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ON THE OCCURRENCE AND THE EXISTENCE OF COMPETITIVE EQUILIBRIA

HISAKAZU NISHINO

INTRODUCTION

Mathematical Theory of Economic Equilibrium:

In his *Recherches sur les principes mathématiques de la théorie des richesses*, a pioneering work in mathematical economics, A. Cournot has duly recognized that, for a complete and rigorous solution of problems relative to some parts of the economic system, it is indispensable to take the entire system into consideration; he believed, however, that this would surpass the powers of mathematical analysis as well as the practical methods of calculation in his days.

It was L. Walras [23] who first succeeded in an attempt to elucidate the mechanism of how the interactions of economic elements are related to the working of a whole economic system. In his formulation, an economy consists of finitely many kinds of commodities and finitely many economic agents, each of which is thought of as a maximizer of utility or of profit; conflicting interests of the agents are adjusted through the competitive market mechanism, where the term "competitive" means that every individual agent has only a negligible influence on the determination of prices of commodities in the market, so that he is forced to regard them as just a datum. Therefore, once a set of prices has been given in the system, the total amounts of demands and supplies are determined through the maximizing behaviors of the agents, that is, excess demands in the market are represented in the form of a vector valued function of prices. Walras has called such a system in equilibrium when excess demands vanish simultaneously.

In order to guarantee the effectiveness of the working of a Walrasian economy, we have to establish *the existence and the social optimality of an equilibrium.* The first studies of the above two problems have been made by Walras himself and V. Pareto.

Neither of these men, however, has been able to achieve a complete solution to the problem of the existence of equilibrium. Walras has asserted that his system of simultaneous equations to determine a set of equilibrium prices contains a dependence relation which is now called "the Walras law". He has taken, at the same time, notice of the zero'th homogeneity of the excess demand function. Hence, the price of abritrary one of the commodities can be fixed as a standard of value. He, then, has concluded that his economic model has an equilibrium since the model is described by a system of equations number of which equals that of unknowns to be determined. But, it is clear that the existence of an economically

meaningful solution cannot be ensured merely by an equality between the number of equations and that of unknowns.

The first rigorous solution of the problem has been shown in the paper of A. Wald [22]. He has proved analytically an existence theorem of equilibrium for a Casselian system, which is a special model of Walrasian economies, with several restrictive assumptions on consumers' demands. J. von Neumann [20] has published, at almost the same time, a paper on an economic growth system,in which he has demonstrated, by means of Brouwer's fixed point theorem, the existence of a balanced growth solution. By extending Brouwer's result to the case of upper-semicontinuous point-to-set mapping, S. Kakutani [9] has simplified the proof of the Neumann's theorem. While the Neumann model itself is not so closedly related to our problem, Kakutani's fixed point theorem has played an essential role in improving the proof of the existence theorem of equilibrium. As a matter of fact, K. J. Arrow and G. Debreu [1], L. W. McKenzie [12], D. Gale [7], H. Nikaido [13], H. W. Kuhn [11] and Debreu [4] have developed much more general approaches to the problem than Wald did, by making skillful use of the Kakutani's theorem. Thus, the existence problem of equilibrium for a static Walrasian economy was settled at the end of the decade of 1950.

As for the problem of the optimality, Pareto [16] has suggested that an equilibrium in a Walrasian economy achieves social efficiency in a relative sense, which is called "the Pareto optimality". That is, in a equilibrium it is impossible to raise the utility level of an agent without lowering that of at least one other agent.

In his work [6], F. Y. Edgeworth first developed, in different contexts, a socio-economic approach to equilibrium in which the optimality problem is practically involved. He has observed that a set of actually available allocations of commodities in an economy without a price adjustment mechanism would be reduced to a set of equilibria as the number of the agents increases. Recently, his presumption has been taken up by Debreu and H. Scarf [5] and others in which the game theoretical concept of the core was used in order to represent the set of attainable allocations. It seems, unfortunately, that it is not completely satisfactory as the problem has been merely solved in a quite artificial way.

SUMMARY OF THE THESIS

The present thesis deals with the preceding two problems from a topological viewpoint. We first set up an introductory chapter which aims at formulating the scheme of Walrasian economy in terms of its convexity, its continuity and some of its set-theoretical concepts. Our intention in Chapter 2 is to discuss Edgewroth's presumption in a natural scheme. It is shown incidentally that certain uniform conditions on consumers' preferences and initial resources are needed in order to solve the problem affirmatively. We take up, in Chapter 3, the existence problem of equilibrium. Mathematical economists have, so far, devoted mostly their studies of the problem to analyzing static or balanced growth systems.

In this chapter, however, a generalized dynamic system involving savings, investments and the rate of interest is investigated, in which each producer enlarges (or reduces) the activity level according to productive power evaluated by the current market prices. It is the final goal of the discussion to propose an existence theorem of temporary equilibrium for such an extended system. Finally in Chapter 4, we briefly suggest the open or future problems of the theory presented in this thesis.

Several supplementary propositions which are used in the text are collected in Appendix.

CONTENTS

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Chapter 1

MATHEMATICAL PRESENTATION OF ECONOMIC SYSTEM

1. *An exchange economy and a competitive economy.*

An *economy* consists of, in general, n kinds of *commodities*, m *producers* each of which has a well-defined *production set* $Y_j; j = 1, 2, \ldots, m$, of n-dimensional Euclidean space R^n, and l *consumers* each of which has an *initial holding* of commodities w^i; $i = 1, 2, \ldots, l$, and has a well-defined *preference ordering* $\succsim_i$, on his *preference field* X_i; $i = 1, 2, \ldots, l$, of a set of non-negative vectors of R^n, Ω^n, where n, m and l are arbitary positive integers.

An economy which does not have the price adjustment mechanism is called an *exchange economy*. On the other hand, an economy with the price mechanism is called a *competitive economy* if every individual agent of the economy acts on the market as a price-taker.

In an excahnge economy, all of the initial resources and the *produced resources* $y^j \in Y_j$; $j = 1, 2, \ldots, m$, are redistributed among all the consumers through the barter of commodities, where the k-th commodity is thought of as an *output* (or as an *input*) of the production y^j if y^j_k is positive (or negative). It is assumed that, in the redistribution, the i-th consumer has a right to get a *share* β_{ji}, of y^j, from the j-th producer, where $\beta_{ji} \geqq 0$; $i = 1, 2, \ldots, l$, and $j = 1, 2, \ldots, m$.

$$\sum_{i=1}^{l} \beta_{ji} = 1; \quad j = 1, 2, \ldots, m .$$

By an *allocation*, we mean a collection of all the consumers' *final resources* $\{x^1, x^2, \ldots, x^l\}$; $x^i \in X_i$, $i = 1, 2, \ldots, l$, obtained by the redistribution. We shall abbreviate, for convenience, $\{x^1, x^2, \ldots, x^l\}$ to $\{x^i\}$. Necessarily, every allocation $\{x^i\}$ fulfills

$$\sum_{i=1}^{l} x^i = \sum_{i=1}^{l} w^i + \sum_{j=1}^{m} y^j .$$

An allocation $\{x^i\}$ is said to be *blocked by a coalition* S, of consumers, if it is possible for some y'^j in $Y_j; j = 1, 2, \ldots, m$, to reallocate the amount of resources

$$\sum_{i \in S} \left(\sum_{j=1}^{m} \beta_{ji} y'^j + w^i \right),$$

within S, under the condition that the new final resource is either preferable or indifferent to x^i for every i in S and is strictly preferable to x^i for at least one i in S. A set of allocations which cannot be blocked by any non-empty coalition S is called the *core* of an exchange economy. The core stands for, from an economical viewpoint, a set of unanimously attainable allocations. An allocation is said to be *Pareto optimal* if it is not blocked by the coalition composed of all the consumers. Every allocation in the core is Pareto optimal.

We shall call, on the other hand, the *pair* of an allocation and a collection of

the corresponding productions to be *competitive* if there exists a semi-positive vector p, of R^n such that each y^j in the collection maximizes the inner-product $\langle p, y\rangle$, in Y_j; $j = 1, 2, \ldots, m$, and each x^i in the allocation is a maximum element of

$$\left\{x \in X_i; \langle p, x\rangle \leqq \left\langle p, \sum_{j=1}^{m} \beta_{ji} y^j + w^i \right\rangle\right\},$$

for his preference ordering. We say that $\{x^i\}$ is a *competitive allocation* of an exchange economy, when it forms, together with the collection of the corresponding productions $\{y^j\}$, a competitive pair. Let us interpret the semi-positive vector p annexed with a competitive pair as a set of prices of commodities, then the inner-product $\langle p, y^j\rangle$ can be thought of as the j-th producer's *profit* π_j from the production y^j, since the right side of the identity

$$\langle p, y^j\rangle = \sum_{k=1}^{n} p_k \max(y_k^j, 0) - \sum_{k=1}^{n} p_k \max(-y_k^j, 0)$$

has an economical meaning of "revenues minus costs". Therefore, $\beta_{ji}\langle p, y^j\rangle$ stands for the i-th consumer's income from the distribution of the j-th producer's profit, and his *total income* amounts to

$$\left\langle p, \sum_{j=1}^{m} \beta_{ji} y^j + w^i \right\rangle.$$

Now, the preceding two conditions on the pair to be competitive can be restated by saying that y^j maximizes the j-th producer's profit in Y_j, and x^i maximizes the i-th consumer's utility level under the *budget constraint.*

Once a set of prices of commodities is determined in the market of a competitive economy, every producer chooses a production in his production set which maximizes his profit evaluated by the price system, and every consumer chooses a consumption which maximizes his utility level under the budget constraint evaluated by the price system. The correspondence from a set of price systems to the set of maximal productions (or consumptions) in the above sense is called a supply function of producer (or a demand function of consumer), and is represented by $y^j(p)$; $j = 1, 2, \ldots, m$ (or $x^i(p)$; $i = 1, 2, \ldots, l$).

It is assumed, similarly to the case of exchange economy, that each consumer gets, in a competitive economy, a share $\beta_{ji} \geqq 0$ of π_j from the j-th producer, of which summation over all the consumers equals unity. This means that each π_j is entirely distributed among the consumers.

An *equilibrium* of a competitive economy is defined on the concept of balance of supplies and demands. To be precise, the triplet of $\hat{p} \geq 0$, $\{x^i(\hat{p})\}$ and $\{y^j(\hat{p})\}$ is called a *competitive equilibrium* of the economy if $\{x^i(\hat{p})\}$ and $\{y^j(\hat{p})\}$ form a competitive pair under the price system p.

2. *Economic agents.*

Preference field X_i, of the consumer i is a closed convex region of a non-negative orthant of R^n, Ω^n, such that $x \in X_i$ and $x' \geqq x$ imply $x' \in X_i$. A complete pre-

ordering $\succsim_i$, as a representative of his preference relation is defined on X_i. By $x \succsim_i x'$, we mean that x is preferable or indifferent to x'. We write $x \succ_i x'$ instead of $x \succsim_i x'$ with $x' \not\succsim_i x$, when x is strictly prefered to x'.

It is generally assumed in the mathematical theory of economic equilibrium that each $\succsim_i$ satisfies the followings.

Assumption C-1. Continuity. For every x in X_i, $\{x'; x' \succsim_i x\}$ and $\{x'; x \succsim_i x'\}$ are closed sets of R^n.

Assumption C-2. Convexity. If x and x' are in X_i, then $x \succ_i x'$ implies $\alpha x + (1 - \alpha)x' \succ_i x'$ for any α in the interval (0, 1).

Assumption C-3. Desirability. For every x in X_i, $x' \geq x$ implies $x' \succ_i x$, where $x' \geq x$ denotes $x' \geqq x$ with $x' \neq x$.

The continuity assumption ensures the existence of a continuous utility function $u_i(x)$, defined on X_i such that $x \succsim_i x'$ implies $u_i(x) \geqq u_i(x')$. The convexity assumption is a topological expression of the classical law of diminishing marginal utilities. Economic implication of the desirability assumption would be obvious.

On the other hand, production set Y_j of the producer j is a set of technically possible production plans, that is, it is technically possible to produce y_j if and only if y_j is an element of Y_j.

Each Y_j fulfills, in general, the followings.

Assumption P-1. Y_j is a closed convex region of R^n.

Assumption P-2. Y_j contains the origin of R^n.

The law of diminishing returns is included in the first assumption. The closedness of Y_j implies a certain continuity of the region. The second assumption means that each producer has the possibility of inaction.

As is shown in the subsequent chapters, those restrictions on economic agents are indispensable in order to solve our problems.

Now, we can prove the following

Theorem 1. A competitive allocation $\{x^i\}$ of an exchange economy in which every consumer satisfies *C-3* is an element of the core.

Proof. Assumption *C-3* ensures that

$$\langle p, x^i \rangle = \left\langle p, \sum_{j=1}^{m} \beta_{ji} y_j + w^i \right\rangle$$

holds for every i and for some $p \geq 0$, where $y^j; j = 1, 2, \ldots, m$ denote the corresponding productions.

Let us suppose that S is a blocking coalition for $\{x^i\}$, then there must exist a new allocation $\{x'^i\}$; $i \in S$, within S such that

$$\left\langle p, \sum_{i \in S} x'^i \right\rangle > \left\langle p, \sum_{i \in S} x^i \right\rangle = \left\langle p, \sum_{i \in S} \left(\sum_{j=1}^{m} \beta_{ji} y^j + w^i \right) \right\rangle$$

holds for the above p. Since $\langle p, y^j \rangle \geqq \langle p, y'^j \rangle$ for any y'^j in Y_j; $j = 1, 2, \ldots, m$, we have eventually

$$\left\langle p, \sum_{i \in S} x'^i \right\rangle > \left\langle p, \sum_{i \in S} \left(\sum_{j=1}^{m} \beta_{ji} y'^j + w^i \right) \right\rangle$$

for any y'^j in Y_j; $j = 1, 2, \ldots, m$. This contradicts, however, the condition that $\{x'^i\}$; $i \in S$, must fulfill for some y'^j in Y_j; $j = 1, 2, \ldots, m$,

$$\sum_{i \in S} x'^i = \sum_{i \in S} \left(\sum_{j=1}^{m} \beta_{ji} y'^j + w^i \right).$$

Q.E.D.

The preceding assumptions *C–1* to *P–2* lead to the upper-semicontinuities of $y^j(p)$; $j = 1, 2, \ldots, m$, and $x^i(p)$; $i = 1, 2, \ldots, l$, so that, by applying the Kakutani's theorem, we can establish the existence of competitive equilibrium under some additional assumptions.

Theorem 2. Let a competitive economy satisfy, in addition to *C–1* to *P–2*, the followings, then there exists a competitive equilibrium.

Assumption C–4. There is an $\overset{i}{x}$ in X_i such that $\overset{i}{x} < w^i$; $i = 1, 2, \ldots, l$.

Assumption P–3. Let Y be defined by

$$Y = \left\{ y; y = \sum_{j=1}^{m} y^j, y^j \in Y_j, \ j = 1, 2, \ldots, m \right\},$$

then $Y \cap (-Y) = \{0\}$ and $-Y \supset \Omega^n$ hold, where $-Y = \{y; -y \in Y\}$.

