3) are obvious. Equation (4) is the identity of the balance of international payments, the term $\dot{a} + n_1 a$ representing the balance on capital account and the right-hand side plus $n_1 a$ the balance on current account. Assumption 2 means that the marginal revenue of foreign investment is nonincreasing in the per capita asset invested abroad.(3)

If the system (1) through (4) is to be economically meaningful, all the quantities except for $a(t)$ must be nonnegative:

$$k(t) \geqq 0,\ k_i(t) \geqq 0,\ l_i(t) \geqq 0,\ z(t) \geqq 0,\ c_i(t) \geqq 0,\quad (i = 1, 2)\,. \tag{5}$$

It will be assumed furthermore that the deficit of current account cannot exceed a certain proportion of the national income: for a given $m > 1$,

$$pc_1(t) + c_2(t) + z(t) \leqq m\{pl_1(t)f_1(k_1(t)) + l_2(t)f_2(k_2(t)) + g(a(t))\}\,. \tag{6}$$

Finally, the initial stocks of domestic capital and capital invested abroad are historically given:

$$k(0) = k_0,\quad a(0) = a_0\,. \tag{7}$$

The path $\{(k(t), a(t), k_1(t), k_2(t), l_1(t), l_2(t), z(t), c_1(t), c_2(t)):\ t \geqq 0\}$ is said to be *feasible* if (1) through (7) hold for all $t \geqq 0$.

III. BOUNDEDNESS OF FEASIBLE PATHS

In this section, it will be shown that $k(t)$ and $a(t)$ are bounded for any feasible path, so that, by (5) and (6), any feasible path itself is also bounded. To this end, let us first consider the problem of maximizing the domestic product

$$x = pl_1 f_1(k_1) + l_2 f_2(k_2)$$

subject to (1) and (2).

We note that, by Assumption 1, the functions

(3) At this stage, we do not rule out the case where the international rate of interest is constant so that the economy is confronted with a perfectly competitive capital market. As will be seen later, even if the rate of interest is constant over a finite interval of a, the boundedness of feasible paths can be shown and the sufficient conditions for optimality can be obtained. Furthermore, even if the rate of interest is constant over an finite interval of positive a, it can be shown that there is an optimal balanced growth path.

$$\omega_i(k_i) = \frac{f_i(k_i)}{f_i'(k_i)} - k_i, \quad k_i(\omega) = \omega_i^{-1}(\omega), \quad \text{and} \quad q(\omega) = \frac{f_2'(k_2(\omega))}{f_1'(k_1(\omega))}$$

are well-defined and satisfy

$$\omega_i'(k_i) > 0 \quad \text{for all} \quad k_i > 0, \quad \omega_i(0) = 0, \quad \omega_i(\infty) = \infty\,,$$

$$k_i'(\omega) > 0 \quad \text{for all} \quad \omega > 0, \quad k_i(0) = 0, \quad k_i(\infty) = \infty\,,$$

$$\frac{q'(\omega)}{q(\omega)} = \frac{k_2(\omega) - k_1(\omega)}{(k_1(\omega) + \omega)(k_2(\omega) + \omega)}\,.$$

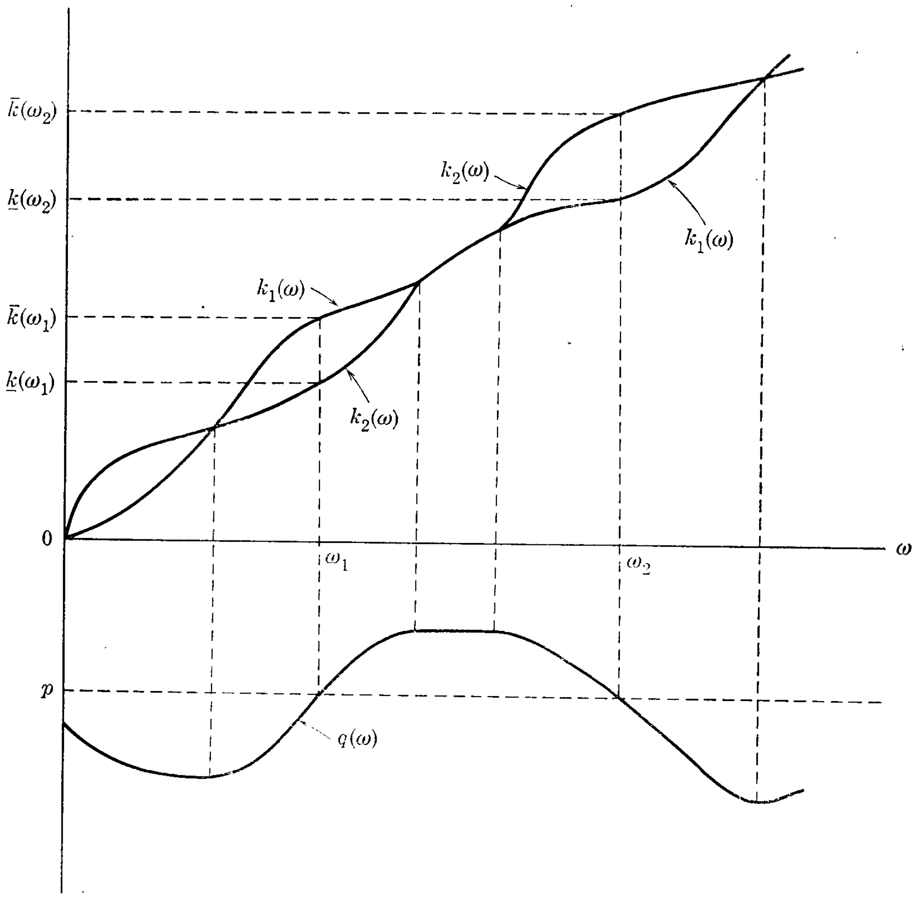

Fig. 1

Let

$$\bar{p} = \sup_{\omega} q(\omega) \quad \text{and} \quad \underline{p} = \inf_{\omega} q(\omega)\,.$$

In the present paper, we shall consider the case where the terms of trade, p, falls into the open interval $]\underline{p}, \bar{p}[$[(4)]. Then we can define a nonempty set

$$\Omega(p) = \{\omega\colon\, p = q(\omega)\}\,.$$

(4) Otherwise, the domestic production will be always specialized completely to one commodity, so that the choice of an optimal policy on production will reduce to a trivial problem.

Further, define

$$\underline{k}(\omega) = \min(k_1(\omega), k_2(\omega)) \quad \text{and} \quad \bar{k}(\omega) = \max(k_1(\omega), k_2(\omega)) .$$

Then we can state and prove the following two lemmas.

LEMMA 1. *If $\omega_1 \in \Omega(p)$, $\omega_2 \in \Omega(p)$, and $\omega_1 < \omega_2$, then $\bar{k}(\omega_1) < \underline{k}(\omega_2)$.*

Proof. By hypothesis, $p = q(\omega_1) = q(\omega_2)$. Hence there is an $\tilde{\omega}$ such that $\omega_1 \leqq \tilde{\omega} \leqq \omega_2$ and $q'(\tilde{\omega}) = 0$, i.e., $k_1(\tilde{\omega}) = k_2(\tilde{\omega})$. Since $k_i'(\omega) > 0$ for all $\omega > 0$, it follows that $\bar{k}(\omega_1) = \max(k_1(\omega_1), k_2(\omega_1)) \leqq k_1(\tilde{\omega}) = k_2(\tilde{\omega}) \leqq \min(k_1(\omega_2), k_2(\omega_2)) = \underline{k}(\omega_2)$. But at least one of the above two inequalities must hold strictly, since $\omega_1 < \tilde{\omega}$ or $\tilde{\omega} < \omega_2$.

LEMMA 2. *Suppose that $\omega_1 \in \Omega(p)$, $\omega_2 \in \Omega(p)$, and $\omega \notin \Omega(p)$ for any $\omega \in]\omega_1, \omega_2[$. Then either* (i) $[\bar{k}(\omega_1), \underline{k}(\omega_2)] = [k_1(\omega_1), k_1(\omega_2)]$ *or* (ii) $[\bar{k}(\omega_1), \underline{k}(\omega_2)] = [k_2(\omega_1), k_2(\omega_2)]$.

Proof. By definition, either $\bar{k}(\omega_1) = k_1(\omega_1)$ or $\bar{k}(\omega_1) = k_2(\omega_1)$. Suppose the former. Then $k_1(\omega_1) \geqq k_2(\omega_1)$. If the strict inequality holds, $q'(\omega_1) < 0$ and so $q(\omega) < q(\omega_1)$ for any ω which is greater than but arbitrarily close to ω_1. Since $\omega \notin \Omega(p)$ for any $\omega \in]\omega_1, \omega_2[$ by hypothesis and q is continuous, this implies $q'(\omega_2) \geqq 0$, so that $k_2(\omega_2) \geqq k_1(\omega_2)$. Hence $\underline{k}(\omega_2) = k_1(\omega_2)$. That is, $k_1(\omega_1) > k_2(\omega_1)$ implies $\underline{k}(\omega_2) = k_1(\omega_2)$; hence case (i) holds. If $k_1(\omega_1) = k_2(\omega_1)$ then trivially either (i) or (ii). Therefore if $\bar{k}(\omega_1) = k_1(\omega_1)$ then either (i) or (ii). The proof for the case $\bar{k}(\omega_1) = k_2(\omega_1)$ is similar.

We are now in a position to prove

THEOREM 1. *Suppose that $\omega_1 \in \Omega(p)$, $\omega_2 \in \Omega(p)$, and $\omega \notin \Omega(p)$ for any $\omega \in]\omega_1, \omega_2[$. Suppose further that $\bar{k}(\omega_1) \leqq k \leqq \underline{k}(\omega_2)$. If $[\bar{k}(\omega_1), \underline{k}(\omega_2)] = [k_1(\omega_1), k_1(\omega_2)]$ then $k_1 = k$ and $l_1 = 1$, while if $[\bar{k}(\omega_1), \underline{k}(\omega_2)] = [k_2(\omega_1), k_2(\omega_2)]$ then $k_2 = k$ and $l_2 = 1$.*

Proof. We prove the first statement. The proof of the second statement is symmetric. Since $f_2(k_1(\omega_1)) < f_2(k_2(\omega_1)) + (k_1(\omega_1) - k_2(\omega_1))f_2'(k_2(\omega_1)) = (\omega_1 + k_1(\omega_1))f_2'(k_2(\omega_1)) = pf_1(k_1(\omega_1))$ and similarly $f_2(k_1(\omega_2)) < pf_1(k_1(\omega_2))$, it must be that $f_2(k) < pf_1(k)$ for all $k \in [\bar{k}(\omega_1), \underline{k}(\omega_2)]$. (See Figure 2.) Since $k \in [\bar{k}(\omega_1), \underline{k}(\omega_2)]$ by hypothesis, if $k_1 = k_2$ then $k_1 = k$ and the maximization of x implies $l_1 = 1$. If $k_1 \neq k_2$, we have the following three cases: (a) $k_1 = k$, (b) $k_2 = k$, and (c) $k_1 \neq k \neq k_2$. In case (a), $l_2k_2 = l_2k$. Hence if $l_2 > 0$ then $k_1 = k_2$, reducing to the case treated above, while if $l_2 = 0$ then $l_1 = 1$ so that $k_1 = k$. In case (b), $l_1k_1 = l_1k$. If $l_1 = 0$ then $l_2 = 1$ so that $x = f_2(k)$; but since $f_2(k) < pf_1(k)$ this cannot be optimal. Thus $l_1 > 0$ and hence $k_1 = k_2$, reducing again to the case treated above. Therefore $k_1 = k$ and $l_1 = 1$ for all cases except for (c).

