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THE SHRINKING PROCESS OF THE CORE AND THE OCCURRENCE OF THE COMPETITIVE EQUILIBRIUM OF AN EXCHANGE ECONOMY WITH MONEY

YASOI YASUDA*

ABSTRACTS

This paper considers a model of an exchange economy with money. The shrinking process of the core of an economy is shown, and further the mechanism of the occurrence of a competitive equilibrium is verified. The core, herein, is the organizationally stable imputation in the sense that it consists of the imputation which is not blocked by a coalition of the economy.

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1. INTRODUCTION

Competitive markets are often assumed in the economic theory, and discussions are advanced under the condition that such markets already exist. However there have been only a few successful attempts to analyze exactly the mechanism of the occurrence of the competitive equilibrium of markets.

Recently, Debreu and Scarf [6] have succeeded to provide a mathematical proof of the problem as a limit theorem on the core of an economy. Originally, the core is a solution concept of the theory of games first introduced by Gillies [8] and Shapley. The *core* is defined as the set of imputations which is not blocked by coalitions. It coincides with the *actually attainable allocations* in the pure exchange economy that are discussed in Mathematical Psychics [7] by Edgeworth. He named this allocation *the contract curve.*

Many economists regard a limit theorem on the core as a modern interpretation of the Edgeworth's conjecture, "the quantity of final setlement is diminished as the number of competitors is increased" (p. 40 [7]). I wish to point out, however, that the above conjecture is only a part of Edgeworth's conjecture. It seemes to me that Edgeworth's main interest is *competition* in the market of small size;

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he felt that as the size of the market grows larger *economic conflict* decreases and competition plays an increasingly priviledged role, giving a deterministic optimal allocation. In other words, his concern seemed to be the relation between economic conflict and the size of the market. Using the concept of *contract* he insisted that in a small size market there are few possibilities that final settlements are determined by competition, and that increasingly economic conflict can be resolved only by *arbitration.*

He regards the notion of contract as playing an important role in the analysis of the economic systems. He wrote, for example, that "A *settlement* is a contract which can not be varied with the consent of all the parties to it," and "A *final settlement* is a settlement which can not be varied by recontract within the field of competition".

No proposed allocation of goods would be finally acceptable to the market as a whole if any subset of traders could do better by exchanging goods only among themselves. In such cases, there is the possibility of *recontracting.* In this sense, the contract curve are the final settlements that have no possibility of recontracting.

Edgeworth conjectured that if the number of traders in the market were increased, more and more allocations would be ruled out, and eventually only competitive allocations would remain. In other words, contract with perfect competition is perfectly determinate and is without economic conflict.

The shrinking process of the core probably has a double meaning, that is, on the one hand it implies that economic conflict decreases as the size of the market grows, and on the other it means that competitive equilibrium plays the priviledged role in the large market. The shrinking process of the core depends on the possibility of recontracting. The recontracting principle corresponds closely to the notion of *domination* in the theory of games.

Shubik [17] first pointed out that the problem can be studied from the point of view of the *n*-person game theory. He formulated a game-theoretic model of the Edgeworth markets with two goods and two types of traders. This model showed the remarkable results on the core obtained by the Edgeworth theory, and it further showed von Neumann-Morgenstern solutions of the Edgeworth market games.

In 1962 Scarf [10] showed a limit theorem on the core. Debreu [5] simplified the proof of Scarf and weakened his assumptions. In a paper [6], Debreu and Scarf established an elegant proof of a limit theorem on the core. They used a model of an economy with an arbitrary finite number of types of traders and an identical n participants of each type. They proved the limit theorem as the case n becomes infinite without considering the shrinking process. From the view point of game theory, the model by Debreu and Scarf is a cooperative game without side payments; this theory has been developed by Aumann [1], [2] and others.

Vind [18] considered the approximate properties of the core of an economy with finite number of trades.

Aumann [3] has adopted a continuous market model, in which traders are represented by points in an finite set, and are individually insignificant. The limiting process is therefore bypassed. Without the assumption of convexity of preferrences, it was demonstrated that the core coincides with the competitive allocation. The proof of the existence of the competitive equilibrium in a continuous market was followed by Aumann [4]. Other aspects of a continuous market have been considered by Vind [18].

With respect to the classical market games with money, remarkable results were established by Shapley and Shubik. (see references [12]–[17]). In one paper [16], Shapley and Shubik established a covergence theorem on the core in a *replicated* (km, kn) *market*.

This paper intends to refine and extend the analysis of the Edgeworth market games. The bounds of the core of the (m, n) market in the explicit form are established and the shrinking process of the core from the viewpoint of the relation between *economic conflict* and the *size of the market* is analyzed.

First, a model of an exchange economy with money is discussed. This market is said to be an $(m_1, \ldots, m_l)$ market, where l is the number of types of traders. Next the analysis concentrates on an (m, n) market, where m, n is the number of traders of each type, respectively. The characteristic function of an (m, n) market is employed. In section 3, the competitive solutions to an (m, n) market are defined. In section 4, the set of coalitions is classified into three classes by the relative composition of the market in order to obtain the core. Further, the upper and lower bounds of the core are represented in the explicit form. At the same time an extension of a theorem of classical welfare economics, i.e., a theorem that a competitive imputation is in the core, is demonstrated. Then a limit theorem is shown on the core of an Edgeworth market game. Next three sections are devoted to special cases. In section 8, (n, n) markets are considered and the core is represented in the explicit form. In section 9 the replicated (km, kn) markets are considered. In the section 10, the Edgeworth market with one trader of the first type and any of the second, i.e., the $(1, n)$ market is considered. In the next section some experimental results of the calculation of the shrinking process of the core are shown.

2. A MODEL OF AN EXCHANGE ECONOMY WITH MONEY

An exchange economy with n traders and m goods is now considered. An *allocation* will be written:

$$x = (x^1, \ldots, x^n), \text{ where } x^i = (x_1^i, \ldots, x_m^i) \text{ and all } x_j^i \geq 0 .$$

Here x_j^i is the amount of the j-th good allocated to the j-th trader. The *utility function* of the j-th trader is assumed to have the following separable form:

$$U^i(x^i, \xi^i) = u^i(x^i) + \xi^i \tag{2.1}$$

where u^i is concave and differentiable. Here ξ^i represents the amount of a kind of *money* which implies a net change from the initial level. This kind of money was first introduced into a mathematical model of an exchange economy by Shapley and Shubik [15]. For an exact discussion of this money the paper (15) or the forthcoming book by Shapley and Shubik should be consulted. Here it should be noted that a transfer of money between individuals leaves the sum of their utilities unchanged.

The *initial allocation* of goods will be denoted by a. We shall assume that every good is present in some amount, i.e.,

$$\sum_i a^i_j > 0 \quad \text{for all} \quad j(j = 1, \ldots, m)\,. \tag{2.2}$$

If consenting players are permitted to transfer goods and money at will, this economic model becomes a *cooperative n-person game with side payments* by von Neumann and Morgenstern [20]. Let N denote the set of all players, and S any subset of N. The potential worth of the coalition S, that is, total utility of S is given by

$$V(S) = \max_x \sum_S u^i(x^i) \tag{2.3}$$

where the maximization subject to $\sum_S x^i = \sum_S a^i$ and $x \geq 0$. This is known as the *characteristic function* of the game in the theory of cooperative games. The continuity of the u^i and the compactness of the range of X give the achievement of maximum. There may be some allocation since the u^i need not be strictly concave.

In particular, taking $S = N$, there may be at least one *optimal allocation for the whole economy*: let's denote it by b. Thus we have

$$\sum_N u^i(b^i) = V(N) = \max_x \sum_N u^i(x^i) \tag{2.4}$$

where maximization subject to

$$\sum_N x^i = \sum_N a^i \quad \text{and} \quad x^i \geq 0\,.$$

We can define the *competitive prices* as

$$\pi_j = \partial u^i(b^i)/x_j \quad \text{for all } i \text{ such that} \quad b^i_j > 0\,, \quad \text{for} \quad j = 1, \ldots, m\,. \tag{2.5}$$

$$\pi_j = \partial u^i(b^i)/x_j \quad \text{for all } i \text{ such that} \quad b^i_j = 0\,, \quad \text{for} \quad j = 1, \ldots, m\,. \tag{2.6}$$

If the traders use these prices to by and sell their way from the initial allocation, a, to an optimal allocation, b, then their net receipts of money are given by $\pi \cdot (a^i - b^i) = \sum_j \pi_j \cdot (a^i_j - b^i_j)$, $i = 1, \ldots, n$. Then, their final utility levels are given by

$$\omega_i = u^i(b^i) + \pi \cdot (a^i - b^i)\,, \qquad i = 1, \ldots, n\,. \tag{2.7}$$

There may be competitive allocations other than b, since the u^i need not be strictly concave. But these payoffs are unique, as are the competitive prices. We call ω the *competitive payoff vector*, or *competitive imputation*. Note that it corresponds to the classical competitive equilibrium solution of the $(m+1)$—goods exchange economy with the separable form of utility functions $U^i(x^i, \xi^i) = u^i(x^i) + \xi^i$ and with no effective lower bound constraint on the ξ^i.