Assumption *C–4* asserts that every consumer's initial resource is located in the interior of his preference field. Assumption *P–3* means the irreversibility and the dissipatability of social production processes. This includes the fact that it is impossible to produce commodities without inputs in an economy.

An elegant proof of Theorem 2 has been given by Debreu ([4], p. 83), in which *C–3* was replaced with a considerably weakened

Assumption C–3′. Insatiability. For every x in X_i, there is an x' in X_i such that $x' \succ_i x$.

It should be noted that *C–2* and *C–3′* imply, for every consumer, the budget equality between $\langle p, x^i \rangle$ and his total income whenever $\{x^i\}$ is competitive. We immediately have, therefore, the following variation of Theorem 1.

Theorem 1′. A competitive allocation $\{x^i\}$ of an exchange economy in which every consumer satisfies *C–2* and *C–3′* is an element of the core.

Chapter 2

THE CORE AND COMPETITIVE ALLOCATIONS OF AN ECONOMY WITH FINITELY MANY CONSTITUENTS

1. *Introduction.*

In this chapter, discussion which may partially contribute to the explanation of the mechanism of how a competitive allocation can be realized in an exchange economy is presented.

As is generally known, the origin of the problem is found in [6] by Edgeworth.

He has confined his argument to a pure exchange economy consisting of two kinds of commodities and two groups of consumers each of which is composed of r of them having the identical preference ordering and the identical amount of their initial resources, and has shown geometrically that a set of acutually attainable allocations, viz. the core, is reduced to the set of competitive ones as *r becomes infinite.*

Recently in their paper [5], Debreu and Scarf have taken a topological approach to the problem based on the mathematical notion of the core, and have proved that the result by Edgeworth can be applied to an exchange economy with n kinds of commodities, a social production set Y being a closed convex cone of R^n with a vertex at the origin, and l groups of consumers each of which is composed of r identical consumers, in which each representative consumer has a strictly positive amount of his initial resource for every commodity and has a complete pre-ordering $\succsim_i$ on Ω^n satisfying *C–1*, *C–3′* and the following

Assumption C–2′. Strong-convexity. If x and x' are points of Ω^n, then $x \succsim_i x'$ and $x \neq x'$ imply $\alpha x + (1 - \alpha)x' \succ_i x'$ for every α in (0, 1).

The above way in which the number of consumers lr, is going to infinity might be quite artificial, and then naturally the question will arise as to whether or not Debreu and Scarf's model can be generalized to allow the differences in their patterns of preferences and in their initial resources for all the consumers.

Aumann [3] and Vind [18] have analyzed the pure exchange model with n kinds of commodities and a *continuum of consumers* in which the assumptions on the patterns of their preferences and of their initial holdings are quite generalized with several reasonable conditions on the measurability. However, it seems to us the assumption that an economy consists of finitely many constituents will be much more meaningful than in the case of the continuum.

In [19], Vind has also analyzed an economy which consists of n kinds of commodities and l consumers who may or may not have different preferences and different initial resources one against one another and whose preferences satisfy *C–1*, *C–2′* and *C–3′*. His theorem states that, for every allocation $\{x^i\}$ in the core and for every positive number e, there exist a semi-positive vector p of R^n and a positive integer M such that the number of consumers each of whose final resources fulfills the condition

$$\{x; x \succ_i x^i\} \cap \{x; \langle p, x - w^i \rangle \geqq -e\} \neq \phi$$

never exceeds M. If M does not diverge as l tends to infinity, this statement is equivalent to saying that every allocation in the core assigns approximately competitive final resources for *almost all* of consumers as l becomes sufficiently large. Unfortunately, however, it seems that either the above-imposed three restrictions on consumer's preference or any slight modification of those restrictions do not necessarily certify the uniform boundedness of M.

Whereas, it is the intention of the present investigation to show that, in an exchange economy with finitely many consumers and producers, every allocation

in the core assigns approximately competitive final resources for *all* of consumers as the number of them becomes sufficiently large, if a certain uniformity on their preferences is imposed together with several ordinary assumptions involving C–1 to P–2. It will be also shown that uniformity is indispensable for our purpose.

2. *The model.*

We first presuppose an infinite set I, of countably many consumers, and take a countable collection $\{I_1, I_2, \ldots, I_\nu, \ldots\}$ of consumers' sets in $\mathscr{F}$, such that $\nu > \nu'$ implies $l_\nu > l_{\nu'}$ (the monotonicity $I_\nu \supset I_{\nu'}$ is not necessarily required), where $\mathscr{F}$ stands for the family of all finite subsets of I and each l_ν denotes a cardinal number of I_ν in $\mathscr{F}$. The economies $\mathscr{E}_\nu$; $\nu = 1, 2, \ldots$, to be considered consist of n kinds of commodities, m producers and consumers sets I_ν; $\nu = 1, 2, \ldots$.

Each consumer in I satisfies the preceding *C–1*, *C–2*, *C–3* and the followings.

Assumption C–4′. Each consumer's initial resource w^i belongs to X_i and fulfills $w^i \leqq h$ for some h in $\bigcap_{i \in I} X_i$, where $\bigcap_{i \in I} X_i$ is nonempty.

The latter half of *C–4′* implies that a collection $\{w^1, w^2, \ldots\}$ of the initial resources has an upper bound for the ordinary vector ordering $\geqq$.

By making use of *the separation theorem* (cf. Appendix), we have directly from *C–1*, *C–2* and *C–3* that, for every i in I and for every x' in X_i, $\{x;\ x \succsim_i x'\}$ has a supporting hyperplane

$$\langle p^i(x'), x - x' \rangle = 0; \quad x \in R^n$$

with the semi-positive normal $p^i(x')$ of length 1, such that $x \succsim_i x'$ implies $\langle p^i(x'), x - x' \rangle \geqq 0$. We assume on the normal the followings.

Assumption C–5. Unform smoothness. For any $\gamma > 0$, there is a $\beta > 0$ independent of i in I and of x in X_i such that, for every z in R^n, $|z| \leqq 1$ and $\langle p^i(x), z \rangle \geqq \gamma$ imply $x + \beta z \in X_i$ and $x + \beta z \succ_i x$, where $|z|$ denotes the absolute value (or length) of z.

Assumption C–6. If x and x' are in X_i, then $x \succ_i x'$ implies the existence of an $x'' \in X_i$ such that $\langle p^i(x), x'' - x \rangle < 0$.

It follows from *C–5* and *C–6* that $x \to p^i(x)$ is a continuous point-to-point mapping of X_i into P, where $P = \{p > 0;\ |p| = 1\}$ (cf. Appendix and Lemma 1). Hence, for any g in $\bigcap_{i \in I} X_i$, there exists a positive number e_i such that, if x is a point of X_i, then $x \leqq g$ implies $p^i_k(x) \geqq e_i$; $k = 1, 2, \ldots, n$. We impose on those numbers the following.

Assumption C–7. Let e be defined by $e = \inf_{i \in I} e_i$, then, for every g in $\bigcap_{i \in I} X_i$, $e > 0$.

Each producer satisfies the preceding *P–1* and *P–2*. We assume, furthermore, on the total production set Y the following.

Assumption P–3′. There is a positive number K such that $y \in Y$ implies

$$\sum_{k=1}^{n} \max(y_k, 0) - K \sum_{k=1}^{n} \max(-y_k, 0) \leqq 0 .$$

It is assumed, for simplicity, that each consumer i in I_ν gets, in $\mathcal{E}\nu$, a share $\beta\nu_i$, of a total production y, that is $\beta\nu_i = \beta\nu_{1i} = \beta\nu_{2i} = \ldots = \beta\nu_{mi}$; $i = 1, 2, \ldots, l$.

Assumption M-1. There is an $L > 0$ such that, for every i in I_ν, $\beta\nu_i \leqq L/l_\nu$ holds independent of ν.

3. *The convergence theorem.*

First, the preceding definitions of the core and a competitive allocation are slightly modified.

The core $C\nu$ of $\mathcal{E}\nu$ is a set of allocations which have no blocking coalition. If a set of consumers S, fulfills the following conditions for some y' in Y and x'^i; $i \in S$, in X_i, S is called a blocking coalition for an allocation $\{x^{\nu 1}, x^{\nu 2}, \ldots, x^{\nu l}\}$. That is to say,

i) $S \subset I_\nu$ and $S \neq \phi$,

ii) $\sum_{i \in S} x'^i = \sum_{i \in S} (\beta\nu_i y' + w^i)$,

iii) $x'^i \gtrsim_i x^{\nu i}$ for all $i \in S$ and $x'^i \succ_i x^{\nu i}$ for at least one $i \in S$.

A pair of an allocation $\{x^{\nu i}\}$ and the corresponding production $y \in Y$ is called a competitive pair if it has a $p \geq 0$ such that

i) $y' \in Y$ implies $\langle p, y - y' \rangle \geqq 0$,

ii) each $x^{\nu i}$ satisfies $\langle p, x^{\nu i} - w^i - \beta\nu_i y \rangle = 0$,

iii) $x \succ_i x^{\nu i}$ implies $\langle p, x - x^{\nu i} \rangle > 0$ for every i in I_ν.

We say that $\{x^{\nu i}\}$ is a competitive allocation of $\mathcal{E}\nu$ if we can find a total production $y(x^\nu)$ in Y which forms a competitive pair togather with the collection of con-

consumers' final resources.

An allocation $\{x^{\nu i}\}$ is said to be *ε-competitive* if all the conditions to be competitive are preserved excepting that each $x^{\nu i}$ satisfies

$$-\varepsilon \leqq \langle p, x^{\nu i} - w^i - \beta\nu_i y(x^\nu) \rangle \leqq \varepsilon$$

instead of the equality $\langle p, x^{\nu i} - w^i - \beta\nu_i y(x^\nu) \rangle = 0$.

Specifically, the final goal of the present discussion is to show that the core $C\nu$ is reduced to a set of competitive allocations under the above-mentioned assumptions as ν tends to infinity. Since the proposition that a competitive allocation is always included in the core has been proved for general cases in Theorem 1 or in Theorem 1′, it is our concern to demonstrate that, for any $\varepsilon > 0$, there is a ν_0 such that the core $C\nu$ is composed of ε-competitive allocations whenever ν exceeds ν_0.

Lemma 1. For every i in I and for every x in X_i, $p^i(x)$ is uniquely determined and is a point of P such that $x' \succ_i x$ implies $\langle p^i(x), x' - x \rangle > 0$.

Proof. Let, for some i in I and x in X_i, a semi-positive $q(\neq p^i(x))$ be also a

normal of length 1 of the supporting hyperplane of $\{z; x + z \succsim_i x\}$ passing through 0 such that $x + z \succsim_i x$ implies $\langle q, z\rangle \geqq 0$. Then simultaneous inequalities

$$\langle p^i(x), z\rangle > 0 \text{ and } \langle q, z\rangle < 0$$

have a non-empty set D, of the solutions. By applying *C–5* to the first inequality, we can find a positive $\beta(z)$ for every z in D such that $x + \beta(z) \in X_i$ and $x + \beta(z) \succ_i x$. On the other hand, $x + \beta(z) \succ_i x$ implies, by the definition of q, $\langle q, \beta(z)z\rangle \geqq 0$. This contradicts the second inequality $\langle q, z\rangle < 0$, and then $p^i(x)$ is uniquely determined for every i in I and for every x in X_i.

Next, we show that $x' \succ_i x$ implies $\langle p^i(x), x' - x\rangle > 0$. It follows directly from the definition of $p^i(x)$ that $x' \succ_i x$ implies $\langle p^i(x), x' - x\rangle \geqq 0$. Suppose that, for some i in I and x in X_i, there is an x' in X_i which is preferable to x and satisfies $\langle p^i(x), x' - x\rangle = 0$.

If $p^i(x) \neq p^i(x')$, then there must exist a z of R^n satisfying both $\langle p^i(x), z\rangle < 0$ and $\langle p^i(x'), z\rangle > 0$. Hence by quite the same argument with the above, we have $x' + \beta z \in X_i$ and $x' + \beta z \succ_i x'$ for some $\beta > 0$. Since $\succ_i$ is transitive, we ultimately see $x' + \beta z \succ_i x$. But, the inequality

$$\langle p^i(x), x' + \beta z - x\rangle = \langle p^i(x), \beta z\rangle < 0$$

contradicts the fact that $x' + \beta z \succ_i x$ implies $\langle p^i(x), x' + \beta z - x\rangle \geqq 0$.

In the case of $p^i(x) = p^i(x')$, the above equality is rewritten with $\langle p^i(x), x\rangle = \langle p^i(x_i), x'\rangle$. Since $x' \succ_i x$, it follows from *C–6* that there is an x'' in X_i such that $\langle p^i(x'), x'' - x'\rangle < 0$, and this leads to a fact that $\langle p^i(x), x'' - x\rangle < 0$. Let $x(\alpha)$ be defined by

$$x(\alpha) = \alpha x'' + (1 - \alpha)\, x' \;;\quad \alpha \in (0, 1)\,,$$

then we have, by *C–1*, $x(\alpha) \succ_i x$ for every sufficiently small $\alpha > 0$. This implies the non-negativity of $\langle p^i(x), x(\alpha) - x\rangle$, which is contradictory to

$$\langle p^i(x), x(\alpha) - x\rangle = \langle p(x), \alpha(x'' - x) + (1-\alpha)(x' - x)\rangle = \alpha\langle p^i(x), x'' - x\rangle < 0.$$

Finally, let us assume that, for some i in I and x in X_i, $p^i(x)$ has a non-positive component, say k, and let us take an x' in X_i such that $x'_k > x_k$ and $x'_{k'} = x_{k'}$ for every $k' \neq k$, then we have both $x' \succ_i x$ and $\langle p^i(x), x' - x\rangle \leqq 0$. The relation $x' \succ_i x$ is, however, incompatible with $\langle p^i(x), x' - x\rangle \leqq 0$.

Q.E.D.

Lemma 2. Every allocation $\{x^{\nu i}\}$ in $C\nu$ has a $p(x^\nu)$ in P such that the corresponding production $y(x^\nu)$ in Y satisfies $\langle p(x^\nu), y(x^\nu) - y\rangle \geqq 0$ for any y in Y and $p(x^\nu) = p^i(x^{\nu i})$ holds for any i in I_ν.