In what follows, it will be shown that case (c) is impossible. Since $k_1 \neq k_2$ we may write

$$x = \frac{k - k_1}{k_2 - k_1} f_2(k_2) + \frac{k_2 - k}{k_2 - k_1} pf_1(k_1) .$$

Hence the maximization of x implies that

$$(k_1 - k)(f_2(k_2) - pf_1(k_1) - (k_2 - k_1)f_2'(k_2)) \leqq 0 \quad \text{with equality if } k_2 > 0\,,$$

$$(k - k_2)(f_2(k_2) - pf_1(k_1) - p(k_2 - k_1)f_1'(k_1)) \leqq 0 \quad \text{with equality if } k_1 > 0\,.$$

If $k_2 = 0$ then $k_1 > 0$ so that $f_1(k_1) - k_1 f_1'(k_1) = 0$. But the last equality can hold only if $k_1 = 0$. This is a contradiction. Hence $k_2 > 0$. Similarly $k_1 > 0$. On the other hand, since $k_1 \neq k$ and $k_2 \neq k$, it follows that

$$f_2'(k_2) = pf_1'(k_1) \quad \text{and} \quad \frac{f_2(k_2)}{f_2'(k_2)} - k_2 = \frac{f_1(k_1)}{f_1'(k_1)} - k_1\,.$$

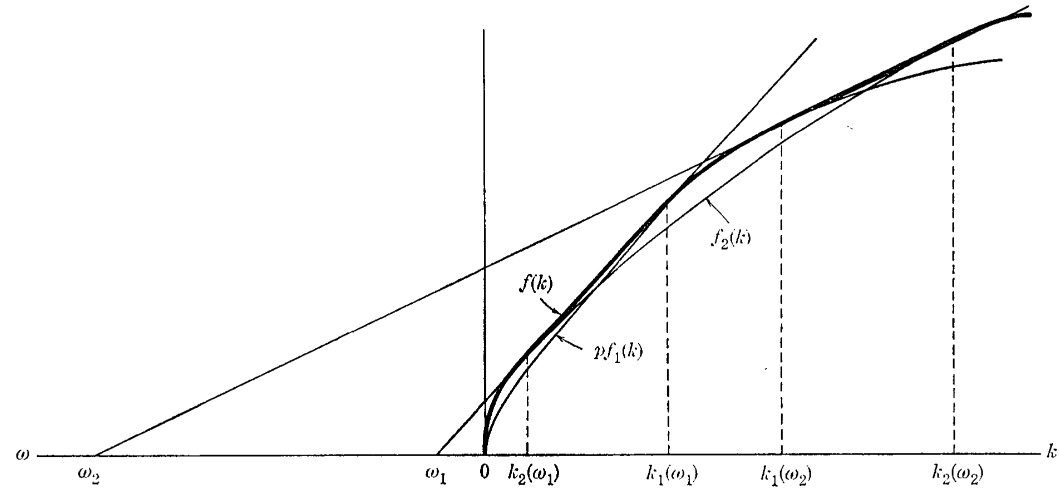

Fig. 2

This implies that there is an $\tilde{\omega}$ such that $\tilde{\omega} \in \Omega(p)$ and $k_1 = k_1(\tilde{\omega})$ and $k_2 = k_2(\tilde{\omega})$. But, by hypothesis, $\tilde{\omega} \notin]\omega_1, \omega_2[$. If $\tilde{\omega} = \omega_1$ then either $k_1(\omega_1) < k < k_2(\omega_1)$ or $k_2(\omega_1) < k < k_1(\omega_1)$. The former contradicts the hypothesis $\bar{k}(\omega_1) = k_1(\omega_1)$, while the latter contradicts the hypothesis $\bar{k}(\omega_1) \leqq k \leqq \underline{k}(\omega_2)$. Thus, $\tilde{\omega} \neq \omega_1$. Similarly $\tilde{\omega} \neq \omega_2$. Therefore either $\tilde{\omega} < \omega_1$ or $\tilde{\omega} > \omega_2$. If $\tilde{\omega} < \omega_1$, then, by Lemma 1, $\bar{k}(\tilde{\omega}) < \underline{k}(\omega_1) \leqq k$ and hence $k_2 = k_2(\tilde{\omega}) < k$ and $k_1 = k_1(\tilde{\omega}) < k$. But this is impossible. Similarly the case $\tilde{\omega} > \omega_2$ cannot occur. Hence case (c) is impossible.

Similarly, we can prove the following two theorems.

THEOREM 2. *Suppose that $\omega_1 \in \Omega(p)$ and $\omega \notin \Omega(p)$ for any $\omega < \omega_1$, and that $k \leqq \bar{k}(\omega_1)$. Then $\underline{k}(\omega_1) = k_1(\omega_1)$ implies $k_1 = k$ and $l_1 = 1$, while $\underline{k}(\omega_1) = k_2(\omega_1)$ implies $k_2 = k$ and $l_2 = 1$.*

THEOREM 3. *Suppose that $\omega_1 \in \Omega(p)$ and $\omega \notin \Omega(p)$ for any $\omega > \omega_1$, and that $k \geqq \underline{k}(\omega_1)$. Then $\bar{k}(\omega_1) = k_1(\omega_1)$ implies $k_1 = k$ and $l_1 = 1$, while $\bar{k}(\omega_1) = k_2(\omega_1)$ implies $k_2 = k$ and $l_2 = 1$.*

Next, we shall prove

THEOREM 4. *If* $\omega \in \Omega(p)$ *and* $\underline{k}(\omega) < k < \bar{k}(\omega)$, *then* $k_1 = k_1(\omega)$ *and* $k_2 = k_2(\omega)$.

Proof. If $k_1 = k_2$ then $x = pl_1f_1(k) + l_2f_2(k) \leqq \max(pf_1(k), f_2(k))$. If $k_1 \neq k_2$ we have the following three cases: (a) $k_1 = k$, (b) $k_2 = k$, and (c) $k_1 \neq k \neq k_2$. In the first two cases, we have again $x \leqq \max(pf_1(k), f_2(k))$. In case (c), $k_1 = k_1(\omega)$ and $k_2 = k_2(\omega)$. Hence $x = (k - k_2(\omega))f_2'(k_2(\omega)) + f_2(k_2(\omega)) = p((k - k_1(\omega))f_1'(k_1(\omega)) + f_1(k_1(\omega))) > \max(pf_1(k), f_2(k))$ for all $k \in]\underline{k}(\omega), \bar{k}(\omega)[$. Hence the maximization of x requires that $k_1 = k_1(\omega)$ and $k_2 = k_2(\omega)$.

By Theorems 1 through 4, the optimal production policy (k_1, k_2, l_1, l_2) which maximizes x is determined as a function of k, provided $\Omega(p)$ is a finite (or countable) set. If $\Omega(p)$ is not countable, it is sufficient for the determination of optimal production policy to note the following obvious fact.

THEOREM 5. *Suppose that* $[\omega_1, \omega_2] \subset \Omega(p)$ *and* $\bar{k}(\omega_1) \leqq k \leqq \underline{k}(\omega_2)$. *Then* (i) *either* $k_1 = k$ *or* $k_2 = k$; (ii) *if* $k_1 \neq k$ *then* $l_2 = 1$ *while if* $k_2 \neq k$ *then* $l_1 = 1$, (iii) $x = pf_1(k) = f_2(k)$.

Let $\omega(k)$ be the value of ω, if any, such that $\omega \in \Omega(p)$ and $\underline{k}(\omega) < k < \bar{k}(\omega)$. Then $\omega(k)$ assumes a constant value on an interval of the domain. By Theorems 1 through 5, the maximum value of x can be written as

$$f(k) = \max(pf_1(k), f_2(k), f_2'(k_2(\omega(k)))k + f_2(k_2(\omega(k))) - k_2(\omega(k))f_2'(k_2(\omega(k)))).$$

Evidently f and f' are continuous. Furthermore, by Assumption 1, f is concave in k and satisfies $f'(0) = \infty$, $f'(\infty) = 0$, and $f(0) = 0$.(5)

We can now prove the central result of this section that $k(t)$ and $a(t)$ are bounded for any feasible path.

THEOREM 6. *If* $g'(-\infty) > 0$ *and* $g'(\infty) < n_1$, *then* $k(t)$ *and* $a(t)$ *are bounded for any feasible path.*

Proof. It suffices to consider the following two cases: Case I. $g'(a) = 0$ for some finite a and Case II. $g'(a) > 0$ for all a.

Case I. Let $\bar{g} = g(a)$ where $g'(a) = 0$. Then, by the concavity of g, $g(a) \leqq \bar{g}$ for all a. Note that $\dot{k} = z - (n_1 + n_2)k \leqq m(f(k) + g(a)) - (n_1 + n_2)k \leqq mf(k) - (n_1 + n_2)k + m\bar{g}$. Let $\phi_1(k) = mf(k) - (n_1 + n_2)k$. Then we can easily see $\phi_1(\infty) = -\infty$. Hence there is a k' such that $0 < k' < \infty$ and $\phi_1(k) + m\bar{g} < 0$ for all $k > k'$. This proves the boundedness of k, in view of the nonnegativity of k.

On the other hand, $\dot{a} \leqq g(a) - n_1a + f(\bar{k})$, where $\bar{k} = \max(k_o, k')$. Let $\phi_2(a) = g(a) - n_1a$. Since $g'(\infty) < n_1$ by hypothesis, it follows that $\phi_2(\infty) = -\infty$. Thus a is bounded from above. Furthermore, $\dot{a} \geqq g(a) - n_1a - m(g(a) + f(k)) = (1 - m)g(a) - n_1a - mf(\bar{k})$. Let $\phi_3(a) = (1 - m)g(a) - n_1a$. Since

(5) So far as there is an interval of k where the specialization of domestic production is incomplete, the function f cannot be strictly concave. This fact will make the subsequent analysis rather complicated.

$g'(-\infty) > 0$ by hypothesis, $\phi_3'(-\infty) < 0$, so that from the convexity of ϕ_3 we get $\phi_3(-\infty) = \infty$. Hence a is bounded from below, completing the proof for Case I.

Case II. Let $mg(\phi_4(k)) + mf(k) - (n_1 + n_2)k = 0$. Since $g'(a) > 0$ for all a it follows that $a < \phi_4(k)$ implies $\dot{k} < 0$. Further, define $f(\phi_5(a)) + g(a) - n_1 a = 0$. Then $k < \phi_5(a)$ implies $\dot{a} < 0$. Let us note that

$$\phi_4'(k) = (n_1 + n_2 - mf'(k))/mg'(\phi_4(k)),$$

$$\phi_4''(k) = -(f''(k) + g''(\phi_4(k))(\phi_4'(k))^2)/g'(\phi_4(k)),$$

$$\phi_5'(a) = (n_1 - g'(a))/f'(\phi_5(a)),$$

$$\phi_5''(a) = -(g''(a) + f''(\phi_5(a))(\phi_5'(a))^2)/f'(\phi_5(a)).$$

Let $mf(\check{k}) = (n_1 + n_2)\check{k}$ and $0 < \check{k} < \infty$. Then $\phi_4(0) = \phi_4(\check{k}) = 0$ and $\phi_4'(\check{k}) > 0$. Since $\phi_4''(\check{k}) \geqq 0$, i.e., ϕ_4 is convex in k, it must be that $\phi_4(k) \geqq (k - \check{k})\phi_4'(\check{k})$ for all $k \geqq 0$. On the other hand, let $\underset{\sim}{a}$ and $\tilde{a}$ be such that $g(\underset{\sim}{a}) = n_1 \underset{\sim}{a}$, $g(\tilde{a}) = n_1 \tilde{a}$, and $\underset{\sim}{a} \leqq \tilde{a}$. Then clearly $\tilde{a} \geqq 0$ and $\phi_5'(\tilde{a}) \geqq 0$. Since $\phi_2(\infty) = -\infty$, it follows that $\phi_5(\tilde{\tilde{a}}) = \infty$ for some $\tilde{\tilde{a}} \leqq \infty$. Hence $f'(\phi_5(\tilde{\tilde{a}})) = 0$ and $0 < g'(\tilde{\tilde{a}}) < n_1$, so that $\phi_5'(\tilde{\tilde{a}}) = \infty$. In view of the fact that ϕ_5 is convex in a, we can find a unique $(\bar{k}, \bar{a})$ such that $\bar{a} = \phi_4(\bar{k})$, $\bar{k} = \phi_5(\bar{a})$, $0 < \bar{k} < \infty$, and $\bar{a} > \tilde{a}$.