The model of an exchange economy with money is modified into a model where traders are classified into several types. Traders are classified into different 1 types distinguished by their utility functions and initial allocations. Traders who belong to the same type have the same utility function and initial allocation. Each type consists of m_i traders, so this market consists of $(m_1, m_2, \ldots, m_l)$ traders. We shall call this market an $(m_1, m_2, \ldots, m_l)$ market.

In particular, markets with two types of traders and two kinds of goods, which Edgeworth considered in his *Mathematical Psychics* [7], are herein investigated. This is henceforth called an *Edgeworth market.*

3. AN EDGEWORTH MARKET MODEL

Traders are classified into two types by their preferences and initial holdings. The first and second types consist of a set M and N of traders, respectively. The utility function of the i-th trader in the first type is denoted by $U^i(x_i, y_i, \xi_i) = u^i(x_i, y_i) + \xi_i$ where x_i and y_i are the quantities of the first and second commodity, and his initial holding is $(a_i, 0)$. Here ξ_i represents the amount of *money* (positive or negative) in his account, normalized so that he starts with zero. The j-th trader in the second type has his utility function $U^j(x_j, y_j, \xi_j) = u^j(x_j, y_j) + \xi_j$ and his initial holding $(0, b_j)$. It is assumed that the u^i and u^j are concave and twice-differentiable. This market is said to be an (m, n) *market.*

Total utility of any coalition is

$$V(S, T) = \max_{X,Y} \left[\sum_S u^i(x_i, y_i) + \sum_T u^j(x_j, y_j)\right] \tag{3.1}$$

for all $(S, T) \subset (M, N)$, $S \subset M$, $T \subset N$,
where the maximization subject to $\sum_S x_i + \sum_T x_j = \sum_S a_i$, $\sum_S y_i + \sum_T y_j = \sum_T b_j$ and $x_i, x_j, y_i, y_j, \geq 0$. It is unique if the u^i and u^j are strictly concave. This is known as a characteristic function of a game.

Our concern is confined to the simple case where all traders have the same utility function of the separable form $u(x, y) + \xi$, where ξ is the net change from the initial money level. Each trader of the first and second types has intially $(a, 0)$, and $(0, b)$, respectively. These two assumptions are fairly drastic, but the mathematical results that are obtained should not be overlooked.

Because of equal tastes, total utility for any coalition is maximized by an equal quantities of the same good, and the characteristic function of the market depends only on the numbers s, t of traders of each type in a coalition. Therefore;

$$V(S, T) \equiv v(s, t) \qquad \text{and}$$

$$v(s, t) = (s + t)u(\sigma a, \tau b) \tag{3.2}$$

where σ, τ is the relative composition of the coalition, that is; $\sigma = s/(s+t)$, $\tau = 1 - \sigma = t/(s+t)$.
The relative composition γ, δ of this market is $\gamma = m/(m+n)$, $\delta = 1 - \gamma = n/(m+n)$, where m and n are the numbers of traders of each type.

In a large market where the number of traders increases infinitely the relative composition of the market is assumed to remain finite. In other words, it is assumed that;

$$\lim_{m+n\to\infty} \gamma = \gamma_0 < \infty\,, \qquad \lim_{m+n\to\infty} \delta = \delta_0 < \infty$$

Theorem 3.1 $v(s, t)$ satisfies the following properties

(3.3) $$v(0, 0) = 0$$

(3.4) $$S_1 \cap S_2 = \phi, \quad T_1 \cap T_2 = \phi, \quad S_1, S_2 \subset M, \quad T_1, T_2 \subset N$$

imply $v(s_1 + s_2, t_1 + t_2) \geq v(s_1, t_1) + v(s_2, t_2)$

(3.5) $v(s, t)$ is a concave function of (s, t).

(3.6) $v(s, t)$ is homogeneous of degree 1, that is;

$v(qs, qt) = qv(s, t)$ for all positive integer q.

Proof. (3.3) and (3.6) are immediate. For super-additivity (3.4), by the concavity of u, we obtain,

$$\begin{aligned}\frac{v(s_1, t_1) + v(s_2, t_2)}{s_1 + s_2 + t_1 + t_2} &= \frac{s_1 + t_1}{s_1 + s_2 + t_1 + t_2} u\Big(\frac{s_1}{s_1 + t_1} a, \frac{t_1}{s_1 + t_1} b\Big) \\ &\quad + \frac{s_2 + t_2}{s_1 + s_2 + t_1 + t_2} u\Big(\frac{s_2}{s_2 + t_2} a, \frac{t_2}{s_2 + t_2} b\Big) \\ &\leq u\Big(\frac{s_1 + s_2}{s_1 + s_2 + t_1 + t_2} a, \frac{t_1 + t_2}{s_1 + s_2 + t_1 + t_2} b\Big)\end{aligned}$$

Hence,

$$\begin{aligned}&v(s_1, t_1) + v(s_2, t_2) \\ &\quad \leq (s_1 + s_2 + t_1 + t_2) u\Big(\frac{s_1 + s_2}{s_1 + s_2 + t_1 + t_2} a, \frac{t_1 + t_2}{s_1 + s_2 + t_1 + t_2} b\Big) \\ &\quad = v(s_1 + s_2 + t_1 + t_2)\end{aligned}$$

For concavity (3.5), for arbitrary $\alpha, \beta \geq 0$, we have

$$\begin{aligned}\alpha v(s_1, t_1) + \beta v(s_2, t_2) &= v(\alpha s_1, \alpha t_1) + v(\beta s_2, \beta t_2) \\ &\leq v(\alpha s_1 + \beta s_2, \alpha t_1 + \beta t_2)\end{aligned}$$

Using the homogeneity. Q.E.D.

From (3.3) and (3.4), the function v forms a characteristic function of a cooperative game with side payments by von Neumann and Morgenstern. In other words, (m, n) market becomes a cooperative game with side payments.

Replicated Market: Let's consider the replicated (km, kn) market where there are km traders of the first type and kn of the second, and k is to be regarded as variable, m and n fixed. The relative composition (γ, δ) of the (km, kn) market will thus remain fixed at $\gamma = m/(m+n)$, $\delta = 1 - \gamma = n/(m+n)$. This replicated market was considered by Shapley and Shubik [15], [16].

4. THE COMPETITIVE EQUILIBRIUM

A *Pareto optimal allocation* is characterized by

$$\sum_N u(c_i, d_i) + \sum_N u(c_j, d_j) = \max_{X,Y} \left[\sum_N u(x_i, y_i) + \sum_N u(x_j, y_j)\right] \tag{4.1}$$

the maximization subject to $\sum_M x_i + \sum_N x_j = ma$, $\sum_M y_i + \sum_N y_j = na$ and x_i, $x_j, y_i, y_j \geq 0$.

In an (m, n) market, Pareto optimality can be achieved by allocating $(\gamma a, \delta b)$ to every trader followed by an arbitrary transfer, by the symmetry and concavity. To support this goods allocation, the *competitive prices* must be

$$\begin{cases} \pi_x = \dfrac{\partial u(\gamma a, \delta b)}{\partial x} & \text{(first good)} \\ \pi_y = \dfrac{\partial u(\gamma a, \delta b)}{\partial y} & \text{(second good)} \end{cases} \tag{4.2}$$

Hence the *competitive imputations* are given by

$$\begin{cases} \omega_1 = u(\gamma a, \delta b) + \delta a \pi_x - \delta b \pi_y & \text{(first type)} \\ \omega_2 = u(\gamma a, \delta b) + \gamma b \pi_y - \gamma a \pi_x & \text{(second type)} \end{cases} \tag{4.3}$$

In these expressions, the first term is the utility of the final holding, the second is the payment received for selling off part of the initial endowment, and the third is the money spent on buying other goods.

There may be competitive allocation other than $(\gamma a, \delta b)$, since u need not be strictly concave. But the competitive imputation ω is unique, as are the prices. In the replicated (km, kn) market, it is noted that ω_1 and ω_2 are independent of k. As the size of the market is changed, the competitive solution remains fixed, i.e., it coincides with that of the (m, n) market.

5. CLASSIFICATIONS OF COALITIONS

In order to obtain the core of an (m, n) market, we classify coalitions as follows. According to the relative composition γ and δ, any coalition $(S, T) \subset (M, N)$ is classified into three classes.

Here we set $\sigma(m, n) = s/(s+t)$, $\tau(m, n) = 1 - \sigma(m, n) = t/(s+t)$, where $s = |S|$, $t = |T|$ is the number of the coalition S and T, respectively.

Class A. The set of coalition (S, T) such that

$$\sigma(m, n) > \gamma \quad \text{and} \quad \tau(m, n) < \delta \,.$$

Class B. The set of coalitions (S, T) such that

$$\sigma(m, n) < \gamma \quad \text{and} \quad \tau(m, n) > \delta .$$

Class C. The set of coalitions (S, T) such that

$$\sigma(m, n) = \gamma \quad \text{and} \quad \tau(m, n) = \delta .$$

Let's define $\underline{\sigma}(m, n)$, $\bar{\tau}(m, n)$, $\bar{\sigma}(m, n)$ and $\underline{\tau}(m, n)$ as follows.

$$(5.1) \qquad \begin{cases} \underline{\sigma}(m, n) = \inf\limits_{(S,T)} (\sigma(m, n); \sigma > \gamma \text{ and } \tau < \delta) \\ \bar{\tau}(m, n) = \sup\limits_{(S,T)} (\tau(m, n); \sigma > \gamma \text{ and } \tau < \gamma) \end{cases}$$

$$\begin{cases} \bar{\sigma}(m, n) = \sup\limits_{(S,T)} (\sigma(m, n); \sigma < \gamma \text{ and } \tau > \delta) \\ \underline{\tau}(m, n) = \inf\limits_{(S,T)} (\tau(m, n); \sigma < \gamma \text{ and } \tau > \delta) \end{cases}$$

In other words, $\underline{\sigma}$ is the infimum of the σ which the coalition belongs to class A, and $\bar{\sigma}$ is the supremum. And $\bar{\tau}$ is the supremum of the τ which the coalition belongs to class B, and σ is the infimum. It is obvious that $\underline{\sigma} + \bar{\tau} = 1$ and $\bar{\sigma} + \tau = 1$.