Proof. Let us first suppose that there are consumers i' and i'' in I_ν for which $p^{i'}(x^{\nu i'}) \neq p^{i''}(x^{\nu i''})$, then we can show that a convex hull H, of $Z_{i'}(x^{\nu i'})^\circ \cup Z_{i''}(x^{\nu i'})^\circ$ contains 0, where $Z_i(x)^\circ = \{z; x + z \succ_i x\}$. If $0 \notin H$, indeed, H can be separated by a hyperplane $\langle p, z\rangle = 0$ with the normal $p > 0$ such that $z \in H$ implies $\langle p, z\rangle > 0$. So, either $x^{\nu i'} + z \succsim_{i'} x^{\nu i'}$ or $x^{\nu i''} + z \succsim_{i''} x^{\nu i''}$

implies, by *C–1*, $\langle p, z\rangle \geqq 0$. Considering the uniqueness of each $p^i(x)$, we have $p^{i'}(x^{\nu i'}) = p^{i''}(x^{\nu i''})$. Hence, $0 \in H$. That is to say, there are a z' in $Z_{i'}(x^{\nu i'})^\circ$, z'' in $Z_{i''}(x^{\nu i''})^\circ$ and an α in [0, 1] such that

$$0 = \alpha z' + (1 - \alpha)z'' .$$

Let a collection $\{x'^i\}$ of all the consumers' final resources be defined by

$$x'^{i'} = \alpha z' + x^{\nu i'},\ x'^{i''} = (1 - \alpha)z'' + x^{\nu i''} \text{ and } x'^i = x^{\nu i} \text{ for all } i \notin \{i', i''\}.$$

Then, $\{x'^i\}$ is an allocation of $\mathscr{E}\nu$, in which either $x'^{i'} \succ_{i'} x^{\nu i'}$ or $x'^{i''} \succ_{i''} x^{\nu i''}$ holds (cf. *C–2*). This contradicts the Pareto optimality of $\{x^{\nu i}\}$.

Let $p(x^\nu)$ be put equal to an arbitrary consumer's $p^i(x^{\nu i})$, then we obtain $p(x^\nu) = p^i(x^{\nu i})$ for any i in I_ν and, by a result of Lemma 1, $p(x^\nu) \in P$.

Now, suppose that a y' in Y exists such that $\langle p(x^\nu), y(x^\nu) - y'\rangle < 0$, then, by letting $y(\alpha)$ be a convex combination $\alpha y' + (1 - \alpha)y(x^\nu)$, of y' and $y(x^\nu)$, we see $\langle p(x^\nu), y(x^\nu) - y(\alpha)\rangle < 0$ for every α in (0, 1]. Hence, we can find, by using *C–5*, an α' in (0, 1] such that $x^{\nu i} + (y(\alpha') - y(x^\nu))/l_\nu \in X_i$ and $x^{\nu i} + (y(\alpha') - y(x^\nu))/l_\nu \succ_i x^{\nu i}$ hold for every i in I_ν. On the other hand,

$$\sum_{i\in I_\nu}\{x^{\nu i} + (y(\alpha') - y(x^\nu))/l_\nu\} = \sum_{i\in I_\nu} w^i + y(x^\nu) + (y(\alpha') - y(x^\nu)) = \sum_{i\in I_\nu} w^i + y(\alpha')$$

can be confirmed through easy calculation. Since *P–1* ensures $y(\alpha') \in Y$, we have a blocking coalition I_ν for $\{x^{\nu i}\}$.

Q.E.D.

Lemma 3. Let $P(e)$ be defined by $P(e) = \{p \in P; p_k \geqq e, k = 1, 2, \ldots, n\}$ for an $e > 0$, then there exists an $\bar{e} > 0$ independent of ν such that $\{x^{\nu i}\} \in C\nu$ implies $p(x^\nu) \in P(\bar{e})$, where $p(x^\nu)$ is a vector in P the existence of which has been established in Lemma 2.

Proof. It is clear from *C–4′* and *P–3′* that

$$\sum_{i\in I_\nu} x^{\nu i} = \sum_{i\in I_\nu} w^i + y(x^\nu) \leqq (1 + K)l_\nu h$$

holds for every ν and for every allocation $\{x^{\nu i}\}$ of $\mathscr{E}\nu$. If there are a ν and an allocation $\{x^{\nu i}\}$ such that none of the consumers' final resources satisfies $x^{\nu i} \leqq n(1 + K)h$, then there must be at least one component k in the set of commodities such that

$$\sum_{i\in I_\nu} x_k^{\nu i} > (1 + K)l_\nu h_k .$$

Hence, every allocation $\{x^{\nu i}\}$ of $\mathscr{E}\nu$ has, for any ν, at least one consumer's final resource, say $x^{\nu i'}$ such that $x^{\nu i'} \leqq n(1 + K)h$. By *C–7*, we have $p^{i'}(x^{\nu i'}) \in P(\bar{e})$ for some $\bar{e} > 0$. Now, it is a direct consequence of Lemma 2 that $\{x^{\nu i}\} \in C\nu$ implies $p(x^\nu) \in P(\bar{e})$ independent of ν. Q.E.D.

Lemma 4. For any $r > 0$, there exists a ν_0 such that $\nu \geqq \nu_0$ and $\{x^{\nu i}\} \in C\nu$ imply

$$\max_{i\in I_\nu} \langle p(x^\nu), z^{\nu i}\rangle < r ,$$

where each $z^{\nu i}$ stands for $x^{\nu i} - w^i - \beta_{\nu i} y(x^\nu)$; $i \in I_\nu$.

Proof. We assume, on the contrary, that there exist an $r' > 0$ and an infinite subsequence $\{\mu\}$, of $\{1, 2, \ldots, \nu, \ldots\}$ on which $\langle p(x_\mu), z^{\mu i_\mu}\rangle \geqq r'$ holds for some $\{x^{\mu i}\}$ in C_μ and i_μ in I_μ.

Let a non-positive vector z_μ of R^n be defined by

$$z_k^\mu = z_k^{\mu i_\mu} \quad \text{if} \quad z_k^{\mu i_\mu} \leqq 0, \quad \text{otherwise } z_k^\mu = 0,$$

then, by letting $z(\alpha)$ be a convex combination $\alpha z^\mu + (1-\alpha)z^{\mu i_\mu}$, of z^μ and $z^{\mu i_\mu}$, we can find an α_μ in [0, 1] such that $\langle p(x^\mu), z(\alpha_\mu)\rangle = r'$.

Let $E_{r'}$ be a set of R^n defined by

$$E_{r'} = \{z;\, z \geqq -L(1+K)h, \langle p, z\rangle \leqq r' \text{ for some } p \in P(\bar{e})\},$$

where $\bar{e}$ is a positive number defined in Lemma 3, then it is easy to see that $E_{r'}$ is a compact set of R^n for every $r' > 0$.

The fact that $z^{\mu i} \geqq -L(1+K)h$ holds for every i in I_μ follows from *C–4′*, *P–3′* and *M–1*. Since $p(x^\mu) \in P(\bar{e})$ by Lemma 3, we see $z(\alpha_\mu) \in E_{r'}$ for every μ. If a positive number d denotes

$$\max_{z \in E_{r'}} |z|,$$

then $|z(\alpha_\mu)| \leqq d$ for every μ. Hence it follows from *C–5* that there is a $\beta > 0$ independent of μ such that $x^{\mu i} + \beta z(\alpha_\mu)/d \in X_i$ and $x^{\mu i} + \beta z(\alpha_\mu)/d \succ_i x^{\mu i}$ for all $i \in I_\mu$. Since $\{\mu\}$ is an infieite sequence, there is a μ_0 in $\{\mu\}$ such that $\mu \geqq \mu_0$ implies $l_\mu - 1 \geqq d/\beta$. This means that each $x^{\mu i} + z(\alpha^\mu)/(l_\mu - 1)$ can be expressed by a convex combination of $x^{\mu i} + \beta z(\alpha_\mu)/d$ and $x^{\mu i}$ for every $\mu \geqq \mu_0$, so that we have, by *C–2*, $x^{\mu i} + z(\alpha_\mu)/(l_\mu - 1) \succ_i x^{\mu i}$ for every $\mu \geqq \mu_0$ and for every i in I_μ. Taking into account the fact that $z^{\mu i_\mu} \geqq z(\alpha_\mu)$ for every μ, we ultimately obtain, by *C–3*,

$$x^{\mu i} + z^{\mu i_\mu}/(l_\mu - 1) \succ_i x^{\nu i}; \quad i \in I_\mu$$

for every $\mu \geqq \mu_0$.

Let a coalition S_μ be defined by $S_\mu = I_\mu \backslash \{i_\mu\}$, then $\{x^{\mu i} + z^{\mu i_\mu}/(l_\mu - 1)\}$; $i \in S_\mu$. satisfies the condition to be an allocation within S_μ

$$\sum_{i \in S_\mu} (x^{\mu i} + z^{\mu i_\mu}/(l_\mu - 1)) = \sum_{i \in S_\mu} x^{\mu i} + z^{\mu i_\mu} = \sum_{i \in I_\mu} x^{\mu i} - w^{i_\mu} - \beta_{\mu_{i_\mu}} y(x^\mu)$$
$$= \sum_{i \in S_\mu} (w^i + \beta_{\mu_i} y(x^\mu)).$$

For every $\mu \geqq \mu_0$, therefore, S_μ is a blocking coalition for $\{x^{\mu i}\}$.

Q.E.D.

It is an immediate consequence of Lemma 3 and Lemma 4 that there exists a compact set E, of R^n in which each $z^{\nu i}$; $i \in I_\nu$, is included independent of ν whenever $\{x^{\nu i}\}$ belongs to C_ν. Let a positive number f be defined by

$$f = \max_{z \in E} |z|,$$

then we have the following lemma on the number.

Lemma 5. For every positive integer M, there exists a ν_0 such that, if $\nu \geqq \nu_0$ and if $\{x^{\nu i}\} \in C_\nu$, then at least one of subsets of I_ν composed of M or more, but not exceeding $n + M$ consumers, say S_ν, fulfills

$$|\sum_{i \in S_\nu} z^{\nu i}| < (n + 2)f .$$

Proof. From the compactness of E, it follows that there exists, for every M, a finite collection $\{a^1, a^2, \ldots, a^{S(M)}\}$ of points of E, $f/2(n + M)$-neighborhood of which covers E, that is,

$$\bigcup_{j=1}^{S(M)} U_j \supset E ,$$

where U_j stands for a $f/2(n + M)$-neighborhood of $a^j \in E$; $j = 1, 2, \ldots, s(M)$.

Let U_{ν_1} be an element of a collection $\{U_1, U_2, \ldots, U_{S(M)}\}$ of the neighborhoods such that the number of consumers N_{ν_1}, for each of which $z^{\nu i} \in U_{\nu_1}$ holds is maximal among all of the numbers corresponding to the neighborhoods of the collection, similarly, U_{ν_2} be an element of the collection such that the number of consumers N_{ν_2}, for each of which $z^{\nu i} \in U_{\nu_2} \backslash U_{\nu_1}$ holds is maximal among all the neighborhoods in the collection, and so on. Necessarily, U_{ν_j} satisfies the condition that the cardinal number N_{ν_j}, of $\{i; z^{\nu i} \in U_{\nu_j} \backslash (U_{\nu_1} \cup U_{\nu_2} \cup \ldots \cup U_{\nu_{j-1}})\}$ is maximal among all the elements of the collection, and $j < j'$ implies $N_{\nu_j} \geqq N_{\nu_{j'}}$. Then we can divide the set $\{1, 2, \ldots, s(M)\}$ of indices into two groups $Q_{\nu_1} = \{j; N_{\nu_j} \geqq M\}$ and $Q_{\nu_2} = \{j; N_{\nu_j} < M\}$.

Let $N^*_{\nu_1}$, $N^*_{\nu_2}$, V_ν, T_ν and $\bar{z}^\nu$ be defined by

$$N^*_{\nu_1} = \sum_{j \in Q_{\nu_1}} N_{\nu_j},\ N^*_{\nu_2} = \sum_{i \in Q_{\nu_2}} N_{\nu_j},\ V_\nu = \bigcup_{j \in Q_{\nu_1}} U_{\nu_j},\ T_\nu = \{i; z^{\nu i} \in V_\nu\} \text{ and } N^*_{\nu_1} \bar{z} = \sum_{i \in T_\nu} z^{\nu i}$$

respectively. From the definitions of $N^*_{\nu_1}$ and $N^*_{\nu_2}$, we immediately have $N^*_{\nu_1} = l_\nu - N^*_{\nu_2}$ and $N^*_{\nu_2} < Ms(M)$ for every ν and for every $\{x^{\nu i}\}$ in C_ν. Hence, we can find a ν_0 such that $\nu \geqq \nu_0$ implies $N^*_{\nu_1} - M^2 s(M)$. This leads to a fact that

$$|\bar{z}| = |\sum_{i \in T_\nu} z^{\nu i}/N^*_{\nu_1}| = |-\sum_{i \ni T_\nu} z^{\nu i}/N^*_{\nu_1}| < f/M$$

for every $\nu \geqq \nu_0$. On the other hand, a convex hull H_ν, of V_ν includes $\bar{z}^\nu$ for every $\nu \geqq \nu_0$ since, by the definitions, the number of consumers in T_ν equals $N^*_{\nu_1}$. Therefore by virtue of the Caratheodory's theorem (cf. Appendix), we see that $\bar{z}^\nu$ is represented by a strictly positive convex combination

$$\sum_{t=1}^{q_\nu} \alpha_{\nu t} c^{\nu t};\ \sum_{t=1}^{q_\nu} \alpha_{\nu t} = 1,\quad \alpha_{\nu t} > 0 ,$$

of at most $n + 1$ points $c^{\nu t}$; $t = 1, 2, \ldots, q_\nu$, of V_ν each of which is included in a different U_{ν_j}; $j \in Q_{\nu_1}$, one against one another, where we can suppose, without loss of generality, that $c^{\nu t} \in U_{\nu_j}$, $c^{\nu t'} \in U_{\nu_{j'}}$ and $t < t'$ imply $j < j'$. Let us denote the U_{ν_j} including $c^{\nu t}$ by U^t_ν; $t = 1, 2, \ldots, q_\nu$, and let $b_{\nu t}$ be the smallest integer greater than or equal to $\alpha_{\nu t} M$, then it is possible to select, without overlapping

each other, $b_{\nu_1} + b_{\nu_2} + \ldots + b_{\nu q_\nu}$ consumers out of I_ν in such a way that the number of consumers satisfying

$$z^{\nu i} \in U_\nu^i \setminus \bigcup_{t'=1}^{t-1} U_\nu^{t'}$$

becomes just $b_{\nu t}$; $t = 1, 2, \ldots, q_\nu$.

Let S_ν be a set of such $b_{\nu_1} + b_{\nu_2} + \ldots + b_{\nu q_\nu}$ consumers, then it is obvious that $M \leqq N(S_\nu) \leqq n + M$ for every $\nu \geqq \nu_0$, where $N(S_\nu)$ denotes the cardinal number of S_ν, viz. $N(S_\nu) = b_{\nu_1} + b_{\nu_2} + \ldots + b_{\nu q_\nu}$. Next, let us divide it into q_ν non-intersecting subsets $S_{\nu t}$; $t = 1, 2, \ldots, q_\nu$, such that $i \in S_{\nu t}$ implies

$$z^{\nu i} \in U_\nu^i \setminus \bigcup_{t'=1}^{t-1} U_\nu^{t'};\ t = 1, 2, \ldots, q_\nu ,$$

then we see

$$\left|\sum_{i \in S_\nu} z^{\nu i} - M\bar{z}^\nu\right| = \left|\sum_{i \in S_\nu} z^{\nu i} - \sum_{t=1}^{q_\nu} M\alpha_{\nu t} c^{\nu t}\right| \leqq \left|\sum_{t=1}^{q_\nu} \left(\sum_{i \in S_{\nu t}} z^{\nu i} - b_{\nu t} c^{\nu t}\right)\right|$$

$$+ \left|\sum_{t=1}^{q_\nu} (b_{\nu t} - M\alpha_{\nu t}) c^{\nu i}\right| < \sum_{t=1}^{q_\nu} b_{\nu t} f/(n + M) + nf \leqq (n + 1)f$$

for every $\nu \geqq \nu_0$. Since $|\bar{z}^\nu| < f/M$ for every $\nu \geqq \nu_0$, we finally have

$$\left|\sum_{i \in S_\nu} z^{\nu i}\right| \leqq \left|\sum_{i \in S_\nu} z^{\nu i} - M\bar{z}^\nu\right| + M|\bar{z}^\nu| < (n + 1)f + f = (n + 2)f$$

for every $\nu \geqq \nu_0$.