Define

$$B_1 = \{(k, a): \ a \geqq \phi_4(k), \ \ k \geqq \max(0, \phi_5(a))\},$$

$$B_2 = \{(k, a): \ a \geqq \phi_4(k), \ \ 0 \leqq k < \max(0, \phi_5(a))\},$$

$$B_3 = \{(k, a): \ a < \phi_4(k), \ \ k \geqq \max(0, \phi_5(a))\},$$

$$B_4 = \{(k, a): \ a < \phi_4(k), \ \ 0 \leqq k < \max(0, \phi_5(a))\}.$$

We now note that the economy can *at most* accumulate its domestic capital and foreign asset subject to

$$\dot{k} = mf(k) - (n_1 + n_2)k + mg(a),$$

$$\dot{a} = f(k) + g(a) - n_1 a.$$

But any solution to this system of differential equations either converges to $(\bar{k}, \bar{a})$ or approaches some point (0, a) such that $a \leqq 0$. This establishes the boundedness of (k, a) from above. (See Figure 3.)

Now let $\bar{\bar{k}}$ be an upper bound of feasible k. Then $\dot{a} \geqq \phi_3(a) - mf(\bar{\bar{k}})$. But as was seen in the proof for Case I, $\phi_3(-\infty) = \infty$ and hence a is bounded from below. This completes the proof for Case II.

In order to guarantee the boundedness of feasible paths, we shall make

Assumption 3. $g'(-\infty) > 0$ and $g'(\infty) < n_1$.

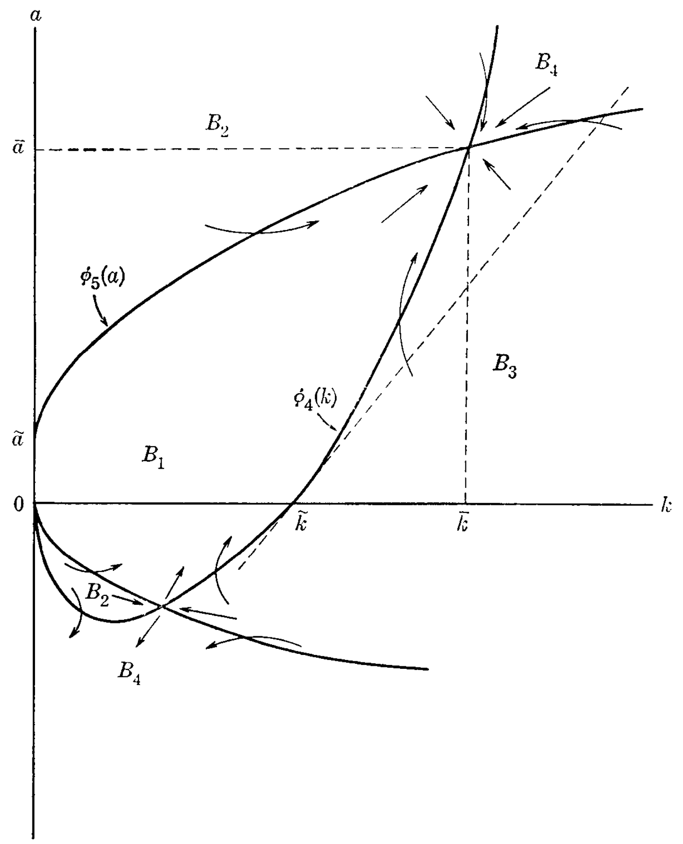

Fig. 3

IV. STATEMENT OF OPTIMIZATION PROBLEM

In what follows, we shall consider the problem of maximizing the social welfare functional

$$\int_0^\infty U(c_1(t), c_2(t))e^{-\delta t}dt, \delta > 0$$

subject to the constraints (1) through (7). As for the instantaneous welfare function U, we make

Assumption 4. $U_1(0, c_2) = \infty$, $U_2(c_1, 0) = \infty$, $U_1(c_1, c_2) > 0$, $U_2(c_1, c_2) > 0$, $U_{11}(c_1, c_2) < 0$, $U_{22}(c_1, c_2) < 0$, and $U_{11}(c_1, c_2)U_{22}(c_1, c_2) - U_{12}^2(c_1, c_2) > 0$ for all $c_1 > 0$ and $c_2 > 0$.

By the first two properties of U, only the case where each commodity is indispensable to consumption will be dealt with. This serves for simplifying the subsequent analysis; for it enables us to define a well-behaved indirect utility function which is strictly concave in consumption expenditure, as will be seen later.

V. INSTANTANEOUS MAXIMIZATION

In order to solve the optimization problem posed above, we utilize the maximum principle of Pontryagin and his associates [4]. Define the Hamiltonian

$$H = U(c_1, c_2) + q_1(z - (n_1 + n_2)k) \\ + q_2(g(a) - n_1 a + p l_1 f_1(k_1) + l_2 f_2(k_2) - p c_1 - c_2 - z)\,.$$

Since H is maximized with respect to c_2 at every point of time along optimal paths, $U_2(c_1, c_2) = q_2$ holds, provided the constraint (6) is ineffective. For the time being, let us assume that $\lim_{t\to\infty} q_2(t) > 0$ for the optimal path which will be considered[6]. As will be seen in the next section, the optimality requires $\dot{q}_2 = (\delta + n_1 - g'(a))q_2 + (q_2 - U_2(c_1, c_2))mg'(a)$, so that q_2 is always positive. Thus the domestic product x must be maximized with respect to k_1, k_2, l_1, and l_2, if H is to be maximized with respect to them. Therefore, Theorems 1 through 5 still state the necessary conditions for optimality. Then the Hamiltonian reduces to

$$H = U(c_1, c_2) + q_1(z - (n_1 + n_2)k) + q_2(g(a) - n_1 a + f(k) - p c_1 - c_2 - z)\,.$$

Furthermore, the constraint (6) becomes

$$p c_1 + c_2 + z \leqq y(k, a)\,, \tag{6'}$$

where $y(k, a) = m(f(k) + g(a))$. Define the Lagrangean

$$G = H + \lambda(y(k, a) - p c_1 - c_2 - z) + \mu z\,.$$

Then the maximization of H with respect to c_1, c_2 and z immediately implies

THEOREM 7. (i) $U_1(c_1, c_2) = pU_2(c_1, c_2)$, (ii) $U_2(c_1, c_2) - q_1 = \mu \geqq 0$, *with equality if* $z > 0$, (iii) $U_2(c_1, c_2) - q_2 = \lambda \geqq 0$, *with equality if* $p c_1 + c_2 + z < y(k, a)$.

Let $c = p c_1 + c_2$. Then from (i) of Theorem 7 we may regard c_1 and c_2 as the functions of c. Thus, write $c_1 = c_1(c)$, $c_2 = c_2(c)$, and $u(c) = U(c_1(c), c_2(c))$. Then we can prove

LEMMA 3. *Suppose that* $U_1(c_1, c_2)U_{22}(c_1, c_2) - U_2(c_1, c_2)U_{12}(c_1, c_2) \leqq 0$ *and* $U_2(c_1, c_2)U_{11}(c_1, c_2) - U_1(c_1, c_2)U_{12}(c_1, c_2) \leqq 0$ *for all* $c_1 > 0$ *and* $c_2 > 0$. *Then* $u'(c) > 0$ *and* $u''(c) < 0$ *for all* $c > 0$ *and* $u'(0) = \infty$.

Proof. Since $u'(c) = U_2(c_1(c), c_2(c))$, evidently $u'(c) > 0$ for all $c > 0$ and $u'(0) = \infty$. On the other hand,

$$u''(c) = \frac{U_2^2(U_{11}U_{22} - U_{12}U_{12})}{U_2(U_2U_{11} - U_1U_{12}) + U_1(U_1U_{22} - U_2U_{12})}\,.$$

The numerator is positive by the strict concavity of U. We must show that the

(6) It will be shown later that there is an optimal balanced growth path with positive q_2. Hence our tentative assumption that $\lim_{t\to\infty} q_2(t) > 0$ is superfluous so far as we consider the path converging to the balanced growth path.

denominator is negative. It is clearly nonpositive by hypothesis. If it were zero, then $U_1U_{22} - U_2U_{12} = 0$ and $U_2U_{11} - U_1U_{12} = 0$, so that $U_{11}U_{22} = U_{12}^2$, contradicting the strict concavity of U. Hence the denominator is negative, which proves the lemma.

The assumption of Lemma 3 is equivalent to saying that both commodities are non-inferior goods; for

$$\frac{dc_1}{dc} = \frac{U_2(U_1U_{22} - U_2U_{12})}{U_2(U_2U_{11} - U_1U_{12}) + U_1(U_1U_{22} - U_2U_{12})},$$

$$\frac{dc_2}{dc} = \frac{U_2(U_2U_{11} - U_1U_{12})}{U_2(U_2U_{11} - U_1U_{12}) + U_1(U_1U_{22} - U_2U_{12})}.$$

In the following, we shall assume the non-inferiority of both goods.

Assumption 5. $U_1(c_1, c_2)U_{22}(c_1, c_2) - U_2(c_1, c_2)U_{12}(c_1, c_2) \leqq 0$ and $U_2(c_1, c_2)U_{11}(c_1, c_2) - U_1(c_1, c_2)U_{12}(c_1, c_2) \leqq 0$ for all $c_1 > 0$ and $c_2 > 0$.
Then from Theorem 7 we immediately obtain

THEOREM 8. (i) $u'(c) - q_1 = \mu \geqq 0$ *with equality if* $z > 0$, (ii) $u'(c) - q_2 = \lambda \geqq 0$ *with equality if* $c + z < y(k, a)$.

We now divide the space of state variables and auxiliary variables into four phases:

$$P_{\mathrm{I}} = \{(k, a, q_1, q_2)\colon \quad q_1 > u'(y(k, a)), \quad q_1 > q_2\},$$

$$P_{\mathrm{II}} = \{(k, a, q_1, q_2); \quad q_1 \leqq u'(y(k, a)), \quad q_2 \leqq u'(y(k, a))\},$$

$$P_{\mathrm{III}} = \{(k, a, q_1, q_2)\colon \quad q_1 = q_2, \quad q_2 > u'(y(k, a))\},$$

$$P_{\mathrm{IV}} = \{(k, a, q_1, q_2)\colon \quad q_1 < q_2, \quad q_2 > u'(y(k, a))\}.$$

These four sets are clearly disjoint. To see that these exhaust the whole space, let us first consider the complement of P_{I}. Then $q_1 \leqq u'(y(k, a))$ or $q_1 \leqq q_2$. This situation can be divided into the following three cases: (i) $q_1 \leqq u'(y(k, a))$ and $q_2 \leqq u'(y(k, a))$, (ii) $q_1 \leqq u'(y(k, a))$ and $q_2 > u'(y(k, a))$, and (iii) $q_1 \leqq q_2$. The first one forms the set P_{II}. The rest can be divided into (i) $q_1 \leqq u'(y(k, a))$, $q_2 > u'(y(k, a))$, (ii) $q_1 < q_2$, $q_2 > u'(y(k, a))$, and (iii) $q_1 = q_2$, $q_2 > u'(y(k, a))$. The last one forms P_{III} and the first two P_{IV}. Therefore the above partition is exhaustive.

We are now in a position to characterize the optimal consumption and domestic investment for each phase.

THEOREM 9. (i) $q_1 = u'(c)$ *and* $z = y(k, a) - c > 0$ *in* P_{I}; (ii) $c = y(k, a)$ *and* $z = 0$ *in* P_{II}; (iii) $q_1 = u'(c)$ *and* $0 \leqq z \leqq y(k, a) - c$ *in* P_{III}; (iv) $q_2 = u'(c)$ *and* $z = 0$ *in* P_{IV}.