Theorem 5.1 As the number of traders in the market increaces, $\underline{\sigma}$, $\bar{\sigma}$ converge to γ_0 and $\bar{\tau}$, $\underline{\tau}$ converge to δ_0. That is;

$$(5.3) \qquad \lim_{m+n\to\infty} \underline{\sigma}(m, n) = \lim_{m+n\to\infty} \gamma = \gamma_0, \ \lim_{m+n\to\infty} \bar{\sigma}(m, n) = \lim_{m+n\to\infty} \gamma = \gamma_0$$

$$(5.4) \qquad \lim_{m+n\to\infty} \bar{\tau}(m, n) = \lim_{m+n\to\infty} \delta = \delta_0, \ \lim_{m+n\to\infty} \underline{\tau}(m, n) = \lim_{m+n\to\infty} \delta = \delta_0 .$$

Proof. From the definition of $\underline{\sigma}(m, n)$, we obtain

$$\underline{\sigma}(m, n) \geq \underline{\sigma}(k, l) \text{ if } k \geq m, l \geq n ,$$

i.e., $\underline{\sigma}(m, n)$ is non-increasing as to (m, n).
By the definition of $\underline{\sigma}(m, n)$, we have

$$m/(m + n - 1) \geq (m, n) > \gamma .$$

Now

$$\lim_{m+n\to\infty} \frac{m}{m+n-1} = \lim_{m+n\to\infty} \frac{m/(m+n)}{1 - 1/(m+n)} = \lim_{m+n\to\infty} \frac{m}{m+n} = \gamma_0 < \infty$$

then, we have

$$\lim_{m+n\to\infty} \underline{\sigma}(m, n) = \lim_{m+n\to\infty} m/(m + n) = \gamma_0 < \infty$$

Using the relation $\bar{\tau} = 1 - \underline{\sigma}$ and $\lim\limits_{m+n\to\infty} \underline{\sigma} = \gamma_0$, we obtain

$$\lim_{m+n\to\infty} \bar{\tau} = 1 - \lim_{m+n\to\infty} \underline{\sigma} = 1 - \gamma_0 = \delta_0 > \infty .$$

As to $\bar{\sigma}$, $\underline{\tau}$, we can verify in the same way.

Q.E.D.

6. THE CORE OF THE (m, n) MARKET

Next, the core of the (m, n) market is considered. In a general multi-person cooperative game, the core may be defined as the set of imputations that are not dominated by other possible imputations by the lack of coalitions. That is, the core of a game is the set of imputations, if any, that satisfy the property of group rationality, for all groups of players.

The core concept may be regarded as an extension of the notion of Pareto optimality, taking into account the possibility of independent optimization by coalitions, as well as by the economy as a whole. That is, group rationaly implies Pareto optimality, for all the player set. Since group rationality also implies individual rationality, the core may be regarded as an extension, to groups, of the individualistic principle that means that one person will not accept any imputation that allows his initial position to worsen, unless compelled to do so. In other words, the core is both Pareto optimal and individually rational. In general the core does not exist, but it can be shown that in an economy in which competitive prices exists, the core is not void. Indeed, all competitive imputations are in the core in such a case.

In our (m, n) market, the *core* is defined as the set of imputations, $C(m, n) = (\alpha_1, \alpha_2) = (\{\alpha_1^i\}, \{\alpha_2^j\})$ such that;

$$\sum_S \alpha_1^i + \sum_T \alpha_2^j \geq v(s, t) \quad \text{for all} \quad (S, T) \subset (M, N). \tag{6.1}$$

In other words, the core is the set of imputations that are not blocked by coalitions. Here, it is said that a coalition $(S, T) \subset (M, N)$ *blocks* an imputation $(\alpha_1, \alpha_2) = (\{\alpha_1^i\}, \{\alpha_2^j\})$ if

$$\sum_S \alpha_1^i + \sum_T \alpha_2^j \leq v(s, t) \tag{6.2}$$

Continuing to the core of the (m, n) market, one obtains the following equation by expanding $u(\sigma a, \tau b)$ a Taylor's series about $u(\gamma a, \delta b)$,

$$v(s, t) = (s + t)[u(\gamma a, \delta b) + (\sigma - \gamma)\, a\pi_x + (\tau - \delta) b\pi_y + 0((\sigma - \gamma)^2)].$$

Using the relation $\sigma - \gamma = \delta - \tau = \sigma\delta - \tau\gamma$ and recalling the definition of ω_1, ω_2, we have

$$v(s, t) = s\omega_1 + t\omega_2 + (s + t)\, 0((\sigma - \gamma)^2). \tag{6.3}$$

Next, the properties of the remainder term $0((\sigma - \gamma)^2)$ are examined;

$$F(\sigma) \equiv 0((\sigma - \gamma)^2) \qquad 0 \leq \sigma \leq 1$$

Theorem 6.1

(6.4) $F(\sigma)$ is non-positive for all σ, i.e.,

$$F(\sigma) \leq 0, \qquad 0 \leq \sigma \leq 1.$$

(6.5) $F(\sigma)$ can be rewritten in the form;

$$F(\sigma) = u(\sigma a, \tau b) - \sigma\omega_1 - \tau\omega_2\,,$$

hence, $F(\sigma)$ is concave as to σ.

(6.6)
$$F(\gamma) = 0$$
$$F(0) = u(0, b) - \omega_2 \leq 0$$
$$F(1) = u(a, 0) - \omega_1 \leq 0$$

(6.7)
$$F(0) \leq F(\sigma) \leq F(\bar{\sigma}) \quad \text{for all } \sigma;\ 0 \leq \sigma \leq \bar{\sigma}\,.$$
$$F(1) \leq F(\sigma) \leq F(\underline{\sigma}) \quad \text{for all } \sigma;\ \underline{\sigma} \leq \sigma \leq 1\,.$$

(6.8)
$$F(\sigma)/(\sigma - \gamma) \leq F(\underline{\sigma})/(\underline{\sigma} - \gamma) \leq 0 \quad \text{for all } \sigma > \gamma\,.$$
$$F(\sigma)/(\sigma - \gamma) \geq F(\bar{\sigma})/(\bar{\sigma} - \gamma) \geq 0 \quad \text{for all } \sigma < \gamma\,.$$

Proof. (6.4) By concavity of the utility function u, it is obvious that the remainder term is either zero or negative. (6.5) From (6.1), we have

$$0((\sigma - \gamma)^2) = \frac{1}{s+t}v(s, t) - \frac{s}{s+t}\omega_1 - \frac{t}{s+t}\omega_2\,.$$

Using $v(s, t) = (s + t)u(\sigma a, \tau b)$, we obtain

$$0((\sigma - \gamma)^2) = u(\sigma a, \tau b) - \sigma\omega_1 - \tau\omega_2\,.$$

Since $u(\sigma a, \tau b)$ is concave, $F(\sigma)$ is concave as to σ. Property (6.6) is obvious from (6.5). (6.7) We can represent the graph of $F(\sigma)$ as in Fig. 1.

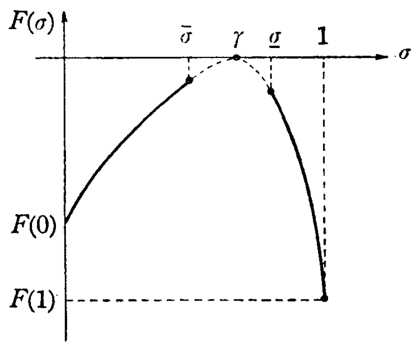

Fig. 6.1

The maximum of $F(\sigma)$ is achieved at point γ, i.e., $\max_{0 \leq \sigma \leq 1} F(\sigma) = F(\gamma) = 0$. Now there may be other points of maximization except for r since u need not be strictly concave. If the maximum of $F(\sigma)$ is achieved at $\sigma^* \neq \gamma$, then $F(\sigma) = 0$ for all $\sigma \in [\gamma, \sigma^*]$. For arbitrary $\alpha, \beta \geq 0, \alpha + \beta = 1$, we obtain

$$0 \geq F(\alpha\gamma + \beta\sigma^*) \geq \alpha F(\gamma) + \beta F(\sigma^*) = 0\,,$$

i.e., $F(\alpha\gamma + \beta\sigma^*) = 0$, since $F(\sigma)$ is concave.