Q.E.D.

Lemma 6. For any $q > 0$, there exists a ν_0 such that $\nu \geqq \nu_0$ and $\{x^{\nu i}\} \in C_\nu$ imply

$$\min_{i \in I_\nu} \langle p(x^\nu), z^{\nu i} \rangle > -q .$$

Proof. Let us suppose that there exist a $q' > 0$ and an infinite subsequence $\{\mu\}$, of $\{\nu\}$ on which $\langle p(x^\mu), z^{\mu i_\mu} \rangle \leqq -q'$ holds for some $\{x^{\mu i}\}$ in C_μ and i_μ in I_μ.

If a set S_μ of consumers fulfills both

$$\left|\sum_{i \in S_\mu} z^{\mu i}\right| < (n + 3)f \text{ and } \langle p(x^\mu), \sum_{i \in S_\mu} z^{\mu i} \rangle \leqq -q'/2 ,$$

it follows from *C–5* that there is a $\beta > 0$ such that $x^{\mu i} - \beta \sum_{i \in S_\mu} z^{\mu i}/(n + 3)f \in X_i$ and $x^{\mu i} - \beta \sum_{i \in S_\mu} z^{\mu i}/(n + 3)f \succ_i x^{\mu i}$ for every i in S_μ.

Let the number M in Lemma 5 be chosen so large as $M \geqq (n + 3)f/\beta$, then by making use of the results of Lemma 4 and Lemma 5, we can find a μ_0 such that, for every $\mu \geqq \mu_0$, the allocation $\{x^{\mu i}\}$ has a set S'_μ composed of M to $n + M$ consumers for which

$$\left|\sum_{i \in S'_\mu} z^{\mu i}\right| < (n + 2)f \text{ and } \langle p(x^\mu), z^{\mu i} \rangle \leqq q'/2(n + M);\ i \in S'_\mu ,$$

hold. By defining S_μ by $S_\mu = S'_\mu$ if $i_\mu \in S'_\mu$ otherwise $S_\mu = S'_\mu \cup \{i_\mu\}$, we see

$$|\sum_{i\in S_\mu} z^{\mu i}| < (n+3)f \text{ and } \langle p(x^\mu), \sum_{i\in S_\mu} z^{\mu i}\rangle \leqq -q'/2 .$$

It is clear from the choice of M that the number of consumers in S_μ, say M_μ, exceeds $(n+3)f/\beta$ for every $\mu \geqq \mu_0$, we have, by means of the convexities of X_i and of $\succsim_i$,

$$x^{\mu i} - \sum_{i\in S_\mu} z^{\mu i}/M_\mu \succ_i x^{\mu i}$$

for every $\mu \geqq \mu_0$ anf for every i in S_μ. Whereas, $\{x^{\mu i} - \sum_{i\in S_\mu} z^{\mu i}/M_\mu\}$; $i \in S_\mu$, satisfies the condition to be an allocation within S_μ

$$\sum_{i\in S_\mu}(x^{\mu i} - \sum_{i\in S_\mu} z^{\mu i}/M_\mu) = \sum_{i\in S_\mu}(x^{\mu i} - z^{\mu i}) = \sum_{i\in S_\mu}(w + \beta_{\mu i} y(x^\mu)) .$$

Thus, we obtain a blocking coalition S_μ for $\{x^{\mu i}\}$. Q.E.D.

Now, the following theorem is a direct consequence of Lemma 4 and Lemma 6.

Theorem 3. For any $\varepsilon > 0$, there exists a ν_0 such that, if $\nu \geqq \nu_0$, then every allocation in the core of $\mathscr{E}\nu$ is ε-competitive.

4. *The necessity of uniform-smoothness.*

Assumption *C–5* imposes a restriction on consumers' indifference hypersurfaces nameby that those have a common positive radius of the inscribed hypersphere without regard to the choice of i in I and of x in X_i, where by "indifference hypersurface of $i \in X_i$" we mean a set $\{x'; x' \succsim_i x \text{ and } x \succsim_i x'\}$. It follows directly from *C–1* and *C–3* that the set forms an at most $n-1$ dimensional hypersurface of R^n.

This signifies, from an economic point of view, that there is neither particularly strong substitution effect nor extreme complimentarity among the commodities.

Although it is really desirable to eliminate such a rigorous restriction on the shape of each indifference hypersurface, the following example would suggest that some sort of assumptions such as this must be placed in order to complete the proof of our convergence theorem.

Example. Next, pure exchange economies consisting of two kinds of commodities and ν consumers; $\nu = 2, 3, \ldots$, in which each individual has a preference field X_i; $i = 1, 2, \ldots, \nu$, a preference ordering $\succsim_i$; $i = 1, 2, \ldots, \nu$, and an initial holding of commodities w^i; $i = 1, 2, \ldots, \nu$, represented by the followings are considered.

$$X_1 = \{x \in \Omega^n; x_1 x_2 \geqq 1/2\} ,$$

$$X_i = \{x \in \Omega^n; \min(x_1(x_2+1), (x_1+1)x_2) \geqq 1/2\}; i = 2, 3, \ldots, \nu .$$

$$x \succsim_1 x' \text{ if and only if } x, x' \in X_i \text{ and } x_1 x_2 \geqq x'_1 x'_2 ,$$

$x \succsim_i x'$ if and only if $x, x' \in X_i$ and $\min(x_1(x_2+1), (x_1+1)x_2) \geqq \min(x'_1(x'_2+1),$

$(x_1' + 1)x_2'$); $i = 2, 3, \ldots, \nu$.

$$w^1 = (5/4, 2/3),\ w^2 = (3/4, 4/3),\ w^i = (1, 1);\ i = 3, 4, \ldots, \nu\,.$$

Regarding a one point set $\{0\}$, of R^n consisting of the origin as the total production set Y, it can be easily shown that the economies satisfy all of the preceding assumptions *C–1* to *M–1* excepting *C–5*. Suppose that each consumer receives a common final resource (1, 1), then we can varify that the collection $\{(1, 1)\}$ of such the ν final resources is an allocation of the economy for every ν.

Let Z_i be defined by $Z_i = \{z \in R^n;\ (1, 1) + z \succ_i (1, 1)\}$; $i = 1, 2, \ldots, \nu$, then a convex hull H_μ, of $\bigcup_{i=1}^{\nu} Z_i$ has a supporting hyperplane $\langle p, z\rangle = 0$, with the normal $p = (1, 1)$.

Let z^i be $(1, 1) - w^i$; $i = 1, 2, \ldots, ,$ then a coalition S must satisfy $S \ni 2$ with $S \not\ni 1$ in order to make $\langle p, -\sum_{i \in S} z^i\rangle$ positive. It is easy to see that, for every S such that $S \not\ni 1$, Z_S is a subset of an open cone $K = \{z;\ c_1(-1/2, 1) + c_2(1, -1/2),\ c_1 > 0,\ c_2 > 0\}$, where $Z_S = \{z;\ z = -\sum_{i \in S} z'^i,\ (1, 1) + z'^i \succsim_i (1, 1)$ for every $i \in S$, $(1, 1) + z'^i \succ_i (1, 1)$ for some $i \in S\}$.

While $-\sum_{i \in S} z^i$ must be in K whenever S is a blocking coalition for the allocation $\{(1, 1)\}$, we cannot find such a coalition seeing that $z^2 = (1/4, -1/3)$ and $z^i = (1, 1)$; $i = 3, 4, \ldots, \nu$. Hence $\{(1, 1)\}$ is an allocation in the core C_ν.

On the other hand, the fact that $\langle p, z^1\rangle = 1/12$ and $\langle p, z^2\rangle = -1/12$ leads to the conclusion that the allocation $\{(1, 1)\}$ does not approach to a competitive one as ν tends to infinity.

5. *Existence of a competitive allocation.*

So far, the mechanism of appearance of the so called perfect competition has been clarified. The above-mentioned convergence theorem would acquire little significance without assurance of either the existence of a competitive allocation or the non-emptiness of the core.

Although an economy does not always have a competitive allocation, it follows directly from Theorem 2 that, for every ν, our economy $\mathscr{E}\nu$ has a competitive allocation if *C–4′* and *P–3′* are strengthened so as to fulfill *C–4* and *P–3* respectively.

As to the non-emptiness of the core, Scarf [17] has recently shown, without using the fixed point theorem, that every balanced game has a non-empty core, and, though the proof is omitted here, it can be verified that our economy satisfies all the conditions to be a balanced game. By taking Scarf's theorem into account, the convergence theorem can be restated as in the following.

Theorem 3′. For any $\varepsilon > 0$, there exists a ν_0 such that, if $\nu \geqq \nu_0$, then $\mathscr{E}\nu$ has an ε-competitive allocation.

Chapter 3

EXISTENCE OF A TEMPORARY EQUILIBRIUM FOR A COMPETITIVE ECONOMY WITH RENEWALS OF THE PRODUCTION FACILITIES

1. *Introduction.*

Conduct of an economic agent assumed in the ordinary theory of equilibrium is extremely idealized, and is not a little at variance with that of an actual entrepreneur or of an actual household.

In this chapter, we take the dynamic aspect of their behavior into consideration. The model of competitive economy to be considered here involves, therefore, the expectations of future prices by agents. On the basis of these expectations, each producer determines the current production, the production plan in future periods, and the renewal of his current production facilities which maximize his long-run profit; each consumer determines the current consumption and the amount of his saving so as to make his long-run satisfaction level maximum.

J. R. Hicks [8] first analyzed such an extention to a dynamic system, but he has not developed a rigorous argument on the existence of a temporary equilibrium of the system.

In their forthcoming book [2], Arrow and F. Hahn have succeeded in demonstrating the existence of a temporary equilibrium for a competitive economy which is rather faithful to the original system presented by Hicks.

It is our main purpose to give some conditions under which the existence theorem of competitive temporary equilibrium of a generalized Hicksian economy including savings, investments and the interest rate can be proved within the effective range of the Kakutani's fixed point theorem.

To be precise, we impose, in addition to slightly strengthened conventional assumptions, a restriction on the level of G. N. I., which, instead of the zero-th homogeneity of an excess demand in an ordinary system, makes the region of current prices to be considered compact, and also imposes several conditions: such as, the continuity and the strict positiveness on the expectations of price systems in the future periods by both consumers and producers, and of compensation rates for their current savings by consumers, which play an important role for the determination of the amount of excess investment demand.

For simplicity, it is assumed that those expectations only concern themselves with the future periods from the next to $\bar{t}$ terms hence.

2. *The model.*

A dynamic competitive economy $\mathscr{E}^*$, consisting of n kinds of current commodities, n kinds of quite the same commotities in the t-th future period; $t = 1, 2, \ldots, \bar{t}$, m producers and l consumers as forecasters is considered. All kinds of services and production factors are involved in the set of commodities.

All the consumers have a common preference field X^* which is identical with

a set of all non-negetive vectors of $R^{(1+\bar{t})n}$, $\Omega^{(1+\bar{t})n}$. The first n components x_k; $k = 1, 2, \ldots, n$, of each x^* in X^* denote a current consumption and the remainders $\bar{x}_k^t$; $k = 1, 2, \ldots, n$, $t = 1, 2, \ldots, \bar{t}$, denote a consumption scheme in the future spread over $\bar{t}$ periods. We denote, for simplicity, current and future consumptions by x and $\bar{x}$ respectively. Necessarily,

$$x^* = (x, \bar{x}) = (x_1, \ldots, x_n, \bar{x}_1^1, \ldots, \bar{x}_n^1, \ldots, \bar{x}_1^{\bar{t}}, \ldots, \bar{x}_n^{\bar{t}}) .$$

A complete pre-ordering $\gtrsim_i^*$, as a representative of the i-th consumer's preference relation; $i = 1, 2, \ldots, l$, is defined on X^*, and satisfies the following *C*–1* to *C*–3*.

Assumption C–1.* Continuity. For every x^* in X^*, $\{x'^*;\ x'^* \gtrsim_i^* x^*\}$ and $\{x'^*;\ x^* \gtrsim_i^* x'^*\}$ are closed sets of $R^{(1+\bar{t})n}$.

Assumption C–2.* Convexity. If x^* and x'^* are points of X^*, then $x^* \succ_i^* x'^*$ implies $\alpha x^* + (1 - \alpha)x'^* \succ_i^* x'^*$ for any α in (0, 1], where $x^* \succ_i^* x'^*$ means $x^* \gtrsim_i^* x'^*$ with $x'^* \not\gtrsim_i^* x^*$.

Assumption C–3.* Desirability. For every x^* in X^*, $x'^* \geq x^*$ implies $x'^* \succ_i^* x^*$.

The consumer i has an initial holding w^i, of current commodities which satisfies the following

Assumption C–4.* $w^i \geq 0$; $i = 1, 2, \ldots, l$, and the total holding w, in the spot market is strictly positive, that is,

$$w = \sum_{i=1}^{l} w^i > 0 .$$

Each producer has a possible activity set Y_j^*; $j = 1, 2, \ldots, m$, of $R^{(2+\bar{t})n}$. An element y^*, of each Y_j^* is composed of a current production $y \in R^n$, a production plan $\bar{y} = (\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^{\bar{t}}) \in R^{\bar{t}n}$ in the future periods and a renewal z, of the current production facilities $u^j \in \Omega^n$, that is, $y^* = (y, \bar{y}, z)$. We assume, as a matter of convenience, that the commodity q is an imput of the activity y^* if $y_q^* < 0$; $q = 1, 2,, \ldots, (2 + \bar{t})n$. Every Y_j^* fulfills the following technical conditions.

Assumption P–1.* Y_j^* is closed and convex.

Assumption P–2.* Y_j^* contains (0, 0, 0).

Assumtpion P–3.* If $y^* \in Y_j^*$, then the correspinding z satisfies $z \leqq u^j$.

Assumtpion P–4.* Let Y_j be defined by $Y_j = \{y;\ (y, 0, 0) \in Y_j^*\} \subset R^n$, then $Y_j = \{y;\ y^* \in Y_j^*\}$.

Assumption P–5.* For every $z (\leqq u^j)$, there is an $a^j(z) \in \Omega^n$ such that $y^* \in Y_j^*$ implies

$$a_k^j(z) \geqq y_k + \sum_{t=1}^{\bar{t}} \max (\bar{y}_k^t, 0);\ k = 1, 2, \ldots, n,$$

and $z \geqq z'$ implies $a^j(z) \leqq a^j(z')$.