Proof. (i) $0 \leqq \mu = u'(c) - q_1 < u'(c) - q_2 = \lambda$, and so $c + z = y(k, a)$; while since $0 \leqq \mu = u'(c) - q_1 < u'(c) - u'(y(k, a))$, we have $z > 0$. (ii) Let $c < y(k, a)$. Then $0 \leqq u'(y(k, a)) - q_2 < u'(c) - q_2 = \lambda$, and hence $c + z =$

$y(k, a)$, so that $z > 0$. Therefore $0 = \mu = u'(c) - q_1 > u'(y(k, a)) - q_1$. But this is a contradiction. Thus $c = y(k, a)$ and hence $z = 0$. (iii) If $c = y(k, a)$ then $u'(c) - q_2 = \lambda < 0$, a contradiction. Hence $c < y(k, a)$. If $\lambda > 0$ then $z > 0$, so that $\mu = 0$. But, since $\lambda = \mu$, this is a contradiction. Hence $\lambda = \mu = 0$ and therefore $q_1 = u'(c)$. (iv) Since $0 \leqq \lambda = u'(c) - q_2 < u'(c) - q_1 = \mu$, it follows that $z = 0$; while if $c = y(k, a)$ then $u'(c) - q_2 = \lambda < 0$, a contradiction. Therefore $c < y(k, a)$ so that $q_2 = u'(c)$. This completes the proof of the theorem.

VI. FUNDAMENTAL DYNAMIC EQUATIONS

For a feasible path to be optimal, the state variables and the auxiliary variables must satisfy the following differential equations.

THEOREM 10.

$$\dot{k} = z - (n_1 + n_2)k\,, \tag{8}$$

$$\dot{a} = g(a) - n_1 a + f(k) - c - z\,, \tag{9}$$

$$\dot{q}_1 = (\delta + n_1 + n_2)q_1 - f'(k)q_2 + (q_2 - u'(c))y_1(k, a)\,, \tag{10}$$

$$\dot{q}_2 = (\delta + n_1 - g'(a))q_2 + (q_2 - u'(c))y_2(k, a)\,. \tag{11}$$

Proof. Equations (8) and (9) are obvious. If $c + z < y(k, a)$, then

$$H_k = -(n_1 + n_2)q_1 + f'(k)q_2\,,$$

$$H_a = (g'(a) - n_1)q_2\,.$$

If $c + z = y(k, a)$, then

$$H_k = -(n_1 + n_2)q_1 + f'(k)q_2 - (q_2 - u'(c))y_1(k, a)\,,$$

$$H_a = (g'(a) - n_1)q_2 - (q_2 - u'(c))y_2(k, a)\,.$$

Since $\dot{q}_1 = \delta q_1 - H_k$ and $\dot{q}_2 = \delta q_2 - H_a$, equations (10) and (11) follow from the fact that $q_2 = u'(c)$ if $c + z < y(k, a)$.

VII. SUFFICIENCY AND UNIQUENESS

Theorems 7 through 10 state the necessary conditions for optimality. In this section, we show that these, together with a transversality condition stated below, are also sufficient for optimality, and furthermore that the optimal path is unique if g is strictly concave.

THEOREM 11. *If a feasible path* $\{(k^o(t), a^o(t), c_1^o(t), c_2^o(t), z^o(t), q_1^o(t), q_2^o(t)): t \geqq 0\}$ *satisfies the conditions stated in Theorems* 7 *through* 10 *and the transversality condition*

$$\lim_{t\to\infty} q_j^o(t)e^{-\delta t} = 0 \qquad (j = 1, 2) \tag{12}$$

then it is optimal. It is also unique, provided g is strictly concave.

Proof. Let $\{(k(t), a(t), c_1(t), c_2(t), z(t)) \colon \ t \geqq 0\}$ be an arbitrary feasible path. Noticing the following relations due to the strict concavity of u and f_i and the (strict) concavity of g:

$$u(c^o) - u(c) \geqq (c^o - c)u'(c^o) \qquad \text{with strict inequality if } c^o \neq c;$$

$$f(k^o) - f(k) \geqq (k^o - k)f'(k^o);$$

$$g(a^o) - g(a) \geqq (a^o - a)g'(a^o) \qquad \text{with strict inequality if } g \text{ is strictly concave and } a^o \neq a,$$

we can prove the series of inequalities:

$$\begin{aligned}
&\int_0^\infty \{u(c^o) - u(c)\}e^{-\delta t}dt \\
&\geqq \int_0^\infty \{(c^o - c)u'(c^o) + (z^o - z)u'(c^o) - (y(k^o, a^o) - y(k, a))u'(c^o) \\
&\qquad + (y(k^o, a^o) - y(k, a))u'(c^o) - (z^o - z)u'(c^o)\}e^{-\delta t}dt \\
&= \int_0^\infty \{((c^o + z^o - y(k^o, a^o)) - (c + z - y(k, a)))(u'(c^o) - q_2^o) \\
&\qquad + y(k^o, a^o) - y(k, a))u'(c^o) - (z^o - z)u'(c^o) \\
&\qquad + ((c^o + z^o - y(k^o, a^o)) - (c + z - y(k, a)))q_2^o\}e^{-\delta t}dt \\
&\geqq \int_0^\infty \{(y(k^o, a^o) - y(k, a))(u'(c^o) - q_2^o) + (z - z^o)(u'(c^o) - q_1^o) \\
&\qquad + (\dot{k} + (n_1 + n_2)k - \dot{k}^o - (n_1 + n_2)k^o)q_1^o \\
&\qquad + (g(a^o) - n_1 a^o + f(k^o) - \dot{a}^o - g(a) + n_1 a - f(k) + \dot{a})q_2^o\}e^{-\delta t}dt \\
&\geqq \int_0^\infty \{(k^o - k)y_1(k^o, a^o)(u'(c^o) - q_2^o) + (a^o - a)y_2(k^o, a^o)(u'(c^o) - q_2^o) \\
&\qquad + (\dot{k} - \dot{k}^o)q_1^o + (n_1 + n_2)(k - k^o)q_1^o + (\dot{a} - \dot{a}^o)q_2^o + n_1(a - a^o)q_2^o \\
&\qquad + (a^o - a)g'(a^o)q_2^o + (k^o - k)f'(k^o)q_2^o\}e^{-\delta t}dt \\
&= \int_0^\infty \{(k - k^o)((\delta + n_1 + n_2)q_1^o - f'(k^o)q_2^o + (q_2^o - u'(c^o))y_1(k^o, a^o) - \dot{q}_1^o) \\
&\qquad + (a - a^o)((\delta + n_1 - g'(a^o))q_2^o + (q_2^o - u'(c^o))y_2(k^o, a^o) - \dot{q}_2^o)\}e^{-\delta t} \\
&\qquad + \lim_{t\to\infty} (k(t) - k^o(t))q_1^o(t)e^{-\delta t} + \lim_{t\to\infty} (a(t) - a^o(t))q_2^o(t)e^{-\delta t} = 0\,,
\end{aligned}$$

where (q_1^o, q_2^o) is associated with the path to which the superscipt zero is attached. Let us note that $U(c_1(c^o), c_2(c^o)) = u(c^o)$ and $u(c) = U(c_1(c), c_2(c)) \geqq U(c_1, c_2)$ for all c, c_1, and c_2 such that $c = pc_1 + c_2$. Thus the path with superscript zero is optimal.

Now suppose that g is strictly concave. If $c^o(t) \neq c(t)$ or $a^o(t) \neq a(t)$ for some t, then, by the piecewise continuity of feasible c and the continuity of feasible a, either $c^o(s) \neq c(s)$ for all s in a nondegenerate time interval including t or $a^o(s) \neq a(s)$ for all s in a nondegenerate time interval including t. Further, if $c_1^o(t) \neq c_1(t)$

or $c_2^o(t) \neq c_2(t)$ then $c^o(t) \neq c(t)$. Thus, if at least one of $c_1^o(t) \neq c_1(t)$, $c_2^o(t) \neq c_2(t)$, $c^o(t) \neq c(t)$, and $a^o(t) \neq a(t)$ holds for some t, then at least one of the first and third inequalities in the above series holds strictly by the strict concavity of u and g.

If $k^o(t) \neq k(t)$ for some t, then $k^o(s) \neq k(s)$ for all s in a nondegenerate open time interval I including t. If f is strictly concave in a neighbourhood of $k^o(s)$ for some $s \in I$, then the third inequality in the above series holds strictly. Then suppose that f is linear on the set $\{k\colon (\exists s)(s \in I, k = k^o(s))\}$. In order to show that at least one of inequalities holds strictly in the above series, it suffices to consider the case where $c^o(s) = c(s)$ and $a^o(s) = a(s)$ for all $s \in I$. Then, for all $s \in I$, $z^o(s) - z(s) = f(k^o(s)) - f(k(s)) = (k^o(s) - k(s))f'(k^o(t))$, so that $\dot{k}^o(s) - \dot{k}(s) = (k^o(s) - k(s))(f'(k^o(t)) - n_1 - n_2)$. If $f'(k^o(t)) = n_1 + n_2$ then $\dot{k}^o(s) = \dot{k}(s)$ for all $s \in I$. But this is impossible since $k^o(s) - k(s)$ must tend to zero as s tends to t_1 or t_2, where $]t_1, t_2[= I$. Hence $f'(k^o(t)) \neq n_1 + n_2$. If $f'(k^o(t)) > n_1 + n_2$ then $\dot{k}^o(s) > \dot{k}(s)$ if and only if $k^o(s) > k(s)$. But this is again impossible since this implies $\lim_{s \to t_2} (k^o(s) - k(s)) \neq 0$. Similarly we cannot have $f'(k^o(t)) < n_1 + n_2$ since it implies $\lim_{s \to t_1} (k^o(s) - k(s) \neq 0$. Thus if $c^o(s) = c(s)$ and $a^o(s) = a(s)$ for all $s \in I$, then f cannot be linear on the set $\{k\colon (\exists s)(s \in I, k = k^o(s))\}$. Hence if $k^o(t) \neq k(t)$ for some t, then we have at least one strict inequality in the above series of inequalities. Thus the uniqueness of the optimal path has been established.

Remark 1. If the optimal path is unique, then the transversality condition (12) is necessary for optimality.

VIII. OPTIMAL BALANCED GROWTH PATH

Before characterizing the dynamic behaviour of the optimal path with an arbitrary initial condition, we show the existence of an optimal balanced growth path. Let us first define $v(k)$, $w(k)$, and $w_1(k)$ by

$$n_1 v(k) + (m-1)(f(k) + g(v(k))) = 0\,,$$

$$g(w(k)) - n_1 w(k) + f(k) - (n_1 + n_2)k = 0 \text{ with } w(k) \leqq a^1\,,$$

$$f(k) + g(w_1(k)) = 0\,,$$

where $g'(a^1) = n_1$. Then

$$v'(k) = \frac{(1-m)f'(k)}{n_1 + (m-1)g'(v(k))}\,, \quad v''(k) = \frac{(1-m)(f''(k) + g''(v(k))(v'(k))^2)}{n_1 + (m-1)g'(v(k))}\,,$$

$$w'(k) = \frac{n_1 + n_2 - f'(k)}{g'(w(k)) - n_1}\,, \quad w''(k) = \frac{f''(k) + g''(w(k))(w'(k))^2}{n_1 - g'(w(k))}\,,$$

$$w_1'(k) = -\frac{f'(k)}{g'(w_1(k))}\,, \quad w_1''(k) = -\frac{f''(k) + g''(w_1(k))(w_1'(k))^2}{g'(w_1(k))}\,.$$

LEMMA 4. *Suppose that* $0 \leqq g'(0) < \infty$ *and* $g'(\infty) \geqq n_1/(1-m)$. *Then there exists a unique* $k' > 0$ *such that* $mf(k') - (n_1 + n_2)k' = -mg(a')$ *and* $v(k) > w(k)$ *for all* k *such that* $0 < k < k'$, *where* $a' = ((1-m)/m)((n_1+n_2)/n_1)k'$.