Thus, we have

$$F(0) \leq F(\sigma) \leq F(\bar{\sigma}) \qquad \text{for all } \sigma;\ 0 \leq \sigma \leq \bar{\sigma}$$

$$F(1) \leq F(\sigma) \leq F(\underline{\sigma}) \qquad \text{for all } \sigma;\ \underline{\sigma} \leq \sigma \leq 1$$

(6.8) Since $F(\sigma)$ is concave as to σ, we obtain that $\sigma \geq \gamma$ implies

$$\frac{F(\sigma) - F(\gamma)}{\sigma - \gamma} \leq \frac{F(\underline{\sigma}) - F(\gamma)}{\underline{\sigma} - \gamma} \leq 0$$

note that $F(\gamma) = 0$ and $F(\sigma) \leq 0$ for all σ, then

$$\frac{F(\sigma)}{\sigma - \gamma} \leq \frac{F(\underline{\sigma})}{\underline{\sigma} - \gamma} \leq 0 \qquad \text{for all } \sigma > \gamma$$

Simultaneously, the second inequality is obtained. Q.E.D.

Theorem 6.2 The core of the (m, n) market gives the same imputations to the trader of the same type. That is, if $(\alpha_1, \alpha_2) = (\{\alpha_1^i\}, \{\alpha_2^j\})$, $i \in M$, $j \in N$ is in the core, then

$$\alpha_1^i = \alpha_1 \qquad \text{for all } i \in M$$

$$\alpha_2^j = \alpha_2 \qquad \text{for all } j \in N$$

Proof. Assume that α is not same, i.e., α gives unequal payoffs to some pair of traders of the same type. Using the concavity of the utility function,

$$\alpha_1^i \equiv u(x_1^i, y_1^i) \geq \min_M u(x_1^i, y_1^i) \equiv \min_M \alpha_1^i \qquad \text{for all } i \in M.$$

$$\alpha_2^j \equiv u(x_2^j, y_2^j) \geq \min_N u(x_2^j, y_2^j) \equiv \min_N \alpha_2^j \qquad \text{for all } j \in N$$

imply

$$(6.9) \qquad \begin{cases} u(\sum_M x_1^i/m, \sum_M y_1^i/m) > \min_M u(x_1^i, y_1^i) \\ u(\sum_N x_2^j/n, \sum_N y_2^j/n) > \min_N u(x_2^j, y_2^j) \end{cases}$$

On the other hand from the definition of the allocation,

$$\sum_M x_1^i + \sum_N x_2^j = ma$$

$$\sum_M y_1^j + \sum_N y_2^j = na$$

then we have

$$(6.10) \qquad \begin{cases} \sum_M x_1^i/m + \sum_N x_2^j/m = a \\ \sum_M y_1^i/n + \sum_N y_2^j/n = b \end{cases}$$

From (6.9) and (6.10) a pair of the worst-treated trader of the first type and the worst-treated trader of the second type can block the imputation α. This is a contradiction. Q.E.D.

From the above theorem, the core of the (m, n) market may be described as the set of imputations,

$$C(m, n) = (\alpha_1, \alpha_2) \text{ such that;}$$

(6.11) $$s\alpha_1 + t\alpha_2 \geq v(s, t) \qquad \text{for all } (S, T) \subset (M, N).$$

and

(6.12) $$m\alpha_1 + n\alpha_2 = v(m, n)$$

Theorem 6.3 The core $C(m, n)$ of the (m, n) market is the set of imputations (α_1, α_2) which satisfy the following conditions.

$$\underline{\alpha}_1 \leq \alpha_1 \leq \bar{\alpha}_1, \quad \underline{\alpha}_2 \leq \alpha_2 \leq \bar{\alpha}_2, \quad m\alpha_1 + n\alpha_2 = v(m, n).$$

where,

(6.13) $$\begin{cases} \underline{\alpha}_1 = \omega_1 + \delta F(\underline{\sigma})/(\underline{\sigma} - \gamma) \\ \bar{\alpha}_1 = \omega_1 + \delta F(\bar{\sigma})/(\bar{\sigma} - \gamma) \end{cases}$$

(6.14) $$\begin{cases} \underline{\alpha}_2 = \omega_2 + \gamma F(\bar{\sigma})/(\underline{\tau} - \delta) \\ \bar{\alpha}_2 = \omega_2 + \gamma F(\underline{\sigma})/(\bar{\tau} - \delta) \end{cases}$$

Proof. Let $\alpha = (\alpha_1, \alpha_2)$ be any imputation in the core. Then from (6.11), we have

$$s\alpha_1 + t\alpha_2 \geq s\omega_1 + t\omega_2 + (s + t)F(\sigma),$$

hence

$$\sigma\alpha_1 + \tau\alpha_2 \geq \sigma\omega_1 + \tau\omega_2 + F(\sigma), \text{ i.e.,}$$

(6.15) $$\sigma(\alpha_1 - \omega_1) + \tau(\alpha_2 - \omega_2) \geq F(\sigma)$$

From (6.12), we also have

$$m\alpha_1 + n\alpha_2 = m\omega_1 + n\omega_2, \text{ i.e.,}$$

(6.16) $$\gamma(\alpha_1 - \omega_1) + \delta(\alpha_2 - \omega_2) = 0$$

By (6.15) $\times\ \delta$ + (6.16) $\times\ (-\tau)$, we have

$$(\delta\sigma - \gamma\tau)(\alpha_1 - \omega_1) \geq \delta F(\sigma)$$

Using the relation $\delta\sigma - \gamma\tau = \sigma - \gamma$, we obtain

(6.17) $$(\sigma - \gamma)(\alpha_1 - \omega_1) \geq \delta F(\sigma)$$

Thus, if σ is strictly larger than γ, then

$$\alpha_1 \geq \omega_1 + \delta F(\sigma)/(\sigma - \gamma)$$

And, if σ is strictly smaller than γ, then

$$\alpha_1 \leq \omega_1 + \delta F(\sigma)/(\sigma - \gamma).$$

If σ is just equal to γ, then (6.17) holds with equality since $F(\gamma) = 0$.

From the property (6.8) of the theorem 6.1, we have

$$\omega_1 + \delta F(\sigma)/(\sigma - \gamma) \leq \omega_1 + \delta F(\underline{\sigma})/(\underline{\sigma} - \gamma) \leq \alpha_1 \qquad \text{for all } \sigma;\ \sigma > \gamma\,.$$

Simultaneously, we have

$$\omega_1 + \delta F(\sigma)/(\sigma - \gamma) \geq \omega_1 + \delta F(\bar{\sigma})/(\bar{\sigma} - \gamma) \geq \alpha_1 \qquad \text{for all } \sigma;\ \sigma < \gamma\,.$$

Rewriting the above inequalites into one,

$$\omega_1 + \delta F(\underline{\sigma})/(\underline{\sigma} - \gamma) \leq \alpha_1 \leq \omega_1 + \delta F(\bar{\sigma})/(\bar{\sigma} - \gamma)$$

This is one of the required results. As to ω_2, we can arrive at it in same way.

Next $(\underline{\alpha}_1, \bar{\alpha}_2)$ and $(\bar{\alpha}_1, \underline{\alpha}_2)$ are shown to indeed be imputations. It is sufficient to show $m\underline{\alpha}_1 + n\bar{\alpha}_2 = v(m, n)$. For,

$$\begin{aligned} m\underline{\alpha}_1 + n\bar{\alpha}_2 &= m\omega_1 + m\delta F(\underline{\sigma})/(\underline{\sigma} - \gamma) + n\omega_2 + n\gamma F(\underline{\sigma})/(\underline{\sigma} - \delta) \\ &= m\omega_1 + n\omega_2 + F(\underline{\sigma})\,[m\delta/(\underline{\sigma} - \gamma) + n\gamma/(\bar{\tau} - \delta)] \end{aligned}$$

Now

$$m\omega_1 + n\omega_2 = v(m, n) \text{ and}$$

$$\begin{aligned} \frac{m}{\underline{\sigma} - \gamma} + \frac{n}{\bar{\tau} - \delta} &= \frac{mn}{m+n}\cdot\frac{1}{\underline{\sigma} - \gamma} + \frac{mn}{m+n}\cdot\frac{1}{\bar{\tau} - \delta} \\ &= \frac{mn}{m+n}\cdot\frac{(\bar{\tau} - \delta + \sigma - \gamma)}{(\underline{\sigma} - \gamma)(\bar{\tau} - \delta)} = 0\,, \end{aligned}$$

since $\underline{\sigma} + \bar{\tau} = 1$ and $\gamma + \delta = 1$.
Hence we obtain

$$m\underline{\alpha}_1 + n\bar{\alpha}_2 = v(m, n). \qquad \text{Q.E.D.}$$

It is well-known that under our assumptions on utility functions and initial holdings there is a competitive equilibrium. It is obvious that there exists a competitive equilibrium in the (m, n) market. Moreover an extension of the familiar argument of welfare economics, namely that a competitive imputation is Pareto optimal is obtained.

Theorem 6.4 A competitive imputation is in the core.