Assumption P–6.* If $y^* \in Y_j^*$, then $(y, \bar{y}^1, \bar{y}^2, \ldots, \bar{y}^{(t-1)}, 0, \bar{y}^{(t+1)}, \ldots, \bar{y}^{\bar{t}}, z) \in Y_j^*$; $t = 1, 2, \ldots, \bar{t}$.

Assumption P–7.* Let $\langle v^*, y^* \rangle = c$; $v^* = (v, \bar{v}, v) \geqq 0$, be a supporting hyperplane of Y_j^* passing through a y'^* in Y_j^*, then $y''^* \in Y_j^*$ and $\langle v^*, y''^* \rangle = c$ imply $\langle v, y' - y'' \rangle = \langle v, z' - z'' \rangle = 0$.

Let $r_i, p^i = (p^{1i}, p^{2i}, \ldots, p^{\bar{t}i})$; $i = 1, 2, \ldots, l$, and $p^j = (p^{1j}, p^{2j}, \ldots, p^{\bar{t}j})$; $j = 1, 2, \ldots, m$, be the expectations of compensation rate for his current saving by the consumer i, of price systems in the future periods by the consumer i and of price systems in the future periods by the producer j, then those are scalar valued or vector valued functions of current price system $p \in \Omega^n$ and interest rate $r \geqq 0$ satisfying the followings.

Assumption E–1.* Let γ and $\gamma^i = (\gamma_i^1, \gamma_i^2, \ldots, \gamma_i^{\bar{t}})$; $i = 1, 2, \ldots, l$ be defined by $\gamma = 1/(1 + r)$ and $\gamma^i = (1/(1 + r_i), 1/(1 + r_i)^2, \ldots, 1/(1 + r_i)^{\bar{t}})$, then $(p, \gamma) \to \gamma^i$, $(p, \gamma) \to p^i$; $i = 1, 2, \ldots, l$, and $(p, \gamma) \to p^j$; $j = 1, 2, \ldots, m$, are point-to-point continuous mapping of $\Omega^n \times [0, 1]$ into $N^{\bar{t}}$ or $\Omega^{\bar{t}n}$, where $N^{\bar{t}}$ denotes a Cartesian product of common $\bar{t}$ semi-closed intervals $(0, 1]$.

Assumption E–2.* If $p \geq 0$, then p^i; $i = 1, 2, \ldots, l$, and p^j; $j = 1, 2, \ldots, m$, are strictly positive for any γ in $[0, 1]$.

3. *Several properties of the national income, supply and demand.*

Let M', s' and r' be the national income, total amount of savings and the interest rate which were realized in the previous period, then money is supplied by the amount to make the current national income $\hat{M}$, equal to $M' + r's'$.

Lemma 1. If $x^{*i} \in X^*$; $i = 1, 2, \ldots, l$, $y^{*j} \in Y_j^*$; $j = 1, 2, \ldots, m$, and

$$y + z + w - x \geqq 0,$$

then there exists an n-dimensional positive vector e, such that

$$y^j > -e, z^j > -e;\ j = 1, 2, \ldots, m, \text{ and } x^i < e;\ i = 1, 2, \ldots, l,$$

where

$$y = \sum_{j=1}^{m} y^j,\ z = \sum_{j=1}^{m} z^j \text{ and } x = \sum_{i=1}^{l} x^i .$$

Proof. Assumption *P*–4* leads to a fact that $y^j \in Y_j$; $j = 1, 2, \ldots, m$, and then we immediately see, by *P*–2*, *P*–3* and *P*–5*, the following inequalities.

$$y^{j'} + \sum_{j=1}^{m} a^j(0) \geqq y \geqq x - w - z \geqq -\left(w + \sum_{j=1}^{m} u^j\right);\ j' = 1, 2, \ldots, m,$$

$$z^{j'} + \sum_{j=1}^{m} u^j \geqq z \geqq x - y - w \geqq -\left(w + \sum_{j=1}^{m} a^j(0)\right);\ j' = 1, 2, \ldots, m,$$

$$x^{i'} \leqq x \leqq y + z + w \leqq \sum_{j=1}^{m} a^j(0) + \sum_{j=1}^{m} u^j + w; \qquad i' = 1, 2, \ldots, l.$$

Let e be put equal to

$$2\left(\sum_{j=1}^{m} a^j(0) + \sum_{j=1}^{m} u^j + w\right),$$

then it follows directly from the above inequalities that all of $e > 0$, $y^j > -e$; $j = 1, 2, \ldots, m$, $z^j > -e$; $j = 1, 2, \ldots, l$, hold. Q.E.D.

Let E, h^j and $F_j(p, \gamma)$ be defined by

$$E = \{y^* \in R^{(2+\bar{t})n};\ y \geqq -e, z \geqq -e\},$$

$$h^j = (\gamma p^{1j}, \gamma^2 p^{2j}, \ldots, \gamma^{\bar{t}} p^{\bar{t}j});\ j = 1, 2, \ldots, m,$$

and $F_j(p, \gamma) = \{y^* \in Y_j^* \cap E;\ \langle p^{*j}, y^* - y'^* \rangle \geqq 0$ for any $y'^* \in Y_j^* \cap E\}$; $j = 1, 2, \ldots, m$, where $p^{*j} = (p, h^j, p)$; $j = 1, 2, \ldots, m$, then each producer chooses an activity y^{*j} which maximizes, under a price-interest system (p, γ), the long-run profit $\langle p^{*j}, y^* \rangle$ in $Y_j^* \cap E$, that is, $y^{*j} \in F_j(p, \gamma)$; $j = 1, 2, \ldots, m$, for every (p, γ) in $\Omega^n \times [0.\ 1]$.

By using P^*–7, we can define $b_j(p, \gamma)$ and $c_j(p, \gamma)$ as single-valued functions of (p, γ) by

$$b_j(p, \gamma) = (1 + r')d_j' - \langle p, y^j \rangle - \max(\langle p, z^j \rangle, 0);\ y^{*j} \in F_j(p, \gamma),$$

and

$$c_j(p, \gamma) = \max(b_j(p, \gamma), 0) + \max(-\langle p, z^j \rangle, 0);\ y^{*j} \in F_j(p, \gamma),$$

where $j = 1, 2, \ldots, m$, and d_j' denotes the amount of depts of the producer j at the end of the previous term.

The current profit $\pi_j(p, \gamma)$, to be divided among consumers and the amount of depts $d_j(p, \gamma)$, of the producer j at the end of the present term are as follows.

$$\pi_j(p, \gamma) = \max(-b_j(p, \gamma), 0),$$

$$d_j(p, \gamma) = \min(c_j(p, \gamma), \langle h^j, \bar{y}^j \rangle);\ y^{*j} \in F_j(p, \gamma).$$

It is assumed that the consumer i gets a share β_{ji} of $\pi_j(p, \gamma)$ from the producer j and receives a repayment for his saving s_i', in the previous term, where $\beta_{ji} \geqq 0$; $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, l$, and

$$\sum_{i=1}^{l} \beta_{ji} = 1;\ j = 1, 2, \ldots, m.$$

Therefore, the current total income $M_{ci}(p, \gamma)$, of the consumer i amounts to

$$M_{ci}(p, \gamma) = \langle p, w^i \rangle + \sum_{j=1}^{m} \beta_{ji}\pi_j(p, \gamma) \quad \text{if } s' = 0,$$

$$= \langle p, w^i \rangle + \sum_{j=1}^{m} \beta_{ji}\pi_j(p, \gamma) + s_i'\{1 + r'$$

$$- \sum_{j=1}^{m} \max(c_j(p, \gamma) - \langle h^j, \bar{y}^i \rangle, 0)/s'\};\ y^{*j} \in F_j(p,), \text{ if } s' > 0.$$

Let M_{ti} denote the expectation of the i-th consumer's income from the distribution of producers' profits and from his initial holding in the t-th future period being discounted by γ_i^t, then the following holds.

Assumption E–3.* $(p, \gamma) \to M_{ti}$; $t = 1, 2, \ldots, \bar{t}$, $i = 1, 2, \ldots, l$, are conti-

nous point-to-point mapping of $\Omega^n \times [1, 0]$ into Ω^1.

Let C, x^{*t}, p^{*ti}, M, $B_i(p, \gamma, M_i)$ and $A_i(p, \mathbf{z}, M_i)$ be defined by

$$C = \{x^* \in X^*; x \leqq e\},$$

$$x^{*t} = (x, \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^t);\ t = 1, 2, \ldots, \bar{t},$$

$$p^{*ti} = (p, \gamma_i p^{1i}, \gamma_i^2 p^{2i}, \ldots, \gamma_i^t p^{ti});\ t = 1, 2, \ldots, \bar{t},\ i = 1, 2, \ldots, l,$$

$$M_i = (M_{ci}, M_{1i}, M_{2i}, \ldots, M_{\bar{t}i});\ i = 1, 2, \ldots, l,$$

$$B_i(p, \gamma, M_i) = \Big\{x^* \in C; \langle p, x\rangle \leqq M_{ci}, \langle p^{*ti}, x^{*t}\rangle \leqq M_{ci} + \sum_{q=1}^{t} M_{qi},$$

$$t = 1, 2, \ldots, \bar{t}\Big\};\ i = 1, 2, \ldots, l,$$

$$A_i(p, \gamma, M_i) = \{x^* \in B_i(p, \gamma, M_i); x^* \succsim_i^* x'^* \text{ for any } x'^* \in B_i\{(p, \gamma, M_i)\}\};$$

$$i = 1, 2, \ldots, l,$$

then each consumer chooses a consumption plan x^{*i} which is, under (p, γ) and M_i, a maximum element of $B_i(p, \gamma, M_i)$ for $\succsim_i^*$, that is, $x^{*i} \in A_i(p, \gamma, M_i)$; $i = 1, 2, \ldots, l$, for every (p, γ) in $\Omega^n \times [0, 1]$ and every M_i in $\Omega^{1+\bar{t}}$. It is quite easy to see the convexity of each $B_i(p, \gamma, M_i)$. The amount of saving $s_i(p, \gamma, M_i, x^i)$ of the consumer i at the end of the present term corresponding to an x^{*i} in $A_i(p, \gamma, M_i)$ is equal to $M_{ci} - \langle p, x\rangle$.

$A_i(p, \gamma, M_i)$; $i = 1, 2, \ldots, l$, have the following properties.

Assumption C-5.* If $p > 0$, $\gamma = 0$ and $M_c > 0$, then there is an i such that $x^* \in A_i(p, \gamma, (M_c, M_{1i}, \ldots, M_{\bar{t}i}))$ implies $\langle p, x\rangle < M_c$.

Assumption C-6.* For every $M_c > 0$ and every (p, γ) in $\Omega^n \times [0, 1]$, there is an i such that $A_i(p, \gamma, (M_c, M_{1i}, \ldots, M_{\bar{t}i})) \subset \{x^* \in X^*; x \geq 0\}$.

Assumption C-7.* If $p \geq 0$ and $\gamma = 1$, then $x^* \in A_i(p, \gamma, M_i)$ implies $\langle p, x\rangle = M_{ci}$ for every i.

Lemma 2. $s' = d'$ and $y^{*j} \in F_j(p, \gamma)$; $j = 1, 2, \ldots, m$, imply

$$(1 + r')s' - \sum_{j=1}^{m} \max(c_j(p, \gamma) - \langle h^j, \bar{y}^j\rangle, 0) \geqq 0$$

for every (p, γ) in $\Omega^n \times [0, 1]$, where $d' = \sum_{j=1}^{m} d'_j$.

Proof. Since the non-negativities of s' and d' are obvious by the definitions of them, it suffice to show that $y^{*j} \in F(p, \gamma)$ and $c_i(p, \gamma) - \langle h^j, \bar{y}^j\rangle \geqq 0$ imply $(1 + r')d'_j - c_j(p, \gamma) + \langle h^j, \bar{y}^j\rangle \geqq 0$ for every j.

We immediately have

$$\begin{aligned}(1 + r')d'_j - c^j(p, \gamma) + \langle h^j, \bar{y}^j\rangle &= (1 + r')\, d'_j - b_j(p, \gamma) + \langle h^j, \bar{y}^j\rangle \\ &= \langle p^{*j}, y^{*j}\rangle \quad \text{if } \langle p, z^j\rangle \geqq 0 \text{ and } b_j(p, \gamma) \geqq 0, \\ &= (1 + \gamma')d'_j + \langle h^j, \bar{y}^j\rangle \\ &\qquad \text{if } \langle p, z^j\rangle \geqq 0 \text{ and } b_j(p, \gamma) < 0,\end{aligned}$$

$$= (1 + r')d_j' - (1 + r')d_j' + \langle p^{*j}, y^{*j} \rangle$$
$$= \langle p^{*j}, y^{*j} \rangle \quad \text{if } \langle p, z^j \rangle < 0 \text{ and } b_j(p, \gamma) \geqq 0,$$
$$= (1 + r')d_j' + \langle p, z^j \rangle + \langle h^j, \bar{y}^j \rangle$$
$$\text{if } \langle p, z^j \rangle < 0 \text{ and } b_j(p, \gamma) < 0.$$

It follows from *P*–2* and *P*–6* that $y^{*j} \in F_j(p, \gamma)$ implies $\langle p^{*j}, y^{*j} \rangle \geqq 0$ and $\langle h^j, \bar{y}^j \rangle \geqq 0$ for every (p, γ) in $\Omega^n \times [0, 1]$. We see, moreover, $\langle p, z^j \rangle + \langle h^j, \bar{y}^j \rangle \geqq 0$ for every (p, γ) in $\Omega^n \times [0, 1]$ and every y^*, in $F_j(p, \gamma)$. In fact, if $\langle p, z^t \rangle + \langle h^t, \bar{y}^j \rangle < 0$ for some $(p, \gamma) \in \Omega^n \times [0, 1]$ and $y^{*j} \in F_j(p, \gamma)$, then $\langle p^{*j}, y^{*j} \rangle < \langle p^{*j}, (y^j, 0, 0) \rangle$. Since $(y^j, 0, 0) \in Y_j^* \cap E$ by *P*–4*, we have, $y^{*j} \notin F_j(p, \gamma)$, a contradiction.

Q.E.D.

Lemma 3. If $s' = d'$, then $p > 0$ implies $M_{ci}(p, \gamma) > 0$; $i = 1, 2, \ldots, l$, for every γ in [0, 1], and $p \geq 0$ implies that, for every γ in [0, 1], there is an i of which $M_{ci}(p, \gamma)$ is strictly positive.

Proof. Taking the non-negativity of each $\pi_j(p, \gamma)$ into account, we see from Lemma 2 $M_{ci}(p, \gamma) \geqq \langle p, w^i \rangle$; $i = 1, 2, \ldots, l$, $(p, \gamma) \in \Omega^n \times [0, 1]$, so that we have, by *C*–4*, the conclusion of the present lemma. Q.E.D.