Proof. Let $\phi_1(k) = mf(k) - (n_1 + n_2)k$ and $\phi_2(k) = -mg\{((1-m)/m)((n_1 + n_2)/n_1)k\}$. Then $\phi_1(0) = \phi_1(k'') = 0$ for some $k'' > 0$, $\phi_1'(0) = \infty$, and $\phi_1''(k) < 0$ for all $k > 0$. On the other hand, $\phi_2(0) = 0$, $0 \leqq \phi_2'(0) < \infty$, and $\phi_2''(k) \geqq 0$ for all $k > 0$. Hence there is a unique $k' > 0$ such that $mf(k') - (n_1 + n_2)k' = -mg(a')$. But, since, as can be easily seen, $mf(k) - (n_1 + n_2)k = -mg\{((1-m)/m)((n_1 + n_2)/n_1)k\}$ if and only if $v(k) = w(k)$, it follows that $v(k') = w(k')$ and $v(k) \neq w(k)$ for all $k > 0$ such that $k \neq k'$.

We now note that, by hypothesis, $g'(\infty) \geqq n_1/(1-m)$ and g is concave, so that $n_1 + (m-1)g'(a) \geqq 0$ for all a. Since $g(0) = 0$ this implies that $n_1a + (m-1)g(a) \geqq 0$ for all $a \geqq 0$; and hence $v(k) < 0$ for all $k > 0$. On the other hand, since $g'(0) \geqq 0$ by hypothesis, $n_1a + (m-1)g(a) < 0$ for all $a < 0$; and therefore $v(0) = 0$.

Since $g(v(k)) - g(w_1(k)) = (n_1/(1-m))v(k)$ by definition, we have

$$v(k) > w_1(k) \text{ for all } k > 0 \quad \text{and} \quad v(0) = w_1(0) = 0 \,. \tag{13}$$

Furthermore, by definition, $f(k) + g(w(k)) \gtreqless 0$ if and only if $w(k) \gtreqless -((n_1 + n_2)/n_1)k$, and hence,

$$w(k) \gtreqless w_1(k) \text{ if and only if } w(k) \gtreqless -((n_1 + n_2)/n_1)k \,. \tag{14}$$

Since $w'(0) = -\infty$, $w(k) < -((n_1 + n_2)/n_1)k$ for sufficiently small $k > 0$, and therefore, by (14), $w(k) < w_1(k)$ for sufficiently small $k > 0$. Since $v'(0) = -\infty$, it follows from (13) that $w(k) < w_1(k) < v(k) < -((n_1 + n_2)/n_1)k$ for sufficiently small $k > 0$. This establishes that $w(k) < v(k)$ for all k such that $0 < k < k'$.

We are ready to state and prove the existence theorem of an optimal balanced growth path.

THEOREM 12. *Suppose that* $g'(-\infty) > \delta + n_1$, $g'(0) \geqq 0$, $g'(\infty) \geqq n_1/(1-m)$, *and that* $mf(k) - (n_1 + n_2)k = -mg(b(k))$ *implies* $g'(b(k)) > \delta + n_1$, where $b(k) = ((1-m)/m)((n_1 + n_2)/n_1)k$. *Then there exists an optimal balanced growth path.*

Proof. Since $f'(0) = \infty$, $f'(\infty) = 0$, and $g'(\infty) < \delta + n_1 < g'(-\infty)$, the sets $K^* = \{k\colon k > 0, f'(k) = \delta + n_1 + n_2\}$ and $A^* = \{a\colon g'(a) = \delta + n_1\}$ are nonempty. It suffices to consider the following two cases: Case I. $k \in K^*$, $a \in A^*$, and $a \geqq v(k)$ for some (k, a); and Case II. $k \in K^*$ and $a \in A^*$ imply $a < v(k)$.

Case I. Let $k^* \in K^*$, $a^* \in A^*$, and $a^* \geqq v(k^*)$. Then $f'(k^*) = \delta + n_1 + n_2$ and $g'(a^*) = \delta + n_1$. Define $c^* = g(a^*) - n_1a^* + f(k^*) - (n_1 + n_2)k^*$, $z^* = (n_1 + n_2)k^*$, and $q_1^* = q_2^* = u'(c^*)$. If we define $\bar{k}$ by $f(\bar{k}) = (n_1 + n_2)\bar{k}$ and $0 < \bar{k} < \infty$, then clearly $f'(\bar{k}) < n_1 + n_2$ so that $k^* < \bar{k}$. Hence $f(k^*) - (n_1 + n_2)k^* > 0$. If $a^* \geqq 0$ then $g(a^*) - n_1a^* \geqq (g'(a^*) - n_1)a^* \geqq 0$ where the first inequality follows from the concavity of g. Thus if $a^* \geqq 0$ then $c^* > 0$. On the other hand, suppose $a^* < 0$. Then $g'(0) < \infty$ and hence, by Lemma 4, there is a unique $k' > 0$ such that $v(k') = w(k')$ and $v(k) > w(k)$ for all k such that $0 < k < k'$. There are two cases: $k' > k^*$ or $k' \leqq k^*$. If $k' > k^*$ then $a^* \geqq v(k^*) > w(k^*)$. Since $g'(a^*) = \delta + n_1 > n_1$, it follows that $c^* = g(a^*) - n_1 a^* + f(k^*) - (n_1 + n_2)k^* > g(w(k^*)) - n_1 w(k^*) + f(k^*) - (n_1 + n_2)k^* = 0$. Conversely, suppose $k' \leqq k^*$. If we define k^1 by $f'(k^1) = n_1 + n_2$, then clearly $k' \leqq k^* < k^1$. Since $w'(k) < 0$ for all $k < k^1$, we have $a^* > b(k') = w(k') \geqq w(k^*)$ and so $c^* > 0$. Therefore $c^* > 0$. The positivity of k^*, z^*, q_1^*, and q_2^* is obvious. Finally, since $a^* \geqq v(k^*)$, it follows that $c^* + z^* = f(k^*) + g(a^*) - n_1 a^* \leqq f(k^*) + g(a^*) - n_1 v(k^*) = f(k^*) + g(a^*) + (m-1)(f(k^*) + g(v(k^*)) \leqq m(f(k^*) + g(a^*))$. The last inequality is due to the fact that $g'(v(k^*)) \geqq g'(a^*) = \delta + n_1 > 0$ and g is concave. Therefore $(k^*, a^*, c^*, z^*, q_1^*, q_2^*)$ is an optimal balanced growth path in P_{III}.

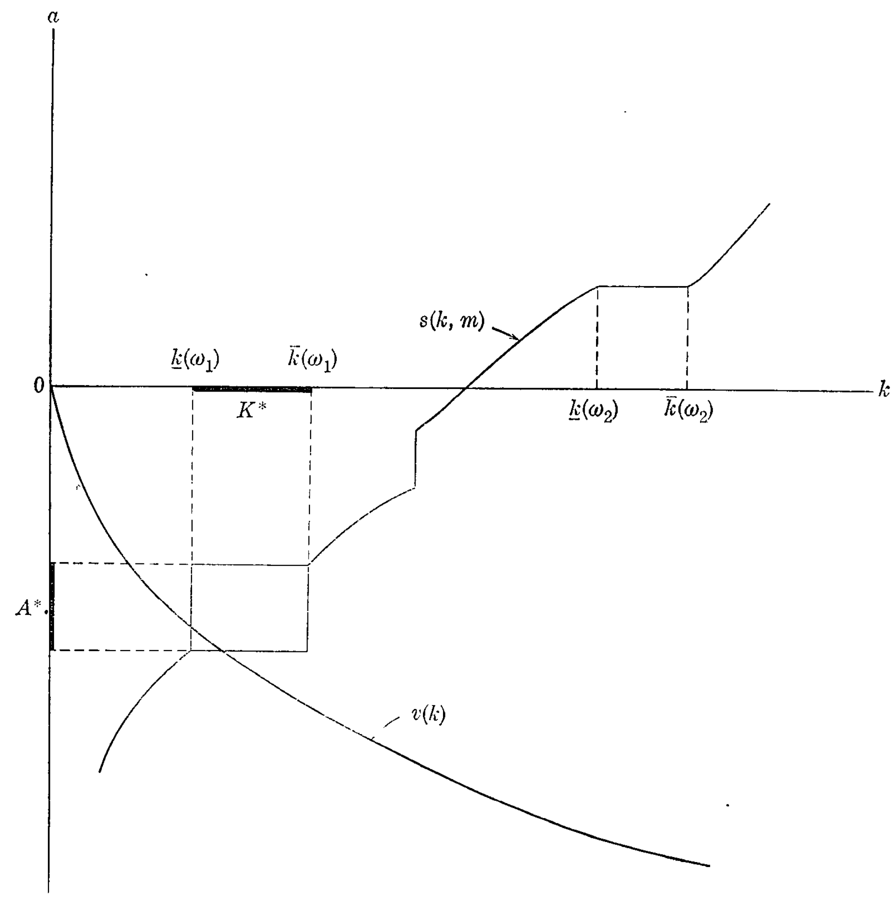

Fig. 4

Case II. Let us define $s(k, m)$ by

$$g'(s(k, m)) = \frac{m}{m-1} \frac{\delta + n_1}{\delta + n_1 + n_2} f'(k) - \frac{\delta + n_1}{m-1}\,.$$

Then the mapping s is not necessarily single-valued. But it is upper semicontinuous and the image set $s(k, m)$ is closed and connected. (See Figure 4 or 5.) It can be easily seen that, for any k' and k'' such that $k' \leqq k''$,

$$\min s(k', m) \leqq \min s(k'', m) \quad \text{and} \quad \max s(k', m) \leqq \max s(k'', m)\,, \tag{15}$$

provided such maxima and minima exist.

Let $\bar{K} = \{k\colon f'(k) = \delta(\delta + n_1 + n_2)/(m(\delta + n_1))\}$. Since $g'(\infty) \geqq n_1/(1 - m)$ by hypothesis, there is a $\bar{k}$ such that $\bar{k} \leqq \min \bar{K}$ and $\lim_{k \to \bar{k} - 0} \min s(k, m) = \infty$. On the other hand, clearly $v(\bar{k}) < 0$. Let $k^* \in K^*$ and $a^* \in A^*$. Then $a^* < v(k^*) < 0$ and $a^* \in s(k^*, m)$. Hence, the continuity of v, the upper semicontinuity of s, and the connectedness of $s(k, m)$ imply that there is a k^{**} such that $v(k^{**}) \in s(k^{**}, m)$. Let $a^{**} = v(k^{**})$. Then

$$(m - 1)(f(k^{**}) + g(a^{**})) + n_1 a^{**} = 0\,,$$

$$g'(a^{**}) = \frac{m}{m-1}\,\frac{\delta + n_1}{\delta + n_1 + n_2}\,f'(k^{**}) - \frac{\delta + n_1}{m - 1}\,.$$

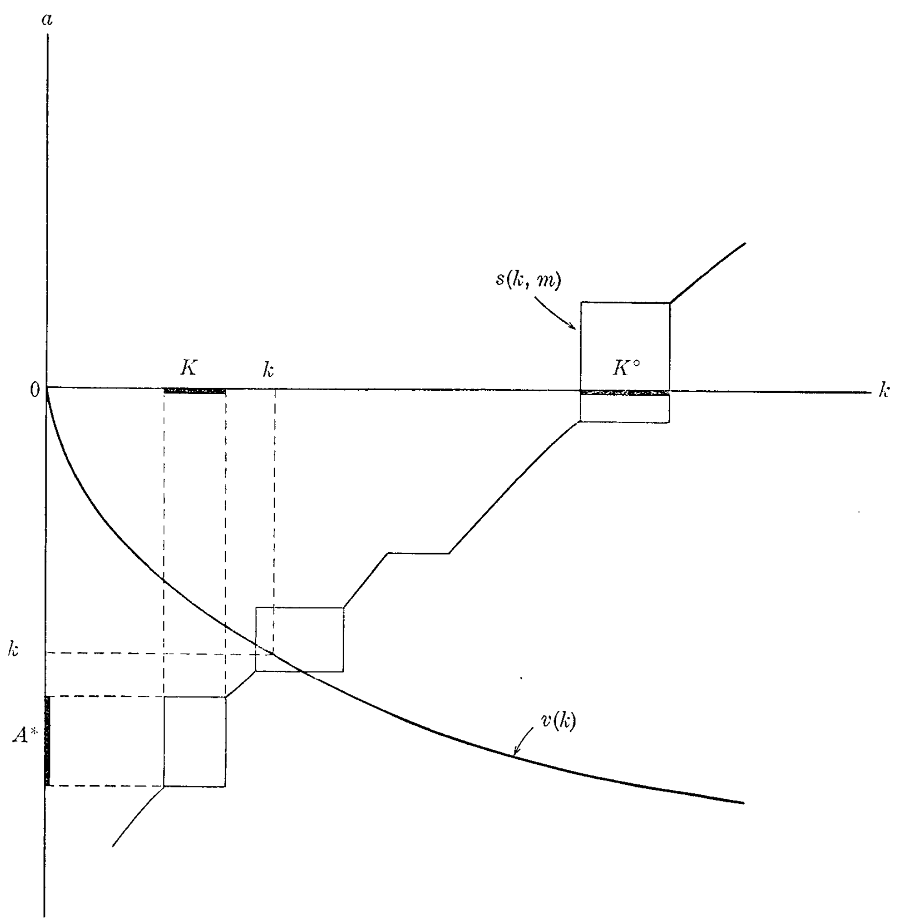

Fig. 5

Further let $c^{**} = g(a^{**}) - n_1 a^{**} + f(k^{**}) - (n_1 + n_2)k^{**}$, $z^{**} = (n_1 + n_2)k^{**}$, $q_1^{**} = u'(c^{**})$, and $q_2^{**} = (\delta + n_1 + n_2 - mf'(k^{**}))q_1^{**}/((1-m)f'(k^{**}))$.