Proof. $\underline{\alpha}_1 = \omega_1 + \delta F(\sigma)/(\underline{\sigma} - \gamma)$. By the definition of $\underline{\sigma}$, we have $\underline{\sigma} - \gamma > 0$. By concavity of u, we have $F(\sigma) = 0((\underline{\sigma} - \gamma)^2) \leq 0$. Hence the second term of $\underline{\alpha}_1$ is non-positive. Then we obtain $\underline{\alpha}_1 \leq \omega_1$. Simultaneously we obtain $\bar{\alpha}_1 \geq \omega_1$. Thus, we have $\underline{\alpha}_1 \leq \omega_1 \leq \bar{\alpha}_1$. In the same way we can obtain

$$\underline{\alpha}_2 \leq \omega_2 \leq \bar{\alpha}_2\,. \qquad \text{Q.E.D.}$$

Another Proof. (Shapley and Shubik)

$$v(s, t) = s\omega_1 + t\omega_2 + (s + t)0((\sigma - \gamma)^2)$$

the remainder term is non-positive by concavity of u. Hence, $v(s, t) \leq s\omega_1 + t\omega_2$

for all $(S, T) \subset (M, N)$. So the competitive imputation ω is an element of the core.

Q.E.D.

7. A LIMIT THEOREM ON THE CORE

The core $C(m, n)$ depends on the number of traders of each type. However the core shrinks as $(m + n)$ increases, that is,

$$C(k, l) \subset C(m, n) \text{ if } k \geq m, l \geq n .$$

Indeed, let $(\alpha_1, \alpha_2) \in C(k, l)$, then, there exists no coalition $(S, T) \subset (K, L)$ which blocks α. Since $\{(S, T): (S, T) \subset (K, L)\} \supset \{(S, T): (S, T) \subset (M, N)\}$, so there exists no coalition $(S, T) \subset (M, N)$ which blocks α. Hence, $(\alpha_1, \alpha_2) \in C(m, n)$. Thus the core shrinks as the number of traders increases.

As the core is interpretted as the economic conflict curve, the above proposition implies that the *economic conflict* decreases as the *size of the market* becomes larger.

Theorem 7.1 In the limit, only the competitive imputation is in the core.

Proof. It is clear that there exists a competitive equilibrium in the limit market since we assumed $\lim\limits_{m+n\to\infty} \gamma = \gamma_0 < +\infty$ and $\lim\limits_{m+n\to\infty} \delta = \delta_0 < +\infty$.
It is sufficient to show that $\lim\limits_{m+n\to\infty} F(\underline{\sigma})/(\underline{\sigma} - \gamma) = 0$ and $\lim\limits_{m+n\to\infty} F(\bar{\sigma})/(\bar{\sigma} - \gamma) = 0$.
From the continity of $F(\sigma)$,

$$\lim_{\underline{\sigma}\to\gamma_0} F(\underline{\sigma}) = F(\gamma_0) = 0 \qquad \text{and} \qquad \lim_{\bar{\sigma}\to\gamma_0} F(\bar{\sigma}) = F(\gamma_0) = 0 .$$

So we obtain

$$\lim_{\underline{\sigma}\to\gamma_0} F(\underline{\sigma})/(\underline{\sigma} - \gamma_0) = \lim_{\underline{\sigma}\to\gamma_0} F'(\underline{\sigma})$$

$$\lim_{\bar{\sigma}\to\gamma_0} F(\bar{\sigma})/(\bar{\sigma} - \gamma_0) = \lim_{\bar{\sigma}\to\gamma_0} F'(\bar{\sigma})$$

After all we must show $\lim\limits_{\underline{\sigma}\to\gamma_0} F'(\underline{\sigma}) = 0$ and $\lim\limits_{\bar{\sigma}\to\gamma_0} F'(\bar{\sigma}) = 0$

Since
$$F(\sigma) = u(\sigma a, \tau b) - \omega\sigma_1 - \tau\omega_2$$
$$= u(\sigma a, (1 - \sigma)b) - \sigma\omega_1 - (1 - \sigma)\omega_2 ,$$

we have
$$F'(\sigma) = \frac{\partial u(\sigma a, \tau b)}{\partial x} \cdot a - \frac{\partial u(\sigma a, \tau b)}{\partial y} \cdot b - \omega_1 + \omega_2 .$$

So
$$F'(\underline{\sigma}) = \frac{\partial u(\underline{\sigma} a, \bar{\tau} b)}{\partial x} \cdot a - \frac{\partial u(\underline{\sigma} a, \bar{\tau} b)}{\partial y} b - \omega_1 + \omega_2 .$$

Using the continuity of the $u(x, y)$,

$$\lim_{\sigma\to\gamma_0} F'(\sigma) = \frac{u(\gamma_0 a, \delta_0 b)}{\partial x} a - \frac{u(\gamma_0 a, \delta_0 b)}{\partial y} b - \omega_1 + \omega_2$$

$$= \pi_x a - \pi_y b - \omega_1 + \omega_2$$

Here $\qquad \omega_1 - u(\gamma_0 a, \delta_0 b) + \delta_0 a \pi_x - \delta_0 b \pi_y$

and $\qquad \omega_2 = u(\gamma_0 a, \delta_0 b) + \gamma_0 b \pi_y - \gamma_0 a \pi_x$

imply $\qquad \omega_1 - \omega_2 = (\delta_0 a + \gamma_0 a)\pi_x - (\gamma_0 b + \delta_0 b)\pi_y$

$$= \pi_x a - \pi_y b \,.$$

Hence $\qquad \lim_{\sigma\to\gamma_0} F'(\sigma) = 0 \,.$

Q.E.D.

Our proof is simple because we could get the upper and lower bounds of the core in the explicit form. In the case where the core is not represented in the explicit form, we can also verify the theorem. This proof can be demostrated according to Shapley and Shubik's work.

The core may be confined to the one-dimensional set P of symmetric Pareto-optimal imputations. We may parametrize this set by distance ω, thus: $P = \{\alpha(G) | -\infty < G < \infty\}$, where

$$\begin{cases} \alpha_1(G) = \omega_1 + \delta G & \text{(first type)} \\ \alpha_2(G) = \omega_1 - \gamma G & \text{(second type)} \end{cases}$$

As we have seen, $\alpha(0) = \omega$ is in the core.

Let Q be a coalition having $m + n - 1$ members, lacking only one trader of the first type. The core α awards Q the amount $v(m, n) - \omega_1 - \delta G$. To estimate the characteristic function of Q, we use

$$v(s, t) = s\omega_1 + t\omega_2 + (s + t)0((\sigma - \gamma)^2) \,,$$

we obtain,

$$v(m - 1, n) = v(m, n) - \omega_1 + 0(1/(m + n))$$

Thus, if G is positive, and if $m + n$ is large enough, then

$$v(m, n) - \omega_1 - \delta G < v(m, n) - \omega_1 = v(m - 1, n)$$

that is, Q can block α. Similary, if G is negative and $m + n$ is large enough, then a coalition lacking just a trader of the second type can block α. In the limit only the competitive imputation ω which corresponds to the case where G is zero remains unblocked. This completes the proof.

8. THE (n, n) MARKET

The solutions of the (m, n) market for special cases, where the number of traders of each type is same, i.e., $m = n.$, are examined.

First, a two-person game, i.e., (1, 1) market is considered. The characteristic function of the two-person game becomes as follows;

$$(8.1)\qquad \begin{cases} v(0,0)=0 \\ v(1,0)=u(a,0) \\ v(0,1)=u(0,b) \\ v(1,1)=2u(a/2,b/2) \end{cases}$$

The competitive prices to this market is given;

$$\pi_x = \frac{\partial u(a/2,b/2)}{\partial x} \qquad \text{(first good)}$$

$$\pi_y = \frac{\partial u(a/2,b/2)}{\partial y} \qquad \text{(second good)}$$

Hence the competitive imputations becomes;

$$(8.2)\qquad \begin{cases} \omega_1 = u(a/2,b/2) + a/2\pi_x - b/2\pi_y & \text{(first type)} \\ \omega_2 = u(a/2,b/2) + b/2\pi_y - a/2\pi_x & \text{(second type)} \end{cases}$$

Note that $\omega_1 + \omega_2 = v(1,1)$.
The core to this market consists of the set of imputations of the form;

$$(8.3)\qquad \begin{cases} \underline{\alpha}_1 \leq \alpha_1 \leq \bar{\alpha}_1 \\ \underline{\alpha}_2 \leq \alpha_2 \leq \bar{\alpha}_2 \end{cases}$$

where,

$$\underline{\alpha}_1 = \omega_1 + \tfrac{1}{2}F(1)/(1-\tfrac{1}{2}) = \omega_1 + \tfrac{1}{2}[u(a,0)-\omega_1]/\tfrac{1}{2} = u(a,0)$$

$$\begin{aligned} \bar{\alpha}_1 &= \omega_1 + \tfrac{1}{2}F(0)/(0-\tfrac{1}{2}) = \omega_1 + \tfrac{1}{2}[u(0,b)-\omega_2]/(-\tfrac{1}{2}) \\ &= \omega_1 + \omega_2 - u(0,b) = v(1,1) - u(0,b) \\ &= 2u(a/2,b/2) - u(0,b) \end{aligned}$$

and

$$\underline{\alpha}_2 = u(0,b)$$

$$\bar{\alpha}_2 = 2u(a/2,b/2) - u(a,0)$$

Or representing by a parameter, the core is;

$$(8.4)\qquad \begin{aligned} (\alpha_1,\alpha_2) = (&2pu(a/2,b/2) - pu(0,b) + (1-p)u(a,0)\,, \\ &2(1-p)u(a/2,b/2) + pu(0,b) - (1-p)au(a,0))\,, \quad 0 \leq p \leq 1\,. \end{aligned}$$

It is noted that the competitive imputation is in the core. In this market the von Neumann-Morgenstern solution coincides with the core.