Lemma 4. If $s' = d'$, $\gamma' > 0$ and $M' - s' > 0$, then there exist positive numbers δ and $\eta (\geqq \delta)$ such that

$$\sum_{i=1}^{l} M_{ci}(p, \gamma) = \hat{M}$$

implies $(p, \gamma) \in P(\delta, \eta) \times [0, 1]$, where $\gamma' = 1/(1 + r')$ and

$$P(\delta, \eta) = \left\{ p \in \Omega^n;\ \delta \leqq \sum_{k=1}^{n} p_k \leqq \eta \right\}.$$

Proof. Assumptions *P*–4* and *P*–5* ensure $\pi_j(p, \gamma) \leqq \max(\langle p, a^j(0) + u^j \rangle - (1 + r')d_j', 0) \leqq \langle p, a^j(0) + u^j \rangle$; $j = 1, 2, \ldots, m$, $(p, \gamma) \in \Omega^n \times [0, 1]$, and then we see

$$\langle p, w \rangle \leqq \sum_{i=1}^{l} M_{ci}(p, \gamma) \leqq \langle p, w \rangle + \left\langle p, \sum_{j=1}^{m} (a^j(0) + u^j) \right\rangle + (1 + r')s';$$
$$(p, \gamma) \in \Omega^n \times [0, 1].$$

Hence, if $p \in \overset{\circ}{\Omega}{}^n$ satisfies either

$$\min_k w_k \sum_{k=1}^{n} p_k > \hat{M}$$

or

$$\max_k \left(w_k + \sum_{j=1}^{m} (a_k^j(0) + u_k^j) \right) \sum_{k=1}^{n} p_k < M' - s',$$

then

$$\sum_{i=1}^{l} M_{ci}(p, \gamma) \geqq \langle p, w \rangle \min_k w_k \sum_{k=1}^{n} p_k > \hat{M}$$

or

$$\sum_{i=1}^{l} M_{ci}(p, \gamma) \leqq \left\langle p,\ w + \sum_{j=1}^{m} (a^j(0) + u^j) \right\rangle + (1 + r')\, s'$$

$$\leqq \max_k \left(w_k + \sum_{j=1}^{m} (a_k^j(0) + u_k^j) \right) \sum_{k=1}^{n} p_k + (1 + r')s' < \hat{M}$$

hold. Let δ and η be defined by

$$\delta = (M' - s')/\max_k \left(w_k + \sum_{j=1}^{m} (a_k^j(0) + u_k^j) \right) (> 0)\,,$$

$$\eta = M/\min_k w_k\, (> 0)\,,$$

then it is easily seen that $(p, \gamma) \notin P(\delta, \eta) \times [0, 1]$ implies $\sum_{i=1}^{l} M_{ci}(p, \gamma) \neq \hat{M}$.

Q.E.D.

Lemma 5. If $p > 0$, then $x^{*i} \in A_i(p, 0, M_i(p, 0))$; $i = 1, 2, \ldots, l$, imply $s(p, 0, \{x^i\}) > 0$, where

$$s(p, \gamma, \{x^i\}) = \sum_{i=1}^{l} s_i(p, \gamma, M_i(p, \gamma), x^i)\,.$$

Proof. From Lemma 3, we immediately have $M_{ci}(p, 0) > 0$; $i = 1, 2, \ldots, l$. Therefore by *C*–5*, we see that $s_i(p, 0, M_i(p, 0), x^i) > 0$ for some i. On the other hand, each $s_i(p, \gamma, M_i, x^i)$ fulfills $s_i(p, \gamma, M_i, x^i) \geqq 0$ for all (p, γ) in $\Omega^n \times [0, 1]$, M_i in $\Omega^{1+\bar{t}}$ and x^{*i} in $A_i(p, \gamma, M_i)$, so that $s(p, 0, \{x^i\}) > 0$. Q.E.D.

Lemma 6. Let Q' be $P(\delta/2, 2\eta) \times (0, 1]$, then there exists a hypercube D, of $R^{(2+\bar{t})n}$ such that $(p, \gamma) \in Q'$ implies $F_j(p, \gamma) \subset D$; $j = 1, 2, \ldots, m$.

Proof. It follows from *P*–5* that $y^* \in Y_j^* \subset E$ implies

$$\bar{y}^t \leqq a^j(z) - y < a^j(-e) + e \leqq \sum_{j=1}^{m} a^j(-e) + e;$$

$$j = 1, 2, \ldots, m,\ t = 1, 2, \ldots, \bar{t}\,.$$

Let us define a set S, of $R^{\bar{t}n}$ and a number ρ by

$$S = \bigcup_{j=1}^{m} \{p^j;\ (p, \gamma) \in P(\delta/2, 2\eta) \times [0, 1]\}\,,$$

$$\rho = \min_{q \in S} \min_t \min_k q_k^t / \sum_{g=1}^{n} q_g^t\,,$$

then we see, from *E*–1* and *E*–2*, that S is a compact set of $R^{\bar{t}n}$ composed of only strictly positive vectors, so that $\rho > 0$.

On the other hand, $y^* \in F_j(p, \gamma)$; $(p, \gamma) \in \Omega^n \times [0, 1]$, implies $\langle p^{tj}, \bar{y}^t \rangle \geqq 0$; $t = 1, 2, \ldots, \bar{t}, j = 1, 2, \ldots, m$. Hence, we obtain

$$\rho \min_t \min_k \bar{y}_k^t + \max_k \left(\sum_{j=1}^n a_k^j(-e) + e_k \right) > 0$$

for every y^* in

$$\bigcup_{(p,\gamma)\in Q'} \bigcup_{j=1}^m F_j(p, \gamma) .$$

Let D be a hypercube of $R^{(2+\bar{t})n}$ defined by

$$D = \left\{ y^* \in R^{(2+\bar{t})n};\ -e \leqq y \leqq \sum_{j=1}^m a^j(0),\ -\max_k \left(\sum_{j=1}^m a_k^j(-e) + e_k \right) \Big/ \rho \right.$$

$$\leqq \bar{y}_k^t \leqq \max_k \left(\sum_{j=1}^m a_k^j(-e) + e_k \right);\ k = 1, 2, \ldots, n, t = 1, 2, \ldots, \bar{t},$$

$$\left. -e \leqq z \leqq \sum_{j=1}^m u^j \right\} ,$$

then we have

$$D \supset \bigcup_{(p,\gamma)\in Q'} \bigcup_{j=1}^m F_j(p, \gamma) .$$

Q.E.D.

Lemma 7. Let Q be $P(\delta/2, 2\eta) \times [0, 1]$, then there exists a hypercube G, of $R^{(1+\bar{t})n}$ such that $(p, \gamma)\in Q$ implies $B_i(p, \gamma, M_i(p, \gamma)) \subset G;\ i = 1, 2, \ldots, l$.

Proof. Let M be a positive number defined by

$$\overline{M} = \max_{(p,\gamma)\in Q} \max_i \left\langle p, w^i + \sum_{j=1}^m \beta_{ji}(a^j(0) + u^j) \right\rangle + (1 + r')s_i' ,$$

then it follows from *P*–4* and *P*–5*, in quite the same manner with the proof of Lemma 4, that $(p, \gamma)\in Q$ implies $M_{ci}(p, \gamma) \leqq \overline{M};\ i = 1, 2, \ldots, l$.

Let us define a vector h^i, of $R^{\bar{t}n}$, a set T, of $R^{\bar{t}n}$ and a number σ, by

$$h^i = (\gamma_i p^{1i}, \gamma_i^2 p^{2i}, \ldots, \gamma_i^{\bar{t}} p_i^{\bar{t}});\ i = 1, 2, \ldots, l ,$$

$$T = \bigcup_{i=1}^l \{h^i;\ (p, \gamma)\in Q\} ,$$

$$\sigma = \min_{h\in T} \min_t \min_k h_k^t ,$$

then we see, from *E*–1* and *E*–2*, that T is a compact set of $R^{\bar{t}n}$ composed of only strictly positive vectors, so that we have $\sigma > 0$ and, by *E*–3*, the fact that there is a positive number $\bar{U}$ such that $(p, \gamma)\in Q$ implies

$$\bar{U} \geqq \sum_{t=1}^{\bar{t}} M_{ti};\ i = 1, 2, \ldots, l .$$

Hence, we obtain $\sigma \bar{x}_k^t \leqq \overline{M} + \bar{U};\ t = 1, 2, \ldots, t, k = 1, 2, \ldots, n$, for every x^* in

$$\bigcup_{(p,\gamma)\in Q} \bigcup_{i=1}^l B_i(p, \gamma, M_i(p, \gamma)) .$$

Let G be a hypercube of $R^{(1+\bar{t})n}$ defined by

$$G=\{x^*\in R^{(1+\bar{t})n};\ 0\leqq x\leqq e,\ 0\leqq \bar{x}^t_k\leqq(\bar{M}+\bar{U})/\sigma,\ t=1,2,\ldots,t,\ k=1,2,\ldots,n,$$

then we have

$$G\supset \bigcup_{(p,\gamma)\in Q}\bigcup_{i}^{l} B_i(p,\gamma,M_i(p,\gamma))\,.$$

Q.E.D.

Lemma 8. $(p,\gamma)\in\Omega^n\times[0,1]$, $M_i\in\Omega^{1+\bar{t}}$ and $x^{*i}\in A_i(p,\gamma,M_i)$ imply

$$\langle p^{*\bar{t}i},x^{*i}\rangle=M_{ci}+\sum_{t=1}^{\bar{t}}M_{ti};\ i=1,2,\ldots,l\,.$$

Proof. Suppose that there exist a point (p,γ) of $\Omega^n\times[0,1]$, an element M_i of $\Omega^{1+\bar{t}}$ and a consumer i such that

$$\langle p^{*\bar{t}i},x^{*i}\rangle<M_{ci}+\sum_{t=1}^{\bar{t}}M_{ti}$$

holds for some x^{*i} in $A_i(p,\gamma,M_i)$. Let x^* be an element of X^* satisfying $x=x^i$, $\bar{x}^t=\bar{x}^{ti}$; $t=1.,2,\ldots,\bar{t}-1$, and $\bar{x}^{\bar{t}}>\bar{x}^{\bar{t}i}$, then we can find an α in $(0,1]$ for which $\alpha x^*+(1-\alpha)x^{*i}\in B_i(p,\gamma,M_i)$ holds.

On the other hand, *C*–2* and *C*–3* assure $\alpha x^*+(1-\alpha)x^{*i}\succ^*_i x^{*i}$. This contradicts $x^{*i}\in A_i(p,\gamma,M_i)$. Q.E.D.

Let us formally extend the domain of x^{*i} on which $s_i(p,\gamma,M_i,x_i)$ is characterized, to $B_i(p,\gamma,M_i)$; $i=1,2,\ldots,l$, and denote the excess supply $y+z+w-x$, and the excess investment demand $d(p,\gamma)-s(p,\gamma,\{x^i\})$, in the current market both of which correspond to $y^{*j}\in F_j(p,\beta)$; $j=1,2,\ldots,m$, and $x^{*i}\in B_i(p,\gamma,M_i(p,\gamma))$; $i=1,2,\ldots,l$, by $o(p,\gamma,\{x^i\},\{y^j\})$ and $o_I(p,\gamma,\{x^i\})$ respectively, then we can establish an extention of the Walras law in the following sense, where $d(p,\gamma)$ denotes the summation of $d_j(p,\gamma)$ over j.

Lemma 9. If $s'=d'$, $r'>0$, then

$$\langle p,o(p,\gamma,\{x^i\},\{y^j\})\rangle+o_I(p,\gamma,\{x^i\})=0$$

for all $(p,\gamma)\in\Omega^n\times[0,1]$, $y^{*j}\in F_j(p,\gamma)$; $j=1,2,\ldots,m$, and $x^{*i}\in B_i(p,\gamma,M_i(p,\gamma))$; $i=1,2,\ldots,l$.

Proof. It is clear from the construction of $s(p,\gamma,\{x^i\})$ that

$$\langle p,x\rangle+s(p,\gamma,\{x\}^i)=\sum_{i=1}^{l}M_{ci}(p,\gamma)$$

always holds.

If $s'=d'=0$, then we immediately have $c_j(p,\gamma)\leqq\langle h^j,\bar{y}^j\rangle$; $j=1,2,\ldots,m$, by the same argument with the proof of Lemma 2. It follows, therefore, that

$$\begin{aligned}\langle p,x\rangle-o_I(p,\gamma,\{x^i\})&=\sum_{i=1}^{l}M_{ci}\,(p,\gamma)-d(p,\gamma)\\&=\langle p,w\rangle+\sum_{j=1}^{m}(\pi_j\,(p,\gamma)-d_j(p,\gamma))\end{aligned}$$

$$= \langle p, w \rangle + \sum_{j=1}^{m} (\pi_j(p, \gamma) - c_j(p, \gamma))$$

$$= \langle p, w \rangle - \sum_{j=1}^{m} (b_j(p, \gamma) + \max(-\langle p, z^j \rangle, 0))$$

$$= \langle p, w + y + z \rangle ,$$

and then $\langle p, o(p, \gamma, \{x^i\}, \{y^j\}) \rangle + o_I(p, \gamma, \{x^i\}) = 0$.

In the case of $s' = d' > 0$, we also have

$$\langle p, x \rangle - o_I(p, \gamma, \{x^i\}) = \langle p, w \rangle + \sum_{j=1}^{m} (\pi_j(p, \gamma) + d'_j/\gamma' - \max(c_j(p, \gamma)$$
$$- \langle h^j, \bar{y}^j \rangle, 0) - d_j(p, \gamma))$$
$$= \langle p, w \rangle + \sum_{j=1}^{m} (\pi_j(p, \gamma) + d'_j/\gamma' - c_j(p, \gamma))$$
$$= \langle p, w \rangle + \sum_{j=1}^{m} (d'_j/\gamma' - b_j(p, \gamma) - \max(-\langle p, z^j \rangle, 0))$$
$$= \langle p, w + y + z \rangle .$$

Q.E.D.

4. *The existence theorem.*

A triplet $((p, \gamma), \{x^{*1}, x^{*2}, \ldots, x^{*l}\}, \{y^{*1}, y^{*2}, \ldots, y^{*m}\})$, of a price-interest system, a collection of consumptions and a collection of productions is called a *temporary equilibrium* of $\mathscr{E}^*$ if all the following conditions are satisfied.

$$p \geq 0, \gamma \in (0, 1], x^{*i} \in A_i(p, \gamma, M_i(p, \gamma)); i = 1, 2, \ldots, l, y^{*j} \in F_j(p, \gamma);$$
$$j = 1, 2, \ldots, m, o(p, \gamma, \{x^i\}, \{y^j\}) = 0, o_I(p, \gamma, \{x^i\}) = 0 ,$$
$$s(p, \gamma, \{x^i\}) < \hat{M} \text{ and } \sum_{i=1}^{l} M_{ci}(p, \gamma) = \hat{M} .$$

Lemma 10. Let $(p, \gamma) \to f_j(p, \gamma)$; $j = 1, 2, \ldots, m$, be a point-to-set mapping of Q into the family of all subsets of $Y_j^* \cap D$, such that $f_j(p, \gamma) = F_j(p, \gamma) \cap D$, then the image $f_j(p, \gamma)$ is non-empty, closed and convex set of $R^{(2+\bar{l})n}$ for every (p, γ) in Q and the mapping is upper semi-continuous.

Proof. Since $Y_j^* \cap D$; $j = 1, 2, \ldots, m$, is non-empty, closed and convex, by *P*-1*, *P*-2* and by the construction of D, it is clear that each $f_j(p, \gamma)$ is non-empty for every (p, γ) in Q.