Since s is nondecreasing in the sense of (15) and v is decreasing, it must be that $k^* \leqq k^{**}$ and $a^* \leqq a^{**} < 0$. Hence $g'(a^{**}) \leqq \delta + n_1$. But if $g'(a^{**}) = \delta + n_1$ then $a^{**} \in A^*$. Therefore $f'(k^{**}) = \delta + n_1 + n_2$ so that $k^{**} \in K^*$. Since $a^{**} = v(k^{**})$, this contradicts the assumption of Case II. Thus $g'(a^{**}) < \delta + n_1$. On the other hand, $a^* < 0$ implies $g'(0) < \infty$. Hence, by Lemma 4, there is a unique $k' > 0$ such that $v(k') = w(k')$ and $v(k) > w(k)$ for all k such that $0 < k < k'$. Since $g'(b(k')) > \delta + n_1$ by hypothesis, $v(k^{**}) = a^{**} > b(k') = v(k') = w(k')$. Hence $k^{**} < k'$ by $v'(k) < 0$. Thus $a^{**} > w(k^{**})$, and so $c^{**} > 0$. The positivity of k^{**}, z^{**}, and q_1^{**} is obvious.

Finally we note that $(\delta + n_1 - g'(a^{**}))q_2^{**} = (q_1^{**} - q_2^{**})mg'(a^{**})$. Since $g'(0) \geqq 0$, $g'(a^{**}) \geqq 0$. Furthermore, $\delta + n_1 - g'(a^{**}) > 0$ as was seen above. Thus $q_2^{**} \leqq 0$ implies $q_1^{**} \leqq q_2^{**}$. But this yields a contradiction since $q_1^{**} = u'(c^{**}) > 0$. Hence $q_2^{**} > 0$ and therefore $q_1^{**} > q_2^{**}$. Thus $(k^{**}, a^{**}, c^{**}, z^{**}, q_1^{**}, q_2^{**})$ is an optimal balanced growth path in P_{I}. This completes the proof of the theorem.

Remark 2. The optimal balanced growth path with one asterisk is a path on which the marginal revenue of foreign investment (or the marginal cost of borrowing) is equal to the net marginal product of domestic capital, while the path with two asterisks does not necessarily satisfy this condition.

In what follows, we shall show that if the function g is strictly concave then the optimal path with two asterisks can rarely satisfy the equality condition of the marginal revenue of foreign investment and the net marginal product of domestic capital. To this end, it is sufficient to prove

LEMMA 5. *Suppose that g is strictly concave. Then*

(i) $s_1(k, m) \geqq 0$ *for all* $k > 0$ *and* $m > 1$,

(ii) $s_2(k, m) \gtreqless 0$ *if and only if* $f'(k) \gtreqless \delta + n_1 + n_2$,

(iii) $s(k, m_o) = h(k)$ *for all* $k > 0$,

(iv) $s(k, m) = h(k)$ *if and only if* $f'(k) = \delta + n_1 + n_2$ *for all* $m > 1$ *such that* $m \neq m_o$,

where $m_o = (\delta + n_1 + n_2)/n_2$ *and* $g'(h(k)) = f'(k) - n_2$.

Proof. Since g is strictly concave, s is single-valued and furthermore

$$s_1(k, m) = \frac{m}{m-1}\,\frac{\delta + n_1}{\delta + n_1 + n_2}\,\frac{f''(k)}{g''(s(k, m))},$$

$$s_2(k, m) = \frac{1}{(m-1)}\,\frac{\delta + n_1}{\delta + n_1 + n_2}\,\frac{\delta + n_1 + n_2 - f'(k)}{g''(s(k, m))}.$$

Hence (i) and (ii) are obvious. Furthermore, by simple substitution, we get $g'(s(k, m_o)) = f'(k) - n_2$, which proves (iii). If $f'(k) = \delta + n_1 + n_2$, then clearly $g'(s(k, m)) = f'(k) - n_2$, so that $s(k, m) = h(k)$. Conversely, if $s(k, m) = h(k)$

then $(\delta + n_1 + n_2 - mn_2)(\delta + n_1 + n_2 - f'(k)) = 0$. Since $m \neq (\delta + n_1 + n_2)/n_2$, it must be that $f'(k) = \delta + n_1 + n_2$. Thus (iv) has been established.

The relation $a = h(k)$ represents the equality of the marginal revenue of foreign investment and the net marginal product of domestic capital. Since $a^{**} = s(k^{**}, m)$ and $f'(k^{**}) < \delta + n_1 + n_2$, it follows from (iv) of Lemma 5 that $a^{**} \neq h(k^{**})$ unless $m = (\delta + n_1 + n_2)/n_2$. But the last condition can seldom be satisfied[7].

Another implications of the strict concavity of g may be summarized as the following two remarks.

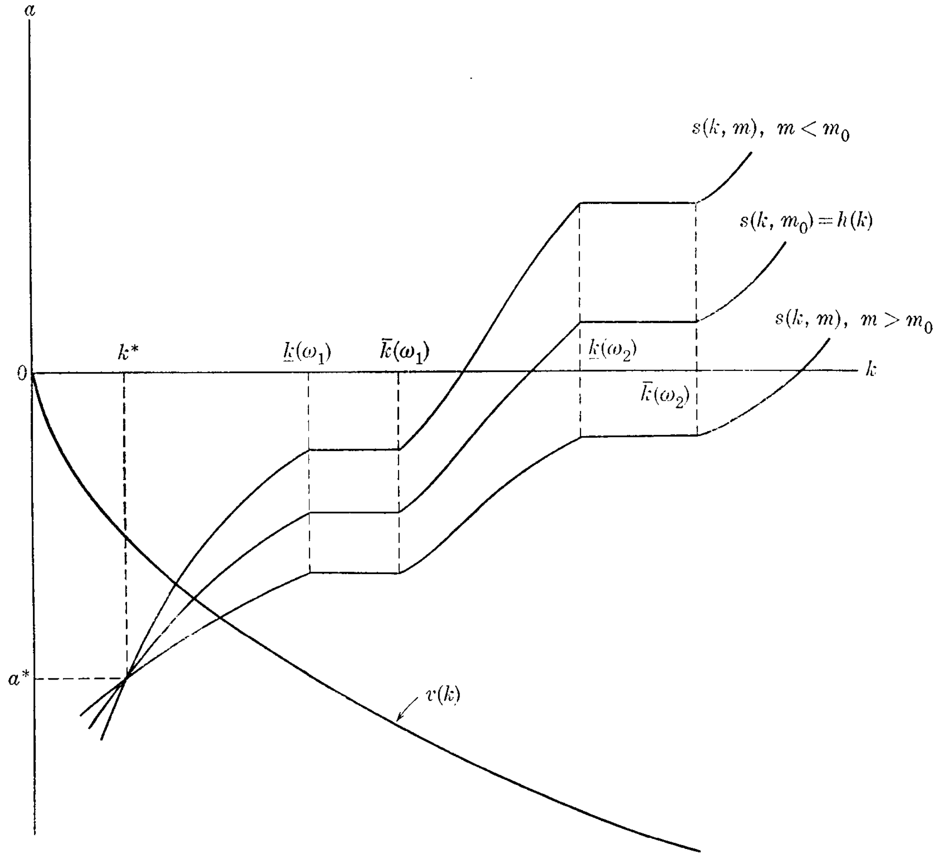

Fig. 6

Remark 3. If g strictly concave then the optimal balanced growth path with two asterisks is unique.

Remark 4. If g is strictly concave and if the domestic production is completely specialized to one of the two commodities, then the optimal balanced growth path is unique.

Remark 3 follows from the fact that s is single-valued and nondecreasing and v is decreasing. Remark 4 is due to the fact that f is strictly concave in the domain of k for which the specialization is complete.

(7) In particular, this condition cannot be satisfied if $n_2 = 0$.

IX. DYNAMIC BEHAVIOUR OF OPTIMAL PATH

In this section, we shall be concerned with the dynamic behaviour of the unique optimal path strarting from an arbitrary initial condition. We strengthen Assumptions 2 and 3 for this purpose.

Assumption 2′. $g''(a) < 0$ for all a.

Assumption 3′. $g'(-\infty) > \delta + n_1$, $g'(0) \geqq 0$, $g'(\infty) \geqq n_1/(1-m)$, and if $mf(k) - (n_1 + n_2)k = -mg(b(k))$ then $g'(b(k)) > \delta + n_1$, where $b(k) = ((1-m)/m)((n_1 + n_2)/n_1)k$.

These are sufficient for guaranteeing both the uniqueness of optimal path and the existence of an optimal balanced growth path.

Let us consider first the case where an optimal balanced growth path lies in P_{III}. We begin by characterizing the path which stays in P_{III}. On such a path, the equality $q_1 = q_2$ continues to hold, so that $\dot{q}_1 = \dot{q}_2$. Hence, by (10), (11), and (iii) of Theorem 9, the relation $a(t) = h(k(t))$ must be satisfied for all t such that the path stays in P_{III}. From (8) and (9),

$$\dot{k} = z - (n_1 + n_2)k = g(h(k)) - n_1 h(k) + f(k) - c - h'(k)\dot{k} - (n_1 + n_2)k\,,$$

while, from (11) and (iii) of Theorem 9,

$$u''(c)\dot{c} = \dot{q}_2 = (\delta + n_1 - g'(h(k)))u'(c) = (\delta + n_1 + n_2 - f'(k))u'(c)\,.$$

Thus

$$\dot{k} = G(k)(R(k) - c)\,, \tag{16}$$

$$\dot{c} = (\delta + n_1 + n_2 - f'(k))\frac{u'(c)}{u''(c)}\,, \tag{17}$$

where

$$G(k) = \frac{1}{h'(k) + 1}\,,$$

$$R(k) = g(h(k)) - n_1 h(k) + f(k) - (n_1 + n_2)k\,.$$

On the other hand,

$$c + z = g(h(k)) - n_1 h(k) + f(k) - h'(k)(z - (n_1 + n_2)k)\,,$$

so that

$$(c + z)(1 + h'(k)) = g(h(k)) - n_1 h(k) + f(k) + (n_1 + n_2)kh'(k) + ch'(k)\,.$$

Hence by the constraint (6),

$$c \leqq T(k) \quad \text{if} \quad h'(k) > 0\,, \tag{18–1}$$

$$a \geqq v(k) \quad \text{otherwise,} \tag{18–2}$$

where

$$T(k) = \frac{1}{h'(k)}\{(m - 1 + mh'(k))(f(k) + g(h(k))) + n_1h(k) - (n_1 + n_2)kh'(k)\}.$$

It should be noted that T is not defined for k such that $\underline{k}(\omega) < k < \bar{k}(\omega)$ where $\omega \in \Omega(p)$, since $h'(k) = 0$ for such k.