It was shown by Shubik (17) that if the number of traders in the first type is the same as the number of traders in the second, then for any size of the market there is a von Neumann-Morgenstern solution analogous to the solution to the two-person game (although such a solution will no longer be the only the solution).

The set of imputations is characterized by a parameter p (as in the two-person market) which can be interprited as a market price. That is, an Edgeworth market denoted by (n, n) for any size n has a solution consisting of all imputations of the form:

$$(8.5)\quad (\{\alpha_i^1\}, \{\alpha_j^2\}) = (2pu(a/2, b/2) - pu(0, b) + (1-p)u(a, 0), \dots, \\ 2(1-p)u(a/2, b/2) + pu(0, b) - (1-p)u(a, 0), \dots), \quad 0 \leq p \leq 1$$

Note, however, that the core of this (n, n) market is strictly included in this solution for any size n such that $n \geq 2$.

The relative composition of this (n, n) market will remain fixed at $\gamma = 1/2$, $\delta = 1/2$ for any size of the market. So the competitive imputations are independent of the size of the market, indeed, they coincide with the competitive imputations of the two-person game.

In an (n, n) market for any size n, it is clear by symmetry that the coalition (s, t) which gives $\underline{\sigma}, \bar{\tau}$ is such that $s = n$, $t = n-1$, and the coalition (s, t) which gives $\underline{\sigma}, \bar{\tau}$ is such that $s = n-1$, $t = n-1$. The distance of the endpoints of the core for the *competitive payoff* coincides one another, i.e., $\bar{\alpha}_1 - \omega_1 = \omega_1 - \underline{\alpha}_1 = \bar{\alpha}_2 - \omega_2 = \omega_2 - \underline{\alpha}_2$.

As the number of the traders becomes infinite, the core monotoneously converges to a single outcome, i.e., the competitive imputation of the (1, 1) market.

9. THE (km, kn) MARKET

Let's consider the replicated market denoted by (km, kn) where there are km traders of the first type and kn of the second, and k is to be regarded as variable, m and n fixed. This (km, kn) market is also a special case of our (m, n) market where m and n are variable. This replicated market was first examined by Shubik (17). He considered an Edgeworth market denoted by (m, n) such that $m = km'$, $n = kn'$ and

$$(9.1)\quad u\left(\frac{ma'}{m'+n'}, \frac{nb'}{m'+n'}\right) = \max_{(s,t)\leq(m,n)}\left(\frac{sa}{s+t}, \frac{tb}{s+t}\right).$$

And he showed the next theorem;

Theorem 9.1 (Shubik)

The imputation $\qquad (\eta_1, \eta_2) = \left(\frac{v(m,n)}{m+n}, \frac{v(m,n)}{m+n}\right)$

is always in the core; and for any ε there exists a $k(\varepsilon)$ such that for all $k \geq k(\varepsilon)$ no imputation $\zeta = (\{\zeta_i\}), \{\zeta_j\})$ with a component

$$(9.3)\quad \begin{cases} \zeta_i < \dfrac{v(m,n)}{m+n} - \varepsilon & i = 1, \dots, n \\ \zeta_j < \dfrac{v(m,n)}{m+n} - \varepsilon & j = 1, \dots, n \end{cases}$$

is in the core.

This (km', kn') market is a special case of a (km, kn) market. The first proposition of the Shubik's theorem corresponds to a special case of the theorem 6.4 in this paper, and the second the limit theorem 7.1.

Shapley and Shubik (16) established the limit theorem on the core of the general (km, kn) market. As the relative composition of this (km, kn) market is $\gamma = m/(m+n)$, $\delta = n/(m+n)$, the (km, kn) market is a case of a *constant relative composition of a market* in the context of our theory.

10. THE $(1, n)$ MARKET

The Edgeworth market with one trader of the first type and any of the second is considered, and the behavior of the competitive solution and the core is examined.

All these solutions will approach a single imputation at which the single trader of the first type acts as a perfectly discriminating monopolist and obtains all of the gain from trading.

In the $(1, n)$ market the competitive imputation is represented as follows;

$$\begin{cases} \omega_1 = u(\gamma a, \delta b) + \delta a\pi_x - \delta b\pi_y & \text{(first type)} \\ \omega_2 = u(\gamma a, \delta b) + \delta b\pi_y - \gamma a\pi_y & \text{(second type)} \end{cases}$$

where

$$\pi_x = \frac{\partial u(\gamma a, \delta b)}{\partial x}, \quad \pi_y = \frac{\partial u(\gamma a, \delta b)}{\partial y}$$

and

$$\gamma = 1/((1+n), \qquad \delta = n/(1+n).$$

In other words the *competitive allocation* is $(a/(n+1), nb/(n+1))$ for all traders. In this market we have $\underline{\sigma} = 1/n$, $\bar{\tau} = 1 - 1/n = (n-1)/n$ and $\bar{\sigma} = 0$, $\underline{\tau} = 1$. So it is easy to get the upper and lower bounds of the core. Of course the competitive payoff vector lies in the core.

As the number of traders increases, the competitive imputation and the core converge into a single imputation,

$$(\eta_1, \eta_2) = \left((u(0, b) + a\frac{\partial u(0, b)}{\partial x} - b\frac{\partial u(0, b)}{\partial y}, u(0, b)\right)$$

since

$$\lim_{n\to\infty} \gamma = 0, \qquad \lim_{n\to\infty} \delta = 1.$$

Note that

$$\begin{aligned} \eta_1 &= u(0, b) + a\frac{\partial u(0, b)}{\partial x} - b\frac{\partial u(0, b)}{\partial y} \\ &= v(1, n) - nu(0, b) \\ &= (n+1)u(a/(n+1), nb/(n+1)) - nu(0, b). \end{aligned}$$

It is noted that the competitive imputation (η_1, η_2) in the limit market always lies in the core for any size of the market. Let's verify the above proposition. The characteristic function of the $(1, n)$ market is obtained; it is as follows;

$$v(1, n) = (1 + n)u(a/(1 + n), nb/(1 + n))$$
$$v(1, t) = (1 + t)u(a/(1 + t), tb/(1 + t)) \qquad \text{for any } T \subset N$$
$$v(0, t) = tu(0, b) \qquad \text{for any } T \subset N$$

Hence, we have, using the super-additivity of v,

$$\begin{aligned}\eta_1 + t\eta_2 &= (1 + n)u(a/(1 + n), nb/(1 + n)) - nu(0, b) + tu(0, b) \\ &= (1 + n)u(a/(1 + n), nb/(1 + n)) - (n - t)u(0, b) \\ &= v(1, n) - v(0, n - t) \\ &\geq v(1, t) \qquad \text{for all } T \subset N .\end{aligned}$$

And we have

$$t\eta_2 = tu(0, b) = v(0, t) \qquad \text{for all } T \subset N .$$

Hence, we obtain,

$$s\eta_1 + t\eta_2 \geq v(s, t) \qquad \text{for all } (S, T) \subset (M, N) .$$

This completes the proof.

In conclusion, it has been demonstrated that as the number of traders facing a single monopolist increases, the competitive imputation and the core will approach a single outcome at which the monopolist is totally discriminating and obtains all the gain to be had from trading.

11. SOME EXAMPLES OF THE CALCULATION EXPERIMENTS

The results of the calculation experiments of the shrinking process of the core in several cases is shown. Here we set $a = 1$, $b = 1$, and $u(x, y) = \sqrt{xy}$, then $u(x, y)$ becomes concave and differentiable. Note that $u(x, y)$ is not bounded.

Case 1. *(n, n) market:* As the relative composition γ, δ of the (n, n) market is fixed for any size n, the competitive imputation is also fixed for any size n, i.e.,

$$\omega_1 = 1/2, \quad \omega_2 = 1/2 \quad \text{for any } n \quad (n = 1, 2, \ldots)$$

The endpoints of the core are given as table 11.1.

The shrinking process of the core is shown as fig. 11.1.