The closedness of $Y_j^* \cap D$ leads to the fact that $(p, \gamma) \in Q$, $y^{*\mu} \to y^*$ and $y^{*\mu} \in f_j(p, \gamma)$; $\mu = 1, 2, \ldots$, imply $y^* \in Y_j^* \cap D$ and $\langle p^{*j}, y^* - y^{*\mu} \rangle = 0$; $\mu = 1, 2, \ldots$. This means $y^* \in f_j(p, \gamma)$, so that each $f_j(p, \gamma)$ is a closed set of $R^{(2+\bar{l})n}$ for every (p, γ) in Q.

Let y^* and y'^* be two elements of $f_j(p, \gamma)$, then we have $\langle p^{*j}, y^* - y'^* \rangle = 0$ and, by the convexity of $Y_j^* \cap D$, $\alpha y^* + (1 - \alpha) y'^* \in y_j^* \cap D$ for every α in $[0, 1]$. Hence, $\alpha y^* + (1 - \alpha) y'^* \in f_j(p, \gamma)$; $\alpha \in [0, 1]$.

Finally, it follows from the continuity of inner-product and from the closedness of $Y_j^* \cap D$ that $(p, \gamma)^\mu \to (p, \gamma)$; $(p, \gamma)^\mu \in Q$, $y^{*\mu} \to y^*$ and $y^{*\mu} \in f_j((p, \gamma)^\mu)$; $\mu = 1, 2, \ldots$, imply both $y^* \in Y_j^* \cap D$ and $\langle p^{*j}, y^* - y'^* \rangle \geqq 0$ for every y'^* in $Y_j^* \cap D$, so that we obtain, $y^* \in f_j(p, \gamma)$, the upper semi-continuity of each $(p, \gamma) \to f_j(p, \gamma)$.

Q.E.D.

Lemma 11. each $(p, \gamma) \to b_j(p, \gamma)$ is a continuous point-to-point mapping of Q into R^1.

Proof. It is clear from *P*–7* that $b_j(p, \gamma)$ is univalent for every j and every (p, γ) in Q.

Let us define a continuous point-to-point mapping $(p, y^*) \to b_j'(p, y^*)$ of $P(\delta/2, 2\eta) \times (Y_j^* \cap D)$ into R^1, by $b_j'(p, y^*) = d_j'/r' - \langle p, y \rangle - \max(\langle p, z \rangle, 0)$, then $y^* \in f_j(p, \gamma)$ implies $b_j'(p, y^*) = b_j(p, \gamma)$ in Q. Therefore, we can interpret $(p, \gamma) \to b_j(p, \gamma)$ as a composite mapping of $(p, \gamma) \to (p, f_j(p, \gamma))$ and $(p, y^*) \to b_j'(p, y^*)$.

Since, by Lemma 10, $(p, \gamma) \to f_j(p, \gamma)$ is an upper semi-continuous mapping such that each $f_j(p, \gamma)$ is a closed convex subset of a compact set $Y_j^* \cap D$, we see that $(p, \gamma) \to b_j(p, \gamma)$ is an upper semi-continuous mapping of Q into a bounded closed interval K, of R^1 (cf. [14] page 71, Theorem 4.6).

The compactness of K leads to a fact that, if $(p, \gamma)^\mu \to (p, \gamma) \in Q$; $(p, \gamma)^\mu \in Q$, $\mu = 1, 2, \ldots$, then $\{b_j((p, \gamma)^\mu)\}$; $\mu = 1, 2, \ldots$, has an infinite subsequence which converges to $b_j(p, \gamma)$. Furthermore, it follows from the univalency of $b_j(p, \gamma)$ in Q that there is no other subsequence of $\{b_j((p, \gamma)^\mu)\}$ which has a limit point $b \in K$ such that $b \neq b_j(p, \gamma)$.

Thus, we ultimately have,

$$b_j(p, \gamma) = \lim_{\mu \to \infty} b_j((p, \gamma)^\mu) ,$$

the continuity of the mapping $(p, \gamma) \to b_j(p, \gamma)$; $j = 1, 2, \ldots, m$.

Q.E.D.

The following assertion is an immediate consequence of the above lemma.

Corollary of Lemma 11. Each of $\pi_j(p, \gamma)$; $j = 1, 2, \ldots, m$, $d_j(p, \gamma)$; $j = 1, 2, \ldots, m$, and $M_{ci}(p, \gamma)$; $i = 1, 2, \ldots, l$, is a single-valued continuous function of (p, γ) in Q.

Lemma 12. Let $(p, \gamma) \to g_i(p, \gamma)$; $i = 1, 2, \ldots, l$, be a point-to-set mapping of Q into the family of all subsets of G, such that

$$\begin{aligned} g(p, \gamma) &= A_i(p, \gamma, M_i(p, \gamma)) \quad \text{if } (p, \gamma) \in \{(p, \gamma) \in Q;\ M_{ci}(p, \gamma) > 0\} , \\ &= B_i(p, \gamma, M_i(p, \gamma)) \quad \text{if } (p, \gamma) \in \{(p, \gamma) \in Q;\ M_{ci}(p, \gamma) = 0\} , \end{aligned}$$

then the image $g_i(p, \gamma)$ is non-empty, closed and convex set of $R^{(1+\vec{t})n}$ for every (p, γ) in Q and the mapping is upper semi-continuous.

Proof. First, the non-emptiness and the closedness of $B_i(p, \gamma, M_i(p, \gamma))$; $i = 1, 2, \ldots, l$, $(p, \gamma) \in Q$, and *C*–1* imply $A_i(p, \gamma, M_i(p, \gamma)) \neq \phi$; $i = 1, 2, \ldots, l$, $(p, \gamma) \in Q$, and the closedness of each $A_i(p, \gamma, M_i(p, \gamma))$, so that each $g_i(p, \gamma)$ is a

non-empty closed set of $R^{(1+\bar{t})n}$ for every (p, γ) in Q.

Since each $B_i(p, \gamma, M_i(p, \gamma))$ is convex, we immediately have, by C^*-2, the convexity of each $g_i(p, \gamma)$.

Taking the contiuities of $M_i(p, \gamma)$ and inner-products into account, we can see that $(p, \gamma)^\mu \to (p, \gamma) \in Q$; $(p, \gamma)^\mu \in Q$, $\mu = 1, 2, \ldots$, $x^{*\mu} \to x^*$ and $x^{*\mu} \in g_i((p, \gamma)^\mu)$; $\mu = 1, 2, \ldots$, imply $x^* \in B_i(p, \gamma, M_i(p, \gamma))$. In order to complete the proof, therefore, it suffice to show that, if $M_{ci}(p, \gamma) > 0$ holds in addition to the above-mentioned conditions, then $x^* \in A_i(p, \gamma, M_i(p, \gamma))$.

Suppose that $x^* \notin A_i(p, \gamma, M_i(p, \gamma))$, then there is an $x'^* \in B_i(p, \gamma, M_i(p, \gamma))$ such that $x'^* \succ_i^* x^*$. Since $M_{ci}(p, \gamma) > 0$, we can find, by means of C^*-1, an $x''^* \in B_i(p, \gamma, M_i(p, \gamma))$ in the neighborhood of x'^* such that

$$\langle p, x'' \rangle < M_{ci}(p, \gamma),$$

$$\langle p^{*ti}, x''^{*t} \rangle < M_{ci}(p, \gamma) + \sum_{q=1}^{t} M_{ti};\ t = 1, 2, \ldots, \bar{t},$$

$$x''^* \succ_i^* x^*.$$

Let us take a sufficiently large μ_0, then, for every $\mu \geqq \mu_0$, we have, by the continuities of $M_i(p, \gamma)$ and $p^{*\bar{t}i}$, $M_{ci}((p, \gamma)^\mu) > 0$ and $x''^* \in B_i((p, \gamma)^\mu, M_i((p, \gamma)^\mu))$, and, by C^*-1, $x''^* \succ_i^* x^{*\mu}$. But those conditions are clearly incompatible with $x^{*\mu} \in g_i((p, \gamma)^\mu)$. Q.E.D.

Let H be a Cartesian product denoted by

$$H = Q \times \underbrace{G \times G \times \ldots \times G}_{l} \times \prod_{j=1}^{m} (Y_j^* \cap D)$$

and θ be an element of H. Necessarily, $\theta = ((p, \gamma), \{x^{*i}\}, \{y^{*j}\})$. It is clear that H is a compact convex set of $R^{(1+n)+(1+\bar{t})nl+(2+\bar{t})nm}$.

The domain on which each of $b_j(p, \gamma)$, $c_j(p, \gamma)$, $\pi_j(p, \gamma)$ and $d_j(p, \gamma)$ were defined is formally shifted, to all points of H, the previous definitions of $s_i(p, \gamma, M_i(p, \gamma), x^i)$ and $M_{ci}(p, \gamma)$ are slightly modified as in the following.

$$M_{ci}(\theta) = \langle p, w^i \rangle + \sum_{j=1}^{m} \beta_{ji} \pi_j(\theta) + s_i' \max \Big\{ 1/\gamma' - \sum_{j=1}^{m} \max (c_j(\theta)$$

$$- \langle h^j, \bar{y}^j \rangle, 0)/s', 0 \Big\};\ i = 1, 2, \ldots, l,$$

$$s_i(\theta) = \max (M_{ci}(\theta) - \langle p, x \rangle, 0);\ i = 1, 2, \ldots, l.$$

Those $d_j(\theta)$, $s_i(\theta)$ and $M_{ci}(\theta)$ are clearly continuous with respect to θ in H.

Let $o(\theta)$, $\lambda(\theta)$ and $o_M(\theta)$ be defined by

$$o(\theta) = y + z + w - x,$$

$$\lambda(\theta) = \sum_{k=1}^{n} \max (- o_k(\theta), 0);\ o_k(\theta) = y_k + z_k + w_k - x_k,\ k = 1, 2, \ldots, n,$$

$$o_M(\theta) = \sum_{k=1}^{1} M_{ci}(\theta) - \hat{M},$$

then those functions of θ are obviously continuous in H. Hence, by choosing a sufficiently small $\varphi > 0$, we obtain that $\theta \in H$ implies

$$1/2 \leqq (1 + \lambda(\theta))/(1 + \lambda(\theta) + \varphi o_M(\theta)) \leqq 2.$$

The following holds for such a $\varphi > 0$.

Lemma 13. Let $q(\theta)(= (q_1(\theta), q_2(\theta), \ldots, q_n(\theta)))$ be defined by

$$q_k(\theta) = (p_k + \max(-o_k(\theta), 0) \sum_{k=1}^{n} p_k)/(1 + \lambda(\theta) + \varphi o_M(\theta)); k = 1, 2, \ldots, n,$$

then $\theta \to q(\theta)$ is a continuous point-to-point mapping of H into $P(\delta/2, 2\eta)$.

Proof. The continuity and the univalency of the mapping follow directly from the above discussion.

On the other hand, each $M_{ci}(\theta)$ fulfills $M_{ci}(\theta) \geqq \langle p, w^i \rangle$; $i = 1, 2, \ldots, l$, for any $\theta \in H$, and then the assertion of Lemma 4 is also valid for $M_{ci}(\theta)$; $i = 1, 2, \ldots, l$, that is, $o_M(\theta) > 0$ (or < 0) holds for every θ in H such that

$$2\eta \geqq \sum p_k > \eta \left(\text{or } \delta > \sum_{k=1}^{n} p_k \geq \delta/2 \right).$$

This proves that $q(\theta) \in P(\delta/2, 2\eta)$ for all points of H. Q.E.D.

Let $o_I(\theta)$ and $q_I(\theta)$ be defined by

$$o_I(\theta) = \sum_{j=1}^{m} d_j(\theta) - \sum_{i=1}^{l} s_i(\theta),$$

$$q_I(\theta) = \{\gamma + \max(-o_I(\theta), 0)\}/(1 + |o_I(\theta)|),$$

then we have the following

Lemma 14. $\theta \to q_I(\theta)$ is a continuous point-to-point mapping of H into the closed interval [0, 1].

Now, by making use of Kakutani's fixed point theorem (cf. Appendix), we can prove the following principal lemma.

Lemma 15. There is an element $\hat{\theta}(= ((\hat{p}, \hat{\gamma}), \{\hat{x}^*\}^i, \{\hat{y}^{*j}\}))$ of H such that $(\hat{p}, \hat{\gamma}) = (q(\hat{\theta}), q_I(\hat{\theta}))$, $\hat{x}^{*i} \in g_i(\hat{p}, \hat{\gamma})$; $i = 1, 2, \ldots, l$, and $y^{*j} \in f_j(\hat{p}, \hat{\gamma})$; $j = 1, 2, \ldots, m$.

Proof. From the preceding lemmata 10, 12, 13 and 14, it follows that $\theta \to ((q(\theta), q_I(\theta)), g_1(p, \gamma), g_2(p, \gamma), \ldots, g_l(p, \gamma), f_1(p, \gamma), f_2(p, \gamma), \ldots, f_m(p, \gamma))$ is an upper semi-continuous point-to-set mapping of a compact convex set H into the family of all closed convex subsets of H. We can apply, therefore, the Kakutani's theorem, which ensures the existence of a $\hat{\theta} \in H$ such that $(\hat{p}, \hat{\gamma}) = (q(\hat{\theta}), q_I(\hat{\theta}))$, $\hat{x}^{*i} \in g_i(\hat{p}, \hat{\gamma})$; $i = 1, 2, \ldots, l$, $\hat{y}^{*j} \in f_j(\hat{p}, \hat{\gamma})$; $j = 1, 2, \ldots, m$.

Q.E.D.

Theorem 4. The triplet $((\hat{p}, \hat{\gamma}), \{\hat{x}^{*i}\}, \{\hat{y}^{*j}\})$ given in Lemma 15 constitutes a temporary equilinrium of $\mathscr{E}^*$.

Proof. First,

$$\hat{p} \geq 0$$

and

$$\underline{\hat{y}^{*j} \in F_j(\hat{p}, \hat{\tau});\ j = 1, 2, \ldots, m,}$$

immediately follow from the above-mentioned properties of $\hat{\theta}$, so that we have $d_j(\hat{\theta}) = d_j(\hat{p}, \hat{\tau});\ j = 1, 2, \ldots, m$, and, by Lemma 2, $M_{ci}(\hat{\theta}) = M_{ci}(\hat{p}, \hat{\tau});\ i = 1, 2, \ldots, l$.

The equality $\hat{\tau} = q_I(\hat{\theta})$ means $|o_I(\hat{\theta})|\hat{\tau} = \max(-o_I(\hat{\theta}), 0)$. Hence, if $o_I(\hat{\theta}) > 0$, then we must have $\hat{\tau} = 0$, which implies $d_j(\hat{\theta}) \leqq \langle 0, \bar{y}^j \rangle = 0;\ j = 1, 2, \ldots, m$. Since $s_i(\theta) \geqq 0;\ i = 1, 2, \ldots, l$, for any $\theta \in H$ by the definition, we have, $o_I(\hat{\theta}) \leqq 0$, a contradiction. Consequently, $o_I(\hat{\theta})$ must be non-positive.