In order to characterize the path which stays in P_{III}, we must investigate the system of differential equations (16) and (17) with the constraint (18). Let us first examine the behaviour of the solutions of the system in the neighbourhood of the balanced growth path (k^*, c^*). Notice that a^* is unique by the strict concavity of g while k^* is not necessarily unique. Thus define $\underline{k}^* = \max\,(v^{-1}(a^*), \min\ K^*)$, $\bar{k}^* = \max\ K^*$, $\underline{c}^* = g(a^*) - n_1a^* + f(\underline{k}^*) - (n_1 + n_2)\underline{k}^*$, and $\bar{c}^* = g(a^*) - n_1a^* + f(\bar{k}^*) - (n_1 + n_2)\bar{k}^*$, where K^* is defined in the proof of Theorem 12. We now prove

THEOREM 13. *Suppose that there is a* (k, a) *such that* $k \in K^*$, $a \in A^*$, *and* $a \geqq v(k)$. *Then there exists in the neighbourhood of the optimal balanced growth paths a unique solution to the system* (16) *and* (17) *such that*

(i) *if* $v^{-1}(a^*) < \underline{k}^*$ *and the initial* k *is smaller than* $\underline{k}^*$ *then the solution converges monotonically to* $(\underline{k}^*, \underline{c}^*)$,

(ii) *if the initial* k *lies in the closed interval* $[\underline{k}^*, \bar{k}^*]$ *then the solution remains at the initial position,*

(iii) *if the initial* k *is greater than* $\bar{k}^*$ *then the solution converges monotonically to* $(\bar{k}^*, \bar{c}^*)$.

Proof. Consider the Taylor expansion of the system in the neighbourhood of the balanced growth path (or one of balanced growth paths). The characteristic equation of the linear system

$$x^2 - G(k^*)R'(k^*)x - G(k^*)h'(k^*)g''(h(k^*))u'(c^*)/u''(c^*) = 0$$

has different two real roots. Hence the convergence of any solution is always monotonic. If $h'(k^*) > 0$ then the two roots have opposite signs, while if $h'(k^*) = 0$ then one root is positive and the other vanishes. If $v^{-1}(a^*) < \underline{k}^*$ then $\lim_{k \to \underline{k}^* - 0} h'(k) > 0$. Thus the point $(\underline{k}^*, \underline{c}^*)$ has the properties of a saddle point in so far as the convergence from below is concerned. Therefore there is a unique solution converging to $(\underline{k}^*, \underline{c}^*)$ from below, which proves (i). The proof of (iii) is similar. If $\underline{k}^* \leqq k \leqq \bar{k}^*$ then $\dot{c} = 0$, so that we can find a stationary solution by putting $c = R(k)$, completing the proof of (ii).

Next, we shall examine the global properties of the solution to the system (16) and (17). Let $c = E(k)$ denote the solution in Theorem 13. Define

$$k' = \sup\{k\colon\ k < k^*,\ T(k) \leqq E(k)\},$$

$$k'' = \inf \hat{K}\ (\text{if } \hat{K} \text{ is empty then } k'' = \hat{k}),$$

$$\underset{\sim}{k} = \max\,(k', k'')\quad (\text{if } k' \text{ is not defined then } \underset{\sim}{k} = k''),$$

$$\tilde{k} = \inf\{k\colon\ k > k^*,\ T(k) \leqq E(k)\}\,,$$

where

$$\hat{K} = \{k\colon\ 0 \leqq k \leqq \hat{k},\ (\forall x)\,(k < x \leqq \hat{k} \Rightarrow h'(x) > 0)\}\,,$$
$$h(\hat{k}) = v(\hat{k})\,.$$

Then we can state

THEOREM 14. *Suppose that $f(k) + g(-((n_1 + n_2)/n_1)k) = 0$ implies $g'(-((n_1 + n_2)/n_1)k) > f'(k) - n_2$ and that there is a (k, a) such that $k \in K^*, a \in A^*$, and $a \geqq v(k)$. Then the solution $(c, k) = (E(k), k)$ with $\underset{\sim}{k} \leqq k \leqq \tilde{k}$ represents the optimal path in P_{III}.*

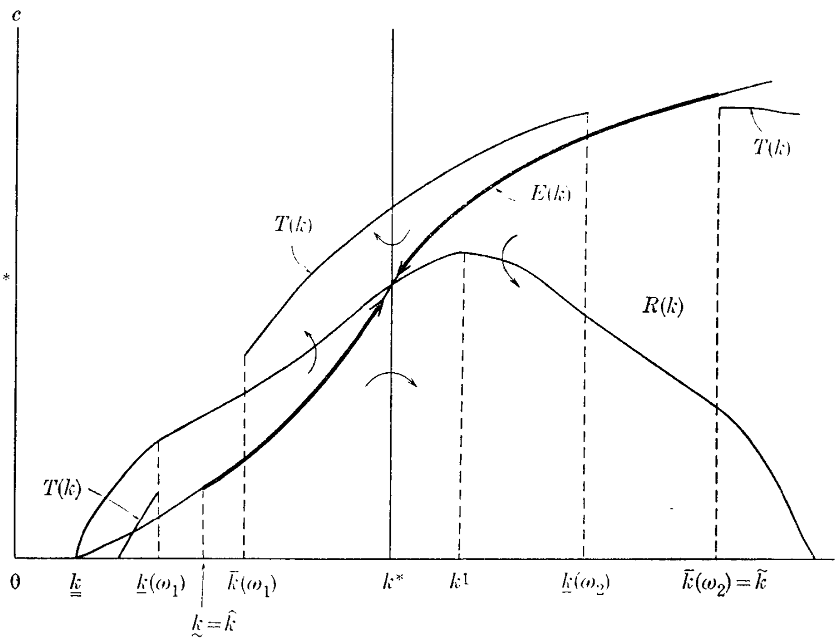

Fig. 7

Proof. It suffices to show that $E(k) > 0$ and (18) is satisfied for all k such that $\underset{\sim}{k} \leqq k \leqq \tilde{k}$. Let $R(\underline{\underline{k}}) = 0$, $\underline{\underline{k}} < k^*$, $h(\underline{k}) = w_1(\underline{k})$, and $\underline{k} > 0$. We shall first show that $\underline{\underline{k}} < \underline{k}$. Let $f(k^1) + g(-((n_1 + n_2)/n_1)k^1) = 0$ and $k^1 > 0$. Then by hypothesis $g'(-((n_1 + n_2)/n_1)k^1) > f'(k^1) - n_2$ so that $h(k^1) > -((n_1 + n_2)/n^1)k^1 = w_1(k^1)$. Since w_1 is decreasing and h is nondecreasing, it follows that $\underline{k} < k^1$. It can be easily seen, from the proof of Lemma 4, that $w_1(\underline{k}) > w(\underline{k})$. Note that $h(\underline{\underline{k}}) = w(\underline{\underline{k}})$. Since h is nondecreasing and $w'(\underline{\underline{k}}) < 0$, we have $\underline{\underline{k}} < \underline{k}$.

Now let us notice that

$$T(k) - R(k) = \frac{h'(k) + 1}{h'(k)}\{(m - 1)(f(k) + g(h(k))) + n_1 h(k)\}\,.$$

If T is defined and $k \geqq \hat{k}$ then $T(k) \geqq R(k)$. On the other hand $E(k) \leqq R(k)$

for all k such that $\underline{\underline{k}} \leqq k \leqq k^*$. As was shown above, $\underline{\underline{k}} < \underline{k}$ and furthermore $\underline{k} < \hat{k}$ since $v(k) > w_1(k)$ for all $k > 0$ and h is nondecreasing. Hence $\underline{\underline{k}} < \hat{k}$ so that (18) is satisfied in the interval $[\hat{k}, k^*]$. Thus, by the definition of $\underset{\sim}{k}$ and $\breve{k}$, the condition (18) is satisfied in the whole interval $[\underset{\sim}{k}, \breve{k}]$.

It remains to show that $E(k) > 0$ in the interval. Since $E(k) > 0$ for all $k > \underline{\underline{k}}$ and $\underline{\underline{k}} < \underline{k}$, it must be that $E(k) > 0$ for all $k \geqq \underline{k}$. If $k'' \geqq \underline{k}$ then $\underset{\sim}{k} \geqq k'' \geqq \underline{k}$ so that $E(k) > 0$ for all $k \geqq \underset{\sim}{k}$. On the other hand, suppose $k'' < \underline{k}$[8]. Since $T(\underline{k}) = \{n_1 h(\underline{k}) - (n_1 + n_2)\underline{k}h'(\underline{k})\}/h'(\underline{k}) < 0$ and $E(\underline{k}) > 0$ it must be that $E(\underline{k}) > T(\underline{k})$. But $E(\hat{k}) \leqq R(\hat{k}) = T(\hat{k})$. Hence there is a k' such that $\underline{k} \leqq k' \leqq \hat{k}$ and $E(k) \geqq T(k)$ for all k such that $\underline{k} \leqq k \leqq k'$. Thus $\underset{\sim}{k} = k' \geqq \underline{k}$, and hence $E(k) > 0$ for all $k \geqq \underset{\sim}{k}$, completing the proof.

Since the optimal path is unique by Assumption 2′ and Theorem 11 there is no optimal path which diverges from the path described in Theorems 13 and 14. We now turn to characterizing the pattern of transitions to this path from other paths.

LEMMA 6. *If $q_1 \geqq q_2$ and $\dot{q}_1 < \dot{q}_2$ then $g'(a) < f'(k)$, while if $q_1 \leqq q_2$ and $\dot{q}_1 > \dot{q}_2$ then $g'(a) > f'(k) - n_2$,* i.e., $a < h(k)$.

Proof. Since $\dot{q}_1 - \dot{q}_2 = (\delta + n_1 + n_2)(q_1 - q_2) + (g'(a) - f'(k) + n_2)q_2 + (u'(c) - q_2)m(g'(a) - f'(k))$, if $q_1 \geqq q_2$ and $\dot{q}_1 < \dot{q}_2$ then either $g'(a) < f'(k) - n_2$ or $g'(a) < f'(k)$; hence $g'(a) < f'(k)$. On the other hand, if $q_1 \leqq q_2$ and $\dot{q}_1 > \dot{q}_2$ then either $g'(a) > f'(k) - n_2$ or $g'(a) > f'(k)$; hence $g'(a) > f'(k) - n_2$.

LEMMA 7. *If $q_1 = q_2$ and $\dot{q}_1 = \dot{q}_2$ then $f'(k) - n_2 \leqq g'(a) < f'(k)$.*

Proof. Since $0 = \dot{q}_1 - \dot{q}_2 = (g'(a) - f'(k) + n_2)q_2 + (u'(c) - q_2)m(g'(a) - f'(k))$, it must be that $g'(a) - f'(k) + n_2 > 0$ implies $g'(a) < f'(k)$ while $g'(a) - f'(k) + n_2 < 0$ implies $g'(a) > f'(k)$. From the latter we get $g'(a) \geqq f'(k) - n_2$, and from the former $g'(a) < f'(k)$; this completes the proof.