TABLE 11.1

n	$\underline{\alpha}_1$	ω_1	$\bar{\alpha}_1$	$\underline{\alpha}_2$	ω_2	$\bar{\alpha}_2$
1	0.00000	0.50000	1.00000	0.00000	0150000	1.00000
2	0.41421	0.50000	0.58579	0.41421	0.50000	0.58579
3	0.44949	0.50000	0.55051	0.44949	0.50000	0.55051
4	0.46410	0.50000	0.53590	0.46410	0.50000	0.53590
5	0.47214	0.50000	0.52786	0.47214	0.50000	0.52786
6	0.47723	0.50000	0.52277	0.47723	0.50000	0.52277
7	0.48074	0.50000	0.51926	0.48074	0.50000	0.51926
8	0.48331	0.50000	0.51669	0.48331	0.50000	0.51669
9	0.48528	0.50000	0.51472	0.48528	0.50000	0.51472
10	0.48683	0.50000	0.51317	0.48683	0.50000	0.51317
11	0.48809	0.50000	0.51191	0.48809	0.50000	0.51191
12	0.48912	0.50000	0.51088	0.48912	0.50000	0.51088
13	0.49000	0.50000	0.51000	0.49000	0.50000	0.51000
14	0.49074	0.50000	0.50926	0.49074	0.50000	0.50926
15	0.49138	0.50000	0.50862	0.49138	0.50000	0.50862

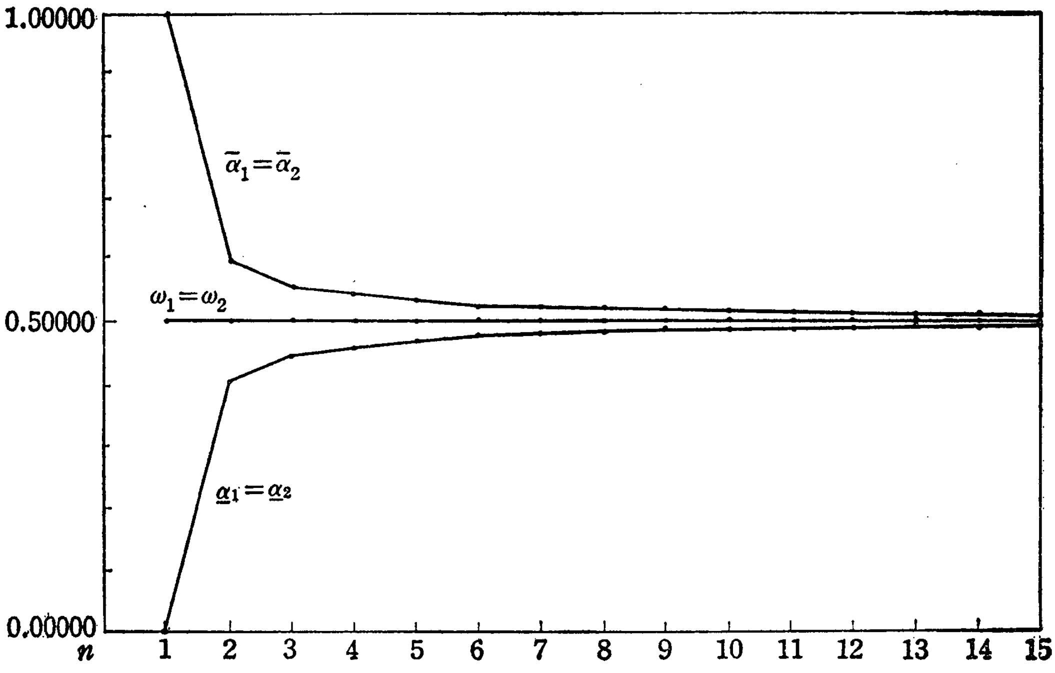

Fig. 11.1

Case 2. (km, kn) market: Here we set $m = 1$, $n = 2$, then $\gamma = 1/3$, $\delta = 2/3$. The endpoints of the core and the competitive imputations are given as table 11.2. The shrinking process of the core is shown as fig. 11.2.

TABLE 11.2

k	$\underline{\alpha}_1$	ω_1	$\bar{\alpha}_1$	$\underline{\alpha}_2$	ω_2	$\bar{\alpha}_2$
1	0.58579	0.70711	1.41421	0.00000	0.35355	0.41421
2	0.65634	0.70711	0.77854	0.31784	0.35355	0.37894
3	0.67490	0.70711	0.74651	0.33385	0.35355	0.36966
4	0.68351	0.70711	0.73434	0.33994	0.35355	0.36535
5	0.68849	0.70711	0.72792	0.34315	0.35355	0.36286
6	0.69173	0.70711	0.72395	0.34513	0.35355	0.36124
7	0.69401	0.70711	0.72125	0.34648	0.35355	0.36010

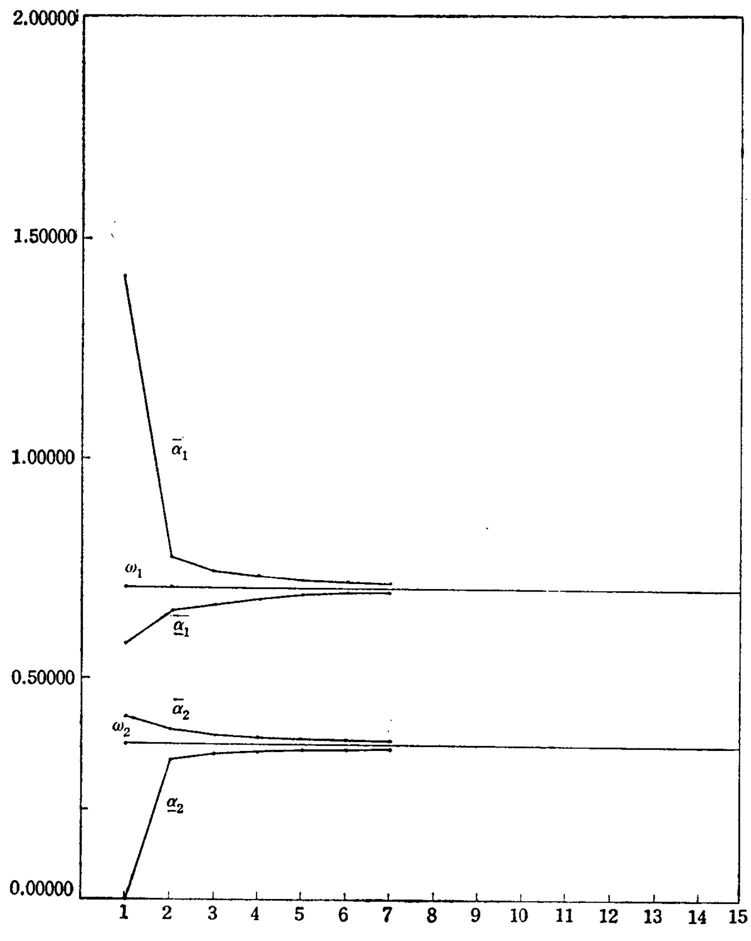

Fig. 11.2

Case 3. (1, *n*) *market.* In this case because of the unboundness of the utility function the competitive imputations do not converge, and so the core diverges as the number of traders increases. The endpoints of the core and the competitive imputations are given as table 11.3. The divergence process of the competitive imputation and the shrinking process of the core are shown as fig. 11.3.

TABLE 11.3

n	$\underline{\alpha}_1$	ω_1	$\bar{\alpha}_1$	$\underline{\alpha}_2$	ω_2	$\bar{\alpha}_2$
1	0.00000	0.50000	1.00000	0.00000	0.50000	1.00000
2	0.58579	0.70711	1.41421	0.00000	0.35355	0.41421
3	0.77854	0.86602	1.73205	0.00000	0.28868	0.31784
4	0.92821	1.00000	2.00000	0.00000	0.25000	0.26795
5	1.05573	1.11803	2.23607	0.00000	0.22361	0.23607
6	1.16896	1.22474	2.44949	0.00000	0.20412	0.21342
7	1.27192	1.32288	2.64575	0.00000	0.18898	0.19626
8	1.36704	1.41421	2.82843	0.00000	0.17678	0.18267
9	1.45586	1.50000	3.00000	0.00000	0.16667	0.17157
10	1.53952	1.58114	3.16228	0.00000	0.15811	0.16228
11	1.61882	1.65831	3.31662	0.00000	0.15076	0.15435
12	1.69439	1.73205	3.46410	0.00000	0.14434	0.14748
13	1.76672	1.80278	3.60555	0.00000	0.13867	0.14145
14	1.83619	1.87083	3.74165	0.00000	0.13363	0.13610
15	1.90312	1.93649	3.87298	0.00000	0.12910	0.13132