On the other hand, the equality $\hat{p} = q(\hat{\theta})$ can be rewritten by

$$(\lambda(\hat{\theta}) + \varphi o_M(\hat{\theta}))\hat{p}_k = \max(-o_k(\hat{\theta}), 0) \sum_{k=1}^{n} \hat{p}_k;\ k = 1, 2, \ldots, n.$$

Multiplying the k-th equality by $o_k(\hat{\theta});\ k = 1, 2, \ldots, n$, and summing over them, we see

$$(\lambda(\hat{\theta}) + \varphi o_M(\hat{\theta}))\langle \hat{p}, o(\hat{\theta}) \rangle = -\left(\sum_{k=1}^{n} \hat{p}_k\right)\sum_{k=1}^{n} (\max(-o_k(\hat{\theta}), 0))^2 \leqq 0.$$

Since $x^{*i} \in g_i(\hat{p}, \hat{\tau});\ i = 1, 2, \ldots, l$, we obtain, by Lemma 9, $\langle \hat{p}, o(\hat{\theta}) \rangle + o_I(\hat{\theta}) = 0$, which, combining with $o_I(\hat{\theta}) \leqq 0$, implies $\langle \hat{p}, o(\hat{\theta}) \rangle \geqq 0$. The above k equalities conclude $(\lambda(\hat{\theta}) + \varphi o_M(\hat{\theta}))\hat{p}_k \geqq 0;\ k = 1, 2, \ldots, n$, so that we have, by $\hat{p} \geq 0$, $\lambda(\hat{\theta}) + \varphi o_M(\hat{\theta}) \geqq 0$. Hence, $\max(-o_k(\hat{\theta}), 0) = 0,\ k = 1, 2, \ldots, n$, and then $\lambda(\hat{\theta}) = 0$. We, thus, obtain $\lambda(\hat{\theta}) + \varphi o_M(\hat{\theta}) = 0$, which ensures

$$\underline{\sum_{i=1}^{l} M_{ci}(\hat{p}, \hat{\tau}) = \hat{M}.}$$

We next show that $\hat{p}$ is strictly positive. Since $o(\hat{\theta}) \geqq 0$, we see from Lemma 1 $\hat{x}^i < e;\ i = 1, 2, \ldots, l$. Let $\hat{p}_k = 0$ for some k and $x^{*i};\ i = 1, 2, \ldots, l$, be points of C such that $x_k^i = e_k$, $x_q^i = \hat{x}_q^i$ for every $q \neq k$, and $\bar{x}^i = \hat{\bar{x}}^i;\ i = 1, 2, \ldots, l$, then we have, from the fact that $\hat{x}^{*i} \in g_i(\hat{p}, \hat{\tau});\ i = 1, 2, \ldots, l$, $x^{*i} \in B_i(\hat{p}, \hat{\tau}, M_i(\hat{p}, \hat{\tau}));\ i = 1, 2, \ldots, l$, and, by C^*–3, $x^{*i} \succ_i^* \hat{x}^{*i};\ i = 1, 2, \ldots, l$. Seeing that $\hat{p} \geq 0$, however, we must have, by Lemma 3, $\hat{x}^{*i} \in A_i(\hat{p}, \hat{\tau}, M_i(\hat{p}, \hat{\tau}))$ for at least one i, which contradicts the existence of an x^{*i} such that $x^{*i} \in B_i(\hat{p}, \hat{\tau}, M_i(\hat{p}, \hat{\tau}))$ and $x^{*i} \succ_i^* \hat{x}^{*i}$. Therefore, $\hat{p} > 0$, and by Lemma 3, we obtain

$$\underline{\hat{x}^{*i} \in A_i(\hat{p}, \hat{\tau}, M_i(\hat{p}, \hat{\tau}));\ i = 1, 2, \ldots, l.}$$

Now, it follows that $o_I(\hat{\theta}) = o_I(\hat{p}, \hat{\tau}, \{\hat{x}^i\})$. Suppose $o_I(\hat{\theta}) < 0$, then we readily see from the equality $|o_I(\hat{\theta})|\hat{\tau} = \max(-o_I(\hat{\theta}), 0)$ that $\hat{\tau} = 1$, which implies, by C^*–7, $s(\hat{p}, \hat{\tau}, \{\hat{x}^i\}) = 0$. But $d(\hat{p}, \hat{\tau}) \geqq 0$ since $\hat{y}^{*j} \in F_j(\hat{p}, \hat{\tau});\ j = 1, 2, \ldots, m$, so that we have, $o_I(\hat{\theta}) \geqq 0$, a contradiction. Hence,

$$\underline{o_I(\hat{p}, \hat{\tau}, \{\hat{x}^i\}) = 0.}$$

Let $\hat{\tau} = 0$, then $s(\hat{p}, \hat{\tau}, \{\hat{x}^i\}) > 0$, by Lemma 5. Since $d(p, 0) = 0$ for any $p \in \Omega^n$,

by the construction, we must have $o_I(\hat{p}, \hat{r}, \{\hat{x}^i\}) < 0$. Hence,

$$\underline{\hat{r} > 0\,.}$$

By Lemma 9, we obtain $\langle \hat{p}, o(\hat{\theta})\rangle + o_I(\hat{\theta}) = 0$, which ensures

$$\underline{o(\hat{p}, \hat{r}, \{\hat{x}^i\}, \{\hat{y}^j\}) = 0}$$

since $\hat{p} > 0$ and $o_I(\hat{\theta}) = 0$.

Finally from *C*–6*, we see

$$\underline{\hat{M} - s(\hat{p}, \hat{r}, \{\hat{x}^i\}) > 0\,.}$$

Thus, we have obtained all the conditions on the triplet to be a temporary equilibrium of $\mathscr{E}^*$.

Q.E.D.

Suppose that a positive amount of money has been supplied in the first period, then it follows inductively from Theorem 4 that, under the assumptions mentioned in this chapter, $\mathscr{E}^*$ has a temporary equilibrium at every period.

5. *Remark.*

So far, the region in which each economic agent maximizes his satisfaction level or his long-run profit has been confined to C or E.

Although the restriction on the maximizing region seems to be quite artificial from an economic viewpoint, the following assertion shows that the limitation can be, substantially, eliminated.

Assertion. Let $F_j'(p, \gamma)$; $j = 1, 2, \ldots, m$, $B_i'(p, \gamma, M_i)$ and $A_i'(p, \gamma, M_i)$; $i=1, 2, \ldots, l$, be defined by

$$F_j'(p,) = \{y^* \in Y_j^*;\ \langle p^{*j}, y^* - y'^*\rangle \geqq 0 \text{ for any } y'^* \in Y_j^*\}\,,$$

$$B_i'(p, \gamma, M_i) = \Big\{x^* \in X^*;\ \langle p, x\rangle \leqq M_{ci},\ \langle p^{*ti}, x^{*t}\rangle \leqq M_{ci} + \sum_{q=1}^{t} M_{qi},$$

$$t = 1, 2, \ldots, \bar{t}\Big\}\,,$$

$$A_i'(p, \gamma, M_i) = \{x^* \in B_i'(p, \gamma, M_i);\ x^* \gtrsim_i^* x'^* \text{ for any } x'^* \in B_i'(p, \gamma, M_i)\}\,,$$

then the triplet $((\hat{p}, \hat{r}), \{\hat{x}^{*i}\}, \{\hat{y}^{*j}\})$ given in Lemma 15 satisfies $\hat{x}^{*i} \in A_i'(\hat{p}, \hat{r}, M_i(\hat{p}, \hat{r}))$; $i = 1, 2, \ldots, l$, and $y^{*j} \in F_j'(\hat{p}, \hat{r})$; $j = 1, 2, \ldots, m$.

Proof. Suppose that there is an i (or a j) such that $x^* \in A_i'(\hat{p}, \hat{r}, M_i(\hat{p}, \hat{r}))$ and $x^* \succ_i^* \hat{x}^{*i}$ (or $y^* \in F_j'(\hat{p}, \hat{r})$ and $\langle \hat{p}^{*j}, \hat{y}^{*j} - y^*\rangle < 0$) hold for some x^* (or for some y^*), then we must have $x^* \notin C$ (or $y^* \notin E$).

Since $\hat{x}^{*i}$ (or $\hat{y}^{*j}$) fulfills $\hat{x}^i < e$ (or $\hat{y}^j > -e$ and $\hat{z}^j > -e$) by Lemma 1, we can find, by the convexity of $B_i'(\hat{p}, \hat{r}, M_i(\hat{p}, \hat{r}))$ (or by *P*–1*), an $\alpha \in (0, 1)$ such that $\alpha x^* + (1-\alpha)\hat{x}^{*i} \in B_i'(\hat{p}, \hat{r}, M_i(\hat{p}, \hat{r})) \cap C = B_i(\hat{p}, \hat{r}, M_i(\hat{p}, \hat{r}))$ (or $\alpha y^* + (1-\alpha)\hat{y}^{*j} \in Y_j^* \cap E$). *C*–2* (or the linearity of inner-products) implies, however, $\alpha x^* + (1-\alpha)\hat{x}^{*i} \succ_i^* \hat{x}^{*i}$ (or $\langle \hat{p}^{*j}, \hat{y}^{*j} - (\alpha y^* + (1-\alpha)\hat{y}^{*j})\rangle < 0$), which con-

tradicts the maximality of $A_i(\hat{p}, \hat{r}, M_i(\hat{p}, \hat{r}))$ in $B_i(\hat{p}, \hat{r}, M_i(\hat{p}, \hat{r}))$, for $\succ_i^*$ (or of $F_j(\hat{p}, \hat{r})$ in $Y_j^* \cap E$, for the inner-product evaluated by $\hat{p}^{*j}$).

Q.E.D.

Chapter 4

CONCLUSION

We have given, in Chapter 2, some uniformity conditions under which the core of an exchange economy shrinks, in a strict sense, to the set of competitive allocations as the number of consumers in the economy tends to infinity. It has been also shown that the lack of assumption *C*–5, which has played an essential role in the process of demonstrating the preceding proposition, leads to an example of exchange economy in which the core includes a non-ε-competitive allocation even if the number of consumers becomes infinite.

As has been proved by Aumann [3], however, only continuity and desirability are the two conditions on consumers' preference relations which are enough to verify the corresponding assertion in a continuum consumers' case, and then convexity is eliminated. The reason why the restrictive assumption such as *C*–5 on the convexity of preference relations is needed in order to solve our problem is an open question. But, it would be natural to presume that the meaning of convergence which we have used in the present discussion is too rigorous to exclude the uniformity condition on the convexity, so that we would be able to obtain conditions of different kinds, under which our problem is solved in a weak sense of convergence without the convexity assumption. At this point, our counter example does not necessarily imply that there is an allocation in the core which assigns non-ε-competitive final resources for considerably many consumers in comparison with the number of all the consumers. It requires, in all events, to fill the gap between the Aumann's model and finitely many consumers' cases.

In Chapter 3, we have turned out the establishment of the existence of temporary equilibrium for a competitive economy in which each agent acts on the current market as a maximizer of his long-run profit or of his long-run satisfaction level based on the forecast of price systems in the future periods.

Investigation on the dynamic property of temporary equilibrium such as its fluctuation or its stability has not been done in the present discussion. In order to develop the microscopic theory of economic dynamics for a generalized Walrasian system, however, many more studies of the fluctuation or the stability of temporary equilibrium are needed. The proof of an existence theorem of temporary equilibrium is merely an initial step in an extention of general equilibrium analysis to economic dynamics.

APPENDIX

Separation theorem ([14] page 28). Let X be a convex set of R^n. If an $x' \in X$ is not an interior point of X, then there exists a hyperplane $\langle p, x - x' \rangle = 0$, passing through x' such that X is included in the half space $\{x; \langle p, x - x' \rangle \geqq 0\}$, of R^n, where $p \neq 0$.

Carathéodory's theorem ([10] page 103). If X is a set of R^n, then each point of the convex hull of X can be represented by a convex combination of at most $n + 1$ points of X.

The continuity of the mapping $x \to p^i(x)$, of X_i into P (Chapter 2 page 41).

Proof. Suppose, to the contrary, that there exists a convergent sequence $\{x^\nu\}$ in X_i with a limit point $x \in X_i$, on which $|p^i(x) - p^i(x^\nu)| \geqq \varepsilon$; $\nu = 1, 2, \ldots$, hold for some $\varepsilon > 0$. Then, clearly there exists a $\gamma > 0$ such that $\langle p^i(x), z^\nu \rangle \geqq \gamma$; $\nu = 1, 2, \ldots$, where $z^\nu = p^i(x) - p^i(x^\nu)$; $\nu = 1, 2, \ldots$.

Let J be a set of R^n defined by $J = \{z \in R^n; |z| \leqq \sqrt{n}, \langle p^i(x), z \rangle \geqq \gamma\}$, then we immediately have $z^\nu \in J$; $\nu = 1, 2, \ldots$. Hence by *C-5*, we can find a $\beta > 0$ such that $x + \beta z^\nu \succ_i x$; $\nu = 1, 2, \ldots$.

On the other hand, $\langle p^i(x^\nu), z^\nu \rangle = \langle p^i(x^\nu), p^i(x) \rangle - 1 = \langle p^i(x), p^i(x^\nu) - p^i(x) \rangle = \langle p^i(x), -z^\nu \rangle \leqq -\gamma$; $\nu = 1, 2, \ldots$, so that there is a ν_0 such that $\nu \geqq \nu_0$ implies $\langle p^i(x^\nu), \beta z^\nu + (x - x^\nu) \rangle = \langle p^i(x^\nu), (x + \beta z^\nu) - x^\nu \rangle < 0$. It follows, therefore, that $x^\nu \succ_i x + \beta z^\nu \succ_i x$ for every $\nu \geqq \nu_0$.

Next, we see from $\langle p^i(x^\nu), -z^\nu \rangle \geqq \gamma$; $\nu = 1, 2, \ldots$, that there is a $\beta' > 0$ such that $x^\nu - \beta' z^\nu \succ_i x^\nu$; $\nu = 1, 2, \ldots$. Since $\langle p^i(x), z^\nu \rangle \geqq \gamma$, there is a ν_1 such that $\nu \geqq \nu_1$ implies $\langle p^i(x), -\beta' z^\nu + (x^\nu - x) \rangle = \langle p^i(x), (x^\nu - \beta' z^\nu) - x \rangle < 0$. Hence, we have $x \succ_i x^\nu - \beta' z^\nu \succ_i x^\nu$ for every $\nu \geqq \nu_1$, which is, however, incompatible with the fact that $x^\nu \succ_i x$ for every $\nu \geqq \nu_0$.

Q.E.D.

The definition of upper semi-continuity. A point-to-set mapping $x \to F(x)$ of X into 2^Y is said to be upper semi-continuous if $x^\nu \to x \in X$; $x^\nu \in X$, $y^\nu \in F(x^\nu)$ and $y^\nu \to y$ imply $y \in F(x)$, where 2^Y denotes the family of allsubsets of Y.

Kakutani's fixed point theorem ([9] Corollary). Let X be a bounded closed convex set of a Euclidian space and $C(X)$ be the family of all closed convex subsets of X. Then, an upper semi-continuous point-to-set mapping $x \to F(x)$ of X into $C(X)$ has a point x°, of X such that $x^\circ \in F(x^\circ)$.

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