THEOREM 15. *Suppose that all the assumptions of Theorem 14 are satisfied. Then every optimal path which leads from the region $\{(k, a)\colon a > h(k)\}$ to the path represented by $(k, a, c) = (k, h(k), E(k))$ and $\underset{\sim}{k} \leqq k \leqq \breve{k}$ lies in P_{I} just before the transition time, with the exception that there may be a path in P_{II} which leads to the point $(\breve{k}, h(\breve{k}), E(\breve{k}))$ if $E(\breve{k}) = y(\breve{k}, h(\breve{k}))$.*

Proof. For any $(k, a, q_1, q_2) \in P_{\text{III}}$ there is a positive number ε such that $q_2 - u'(y(k, a)) = \varepsilon$, while $u'(y(k, a)) \geqq q_2$ in P_{II}. Hence the continuity of q_2 implies that no transition occurs from P_{II} to P_{III}, unless ε is arbitrarily small at the transition point. Therefore, all the transitions to P_{III} must be either from P_{I} or from P_{IV} unless in the exceptional case. But, by Lemma 6, if $a > h(k)$ then $q_1 > q_2$ or $\dot{q}_1 \leqq \dot{q}_2$. Suppose that $\dot{q}_1 \leqq \dot{q}_2$. If $\dot{q}_1 = \dot{q}_2$ then $q_1 \neq q_2$ by Lemma 7. If $q_1 < q_2$ then it must be that $\dot{q}_1 > \dot{q}_2$ in order to enter P_{III} where $q_1 = q_2$.

(8) Note that, in this case, T is defined for all k such that $k'' \leqq k \leqq \hat{k}$, and evidently, for $\underline{k}$.

But this is impossible since $\dot{q}_1 \leqq \dot{q}_2$ by assumption; hence $q_1 > q_2$. If $\dot{q}_1 < \dot{q}_2$ then it is necessary that $q_1 > q_2$ to enter P_{III}. Thus $\dot{q}_1 \leqq \dot{q}_2$ implies $q_1 > q_2$. Therefore $a > h(k)$ implies $q_1 > q_2$. This shows that every path leading to P_{III} from the region such that $a > h(k)$ must be in P_{I} just before the transition time.

THEOREM 16. *Suppose that $n_2 = 0$ in addition to the assumptions of Theorem 15. Then every path which leads to the path represented by $(k, a, c) = (k, h(k), E(k))$ and $\underset{\sim}{k} \leqq k \leqq \hat{k}$ from the region such that $a < h(k)$ lies in P_{IV} just before the transition time, with the exception stated in Theorem 15*(9).

Proof. Symmetrical to the proof of Theorem 15.

The preceding arguments were confined to the case where the region P_{III} has a balanced growth path. The rest of the present section will be devoted to characterizing the unique optimal path converging to a unique balanced growth path in P_{I}. We begin by considering the Taylor approximation of the System (8) through (11) in the neighbourhood of the balanced growth path. The characteristic equation of the linear system can be written as

$$x^4 + a_1 x^3 + a_2 x^2 + a_3 x^1 + a_4 = 0\,,$$

where

$$\begin{aligned}
a_1 &= -2\delta\,,\\
a_2 &= \delta^2 - \{n_1 + (1-m)g'(a^{**})n_2/(\delta + n_1)\}^2\\
&\quad - \delta\{n_1 + (1-m)g'(a^{**})n_2/(\delta+n_1)\}\\
&\quad - (\delta + n_1 + n_2)(f''(k^{**})/f'(k^{**}))(u'(c^{**})/u''(c^{**}))\,,\\
a_3 &= \delta(\delta^2 - a_2)\,,\\
a_4 &= \{m(\delta + n_1)((m-1)g'(a^{**}) + n_1)f''(k^{**})\\
&\quad + (m-1)^2(\delta + n_1 + n_2)f'(k^{**})g''(a^{**})\}u'(c^{**})/u''(c^{**})\,.
\end{aligned}$$

LEMMA 8. *Suppose that $k \in K^*$ and $a \in A^*$ imply $a < v(k)$. Then two of the roots of the above characteristic equation have positive real parts. Furthermore, if $n_1 + (1-m)n_2 \geqq 0$ then the other two roots have negative real parts.*

Proof. Let x_1, x_2, x_3, and x_4 be the characteristic roots. Then

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 = -a_1 > 0\,,\\
&x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = a_2\,,\\
&x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -a_3\,,\\
&x_1x_2x_3x_4 = a_4 > 0\,.
\end{aligned}$$

It is sufficient to consider the following three cases: Case I. All roots are real,

(9) No specification is generally possible of the transitions from or to P_{II} without further restrictions on the model.

Case II. Two roots are real and the other two complex, and Case III. All roots are complex.

Case I. Since both the sum and product of the four roots are positive, either two roots are positive and the other two negative or all the four roots are positive. But, since $g'(a^{**}) < \delta + n_1$, it follows that $n_1 + (1 - m)g'(a^{**})\, n_2/(\delta + n_1) > n_1 + (1 - m)n_2$. Hence if $n_1 + (1 - m)n_2 \geqq 0$ then $\delta^2 > a_2$ so that $a_3 > 0$. Thus not all roots are positive, which implies that two roots are positive and the other two negative.

Case II. Let x_1 and x_2 be real roots and x_3 and x_4 complex roots. Then x_3 and x_4 are conjugate. Let $x_3 = y_3 + iz_3$ and $x_4 = y_3 - iz_3$, where i is the imaginary number. Then $x_1 + x_2 + 2y_3 = - a_1 > 0$, and $x_1x_2(y_3^2 + z_3^2) = a_4 > 0$. By the latter, x_1 and x_2 have the same sign. Hence either $x_1 > 0$ and $x_2 > 0$ or $y_3 > 0$. On the other hand, if $n_1 + (1 - m)n_2 \geqq 0$ then $2x_1x_2y_3 + (x_1 + x_2)(y_3^2 + z_3^2) = - a_3 < 0$ so that either $x_1 < 0$ and $x_2 < 0$ or $y_3 < 0$, completing the proof for Case II.

Case III. In this case, we may write all the roots as $x_1 = y_1 + iz_1$, $x_2 = y_1 - iz_1$, $x_3 = y_2 + iz_2$, and $x_3 = y_2 - iz_2$. Hence $2y_1 + 2y_2 = - a_1 > 0$ and $(y_1^2 + z_1^2)2y_2 + 2y_1(y_2^2 + z_2^2) = - a_3 < 0$. Thus either $y_1 > 0$ or $y_2 > 0$. If $n_1 + (1 - m)n_2 \geqq 0$ then either $y_1 < 0$ or $y_2 < 0$. This proves the lemma.

From this lemma, we immediately have

THEOREM 17. *Suppose that* $k \in K^*$ *and* $a \in A^*$ *imply* $a < v(k)$, *and that* $n_1 + (1 - m)n_2 \geqq 0$. *Then for any* (k, a) *in the neighbourhood of* (k^{**}, a^{**}) *there exists a unique path in* P_{I} *which passes through* (k, a), *converging to* (k^{**}, a^{**}).

As for the transition to P_{I} from other regions, we cannot say anything without further specifications of the model.

X. SUMMARY

In the present paper, we have characterized the behaviour of optimal growth path in an economy which faces both an international (perfectly competitive) commodity market and an international (imperfectly competitive) capital market. Under the usual neoclassical assumptions on the production functions, we have first shown that feasible paths are bounded if the marginal cost of borrowing is positive for sufficiently large value of borrowing and the marginal revenue of foreign investment (lending) is less than the rate of growth of the labour force for sufficiently large value of foreign asset. Since these conditions are very mild, we can meaningfully consider the optimization problem with regard to our model. Secondly, we have obtained sufficient conditions for the existence of an optimal balanced growth path. The conditions obtained are such that:

(i) $g'(-\infty) > \delta + n_1$,

(ii) $g'(0) \geqq 0$,

(iii) $g'(\infty) \geqq n_1/(1-m)$,

(iv) $mf(k) - (n_1 + n_2)k = -mg(b(k))$ implies $g'(b(k)) > \delta + n_1$,

where $b(k) = \dfrac{1-m}{m} \dfrac{n_1 + n_2}{n_1} k$.

Condition (i) means that the marginal cost of borrowing for sufficiently large amount of borrowing is greater than the sum of the subjective rate of discount and the rate of growth of labour force. Conditions (ii) and (iii) are satisfied if the marginal revenue of foreign investment is always nonnegative. Condition (iv) is rather complicated. A simple condition which implies (iv) is given by $g'(0) > \delta + n_1$. But this is too strong to guarantee the existence of an optimal balanced growth path, since $g'(0) \geqq \delta + n_1$ is sufficient as can be easily shown.

It should be noted that the balanced growth path is not necessarily a path which equates the marginal revenue of foreign investment to the net marginal product of domestic investment. But, since $g'(a^*) = \delta + n_1$ where a^* is the foreign asset per capita on the optimal balanced growth path which equates the marginal revenue of foreign investment to the net marginal product of domestic capital, it follows that $a^* \gtrless 0$ if and only if $g'(0) \gtrless \delta + n_1$. That is, if the international rate of interest is relatively low, then the optimal path approaches the balanced growth path with positive debt (borrowing situation); on the other hand, if the rate of interest is relatively high, then the optimal path approaches the balanced growth path with positive asset (lending situation). The critical rate is given by $r(0) = g'(0) = \delta + n_1$, that is, the sum of the subjective rate of discount and the rate of growth of labour force. This result coincides exactly with that by K. Hamada [3].

Although we so far have not been assuming the strict concavity of function g for positive value of foreign asset, we can deduce the uniqueness and further properties of the optimal path by assuming the strict concavity of g on the domain. In what follows, let us focus our attention upon the case for which the optimal balanced growth path is such that the marginal revenue of foreign investment is equal to the net marginal product of domestic capital. Then we can find an optimal path tending to the optimal balanced growth path. By uniqueness, any optimal path tends to the balanced growth path. If the marginal revenue of foreign investment is smaller than the net marginal product of domestic capital, then the domestic absorption (i.e., consumption plus domestic investment) should be maintained at the maximum possible level by importing capital as much as possible. Conversely, if the marginal revenue of foreign investment is greater than the net marginal product of domestic capital, then the domestic investment should be minimized, i.e., the gross investment should be zero, provided the domestic capital stock does not depreciate. In the case with capital depreciation, the last statement is not necessarily true. These results are the same as those of

K. Hamada, so far as the complication caused by capital depreciation is neglected.

Thus, the optimal policy on the international capital movements is not affected by taking into account the international trade explicitly. Furthermore, the pattern of specialization in domestic production is not affected by the international capital movements; that is, it depends only upon the international terms of trade and the amount of capital stock. But, these independences would be merely the consequences of the simplifying assumption of constant terms of trade. Indeed, if we take into account the effects of the international capital movements on the terms of trade, we should have complicated results.

Finally, let us characterize the structure of optimal trade policy. If the domestic production is completely specialized to one commodity, then the other commodity not produced should be imported since both commodities are indispensable for consumption, while the direction of the trade of the produced commodity cannot be specified without further restrictions on the model. If the specialization of domestic production is incomplete, then we cannot tell about the trade policy. Although we can at most say these things about the trade policy in the general model above, we have more things to say if we are dealing with the usual two-sector model in which one commodity is for pure consumption and the other for pure investment. Let us regard the second goods as pure investment goods. It is clear that most of the results obtained remain unchanged by this modification.

Consider first the region where the domestic production is completely specialized to the first goods, i.e., consumption goods. If the debt is relatively small (or the asset is relatively large), i.e., the marginal cost of borrowing is smaller than the net marginal product of domestic capital, then the amount of imports should be maximized, i.e., investment goods should be imported as much as possible. If the debt is relatively large, we cannot generally tell about the direction of trades. But in the absence of capital depreciation, investment goods should be neither exported nor imported, since both gross investment and domestic production of investment goods are zero. The direction of the trade of consumption goods cannot be specified without further restrictions.

Consider second the region where the domestic production is completely specialized to the second goods, i.e., investment goods. Then clearly all of the consumption goods should be imported. If the debt is relatively small, then the direction of the trade of investment goods cannot be determined without further specifications of the model. But, if the balanced growth path lies in this region, some of the output of investment goods will be exported eventually. If the debt is relatively large, we cannot generally specify the direction of the trade of investment goods. If the capital stock does not depreciate, however, all of the output of investment goods should be exported since the domestic gross investment is zero.

In the region where the specialization of domestic production is incomplete, the direction of the trade of consumption goods is indeterminate in general. As for investment goods, we can only say that all the output of them should be exported if the debt is relatively large and the capital stock does not depreciate.

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