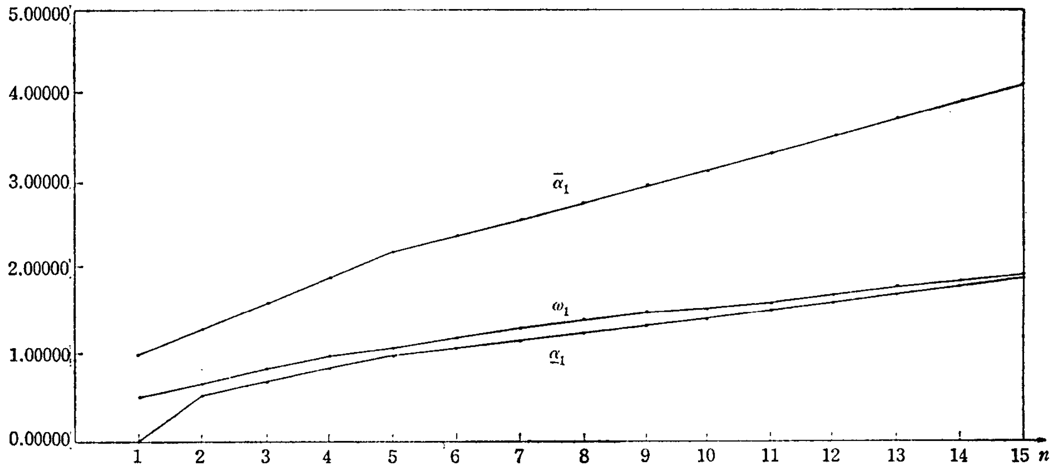

Fig. 11.3–1

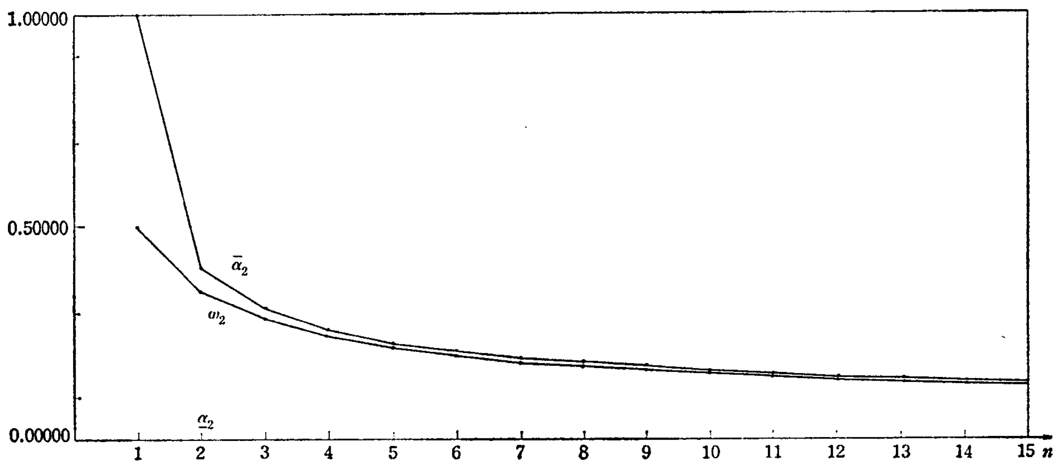

Fig. 11.3–2

Case 4. *(m, n) market*: In this case an (m, n) market where the trader of each type participates reciprocally is considered. The core approaches to the competitive imputation with oscilations. The endpoints of the core and the competitive imputations are given as table 11.4. The shrinking process of the core with vibrations is shown as fig. 11.4.

TABLE 11.4

(m, n)	$m+n$	$\underline{\alpha}_1$	ω_1	$\bar{\alpha}_1$	$\underline{\alpha}_2$	ω_2	$\bar{\alpha}_2$
(1, 1)	2	0.00000	0.50000	1.00000	0.00000	0.50000	1.00000
(2, 1)	3	0.00000	0.35355	0.41421	0.58579	0.70711	1.41421
(2, 2)	4	0.41421	0.50000	0.58579	0.41421	0.50000	0.58579
(3, 2)	5	0.37894	0.40825	0.44949	0.55051	0.61237	0.65634
(3, 3)	6	0.44949	0.50000	0.55051	0.44949	0.50000	0.55051
(4, 3)	7	0.42027	0.43301	0.46416	0.53590	0.57735	0.59434
(4, 4)	8	0.46410	0.50000	0.53590	0.46410	0.50000	0.53590
(5, 4)	9	0.44000	0.44721	0.47213	0.52787	0.55902	0.56803
(5, 5)	10	0.47214	0.50000	0.52786	0.47214	0.50000	0.52786
(6, 5)	11	0.45178	0.45644	0.47722	0.52278	0.54772	0.55331
(6, 6)	12	0.47723	0.50000	0.52277	0.47723	0.50000	0.52277
(7, 6)	13	0.45965	0.46291	0.98074	0.51926	0.54006	0.54386
(7, 7)	14	0.48074	0.50000	0.51926	0.48074	0.50000	0.51926
(8, 7)	15	0.46530	0.46771	0.48331	0.51669	0.53452	0.53727
(8, 8)	16	0.48331	0.50000	0.51669	0.48331	0.50000	0.51669
(9, 8)	17	0.46951	0.47140	0.48528	0.51472	0.53033	0.53241
(9, 9)	18	0.48528	0.50000	0.51472	0.48528	0.50000	0.51472
(10, 9)	19	0.47289	0.47434	0.48683	0.51317	0.52705	0.52866
(10, 10)	20	0.48683	0.50000	0.51317	0.48683	0.50000	0.51317
(11, 10)	21	0.47555	0.47673	0.48808	0.51192	0.52440	0.52570
(11, 11)	22	0.48809	0.50000	0.51191	0.48809	0.50000	0.51191
(12, 11)	23	0.47774	0.47871	0.48912	0.51088	0.52223	0.52330
(12, 12)	24	0.48912	0.50000	0.51088	0.48912	0.50000	0.51088
(13, 12)	25	0.47958	0.48038	0.48999	0.51001	0.52042	0.52129
(13, 13)	26	0.49000	0.50000	0.51000	0.49000	0.50000	0.51000
(14, 13)	27	0.48111	0.48181	0.49074	0.50926	0.51887	0.51963
(14, 14)	28	0.49074	0.50000	0.50926	0.49074	0.50000	0.50926
(15, 14)	29	0.48244	0.48305	0.49137	0.50863	0.51755	0.51820
(15, 15)	30	0.49138	0.50000	0.50862	0.49138	0.50000	0.50862

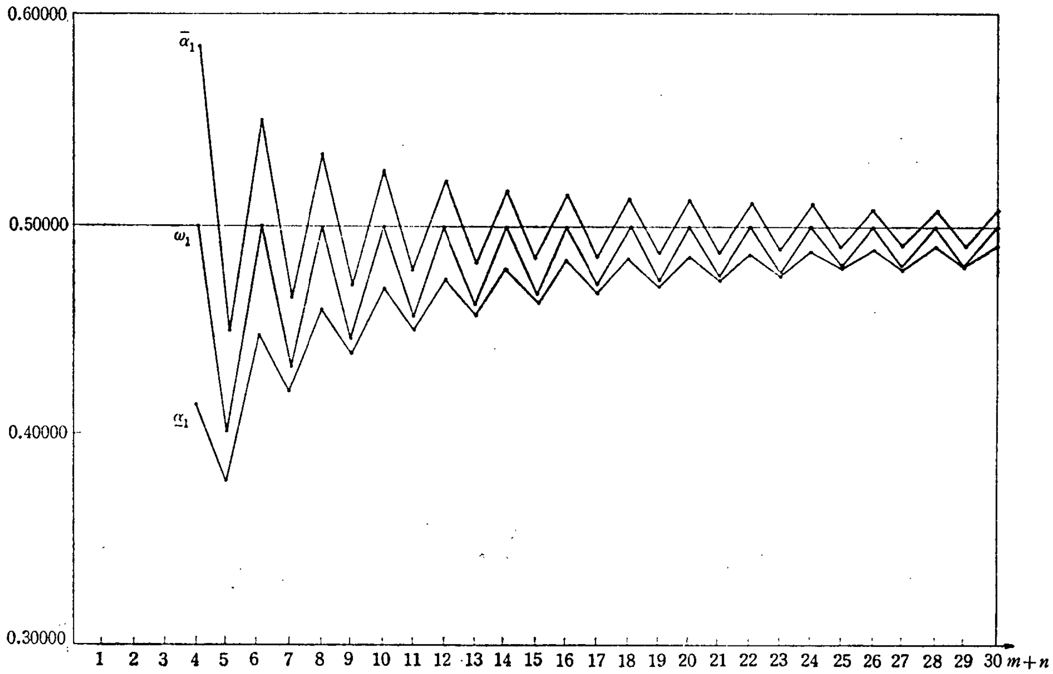

Fig. 11.4

12. CONCLUDING REMARKS

In this paper an analysis of the occurrence of competitive equilibrium of an exchange economy with money was presented. So far, this theme was attacked as a limit theorem on the core of an economy without considering the shrinking process of the core. Heien, this problem from the viewpoint of the relation between economic conflict and the size of an economy was stressed. This interpretation implies that the shrinking process of the core means a decreasing process of economic conflict. Therefore, in the large market competition plays a privileged role by giving determinate optimal allocation.

Approaches to this problem by some economists such as Debreu and Scarf or Shaply and Shubik are unrealistic concerning the traders' participations in the market. In the market of Debreu and Scarf all types of traders participate at the same time, and so the relative composition of the market is fixed $1/l$ for all types (l is the number of types).

In the (km, kn) market of Shapley and Shubik; the market grows two times, three times as k increases, and so the relative composition of the market is fixed at $m/(m + n)$ and $n/(m + n)$ for first and second types, respectively.

On the other hand, in our (m, n) market, traders participate individually, and so the relative composition of the market varies as a trader participates in the market.

Since the core in the explicit form is pesented here, that is, the upper and lower bounds of the core are desribed, the shrinking process of the core as the result of the participation of a trader can be considered. It was shown not in a limited market but in a large market, the core was almost equal to the competitive imputation. Through examples of the caluculation experiments of the shrinking process of the core, the above proposition was verified. The speed of convergence seemes to be exponential considering the number of the combinations of coalitions.

The price system with competitive market requires that participants in the economy possess relatively little information. They need only worry about their own needs and desires. On the other hand the contracting and recontracting market, that is, the market without the competitive price system requires that participants in the economy must possess full information on the outcome of the formation of all coalitions.

It is unclear that a market with full information implies a market with little information as the number of traders becomes infinite; but, from the shrinking process of the core in the finite large market, the effect of information on coalitions for prices become pronounced as the market grows. In other words competitive price systems are efficient in the sence that the amount of required information in the economy is just equal to the number of resources.

At last the results of the calculation experiments must be mentioned. The speed of the convergence is unexpectedly fast in the all examples. This implies that the price system plays a priviledged role in finite large markets giving determinate optimal allocation.

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