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ONE-PERIOD EQUILIBRIUM IN THE VON NEUMANN MODEL*

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1. INTRODUCTION

One of the fundamental notions of modern economic analysis is that of *proportional growth* representing a state in which a competitive economic system grows at a constant rate and maintains an unchanged structure. This notion has been for the first time rigorously analysed in the von Neumann model [5], in which competitive equilibrium is defined directly as a state of proportional growth. The central part of von Neumann's paper consists of the proof of the existence of such an equilibrium.

The essential phenomenon von Neumann explicitly states he wants to study is that "goods are produced from each other" [5, p. 2]. From this approach a concept of price comes out which is different from the traditional (walrasian) one. In fact, if "goods are produced not only from 'natural factors of production', but in the first place from each other" [5, p. 1] their prices must mainly represent a measure of costs of production in terms of the goods themselves.

In order to isolate such a phenomenon, von Neumann leaves out of consideration the primary factors of production, and also the phenomenon of consumption, by introducing hypotheses which allow him to remove them from the scene.

From this point of view, the actual definition of equilibrium may be considered as a means von Neumann uses to emphasize the *circular* nature of the productive process and the corresponding price concept. In fact, if the initial quantities of goods available for the production of goods themselves were arbitrarily given, this would influence prices, which should reflect, at least partly, possible disproportions in such initial quantities. The definition of equilibrium as a state of proportional growth has the effect of transferring initial conditions from the category of data to that of variables and therefore allows the complete elimination of their influence. Now, prices only express the *technical* difficulty of reproducing goods in terms of goods themselves.

Although von Neumann's approach is well suited to focus the properties exclusively deriving from the reproduction mechanism of goods, it gives rise to serious shortcomings as far as the descriptive power of the model is concerned. In order to explain this difficulty, we shall consider how the notion of proportional growth is ordinarily used in the recent models of capital accumulation. At first, on the basis of certain assumptions, a one-period (short-run) equilibrium, depend-

* This is the English translation of a paper appeared, in a slightly different from, in [6].

ing on initial conditions, is defined and only afterwards one asks whether there exists that particular equilibrium represented by the proportional growth, which requires particular initial conditions in order to be realised. The models which are constructed in this way have, at least potentially, a descriptive power, since they are able to determine an equilibrium, *whatsoever conditions inherited from the past may be.*

On the contrary, in the von Neumann model, the first phase is completely neglected and equilibrium is directly defined as a state of proportional growth. It immediately follows that such a model may describe a competitive system if, and only if, initial endowments of goods are in the proportions which are required by conditions of proportional growth. In other words, the von Neumann model, as it is, cannot describe the situations in which initial stocks have a composition different from that compatible with proportional growth. This, of course, automatically deprives the model of any descriptive power, since in general initial stocks of goods, *historically given*, are not available in the proportions consistent with proportional growth.

It is the purpose of this note to overcome such a difficulty by filling the gap left by von Neumann, namely by constructing the one-period model underlying that of proportional growth. In one sense, we shall be pursuing a backward path. Instead of starting from a one-period model and of arriving at its proportional growth solution, as is usually done, we shall start from a state of proportional growth and try to go back to the corresponding one-period model with *given* initial stocks of goods.

This procedure is not without its dangers. In fact, the von Neumann model could, in theory, represent a state of proportional growth corresponding to more than one one-period model. We shall try to remain as close as possible to the spirit of the von Neumann model, by maintaining unaltered the mechanism of accumulation implicit in it. That is to say, we shall show that such a mechanism can determine a competitive equilibrium also *outside* the path of proportional growth.

2. THE ORIGINAL VON NEUMANN MODEL

Let us briefly consider the von Neumann model. Technology is represented by the two nxm non-negative matrices A and B, whose elements a_{ij} and b_{ij} are the quantity used (input) and, respectively, produced (output) of commodity i in activity j. That is, there are n commodities and m activities. An activity is represented by a column of A and by the corresponding column of B. The activities are linear and additive. One supposes that the transformation defined by each activity comes about in one unit of time, that is one supposes that outputs emerge exactly one unit of time after inputs have been introduced into the productive process.

The demand of consumer goods in ruled out as an autonomous element through the drastic assumptions that capitalists do not consume, but reinvest all their income, and that "consuption of goods takes place only through the processes of production which include necessities of life consumed by workers and employees" [5, p. 2].

As far as primary factors of production, including labour, are concerned, it is assumed that they "can be expanded in unlimited quantities" [5, p. 2].

Von Neumann defines as *equilibrium* that state in which the *economic system grows steadily, keeping its structure unchanged.* That is, if y_j indicates the equilibrium level of activity j $(j = 1, 2, \ldots, m)$ in any period, the equilibrium level of the same activity in the following period is αy_j, where α is the constant growth factor of the economy.

The variables to be determined are: $m - 1$ relative activity levels, the growth factor, the $n - 1$ relative prices of goods and the interest factor. In all, $n + m$ variables.

In order to determine such variables, we have $n + m$ relations. The first group of relations is constructed on the basis of the consideration that in every period the quantity of every good used as input cannot be greater than that produced in the previous period. Bearing in mind that the system grows proportionally, we must have

$$\alpha A y \leqq B y \tag{1}$$

where y is the column vector of relative activity levels.

On the other hand, the second group of relations is constructed by considering that in conditions of perfect competition one cannot have positive profits. In other words, taking into account that costs are sustained at the beginning of every period while revenue is obtained at the end, costs cannot be lower than discounted revenue, that is

$$\beta p A \geqq p B \tag{2}$$

where p is the row vector of relative prices of various goods.

Equilibrium levels and prices must, besides, satisfy the two additional competitive conditions: a) if a relation (1) is a strict inequality, that is if a good is redundant, then its price must be zero; b) if a relation (2) is a strict inequality, that is if the operation of an activity implies costs higher than discounted revenue, its level must be zero.

In order to prove the existence of economically meaningful[1] solutions—proof based on a generalisation of Brouwer's fixed-point theorem—von Neumann has made a very unrealistic assumption concerning the technology. That is, he has assumed that every good appears in every activity as input and/or as output ($a_{ij} + b_{ij} > 0$ for all i and j). Only much later, Gale [2] and, at the same time, Kemeny, Morgenstern and Thompson [4] proved the existence of solutions by sub-

(1) Activity levels and prices must be non-negative with at least one postitive level and one positive price.

stituting the above assumption with the following perfectly acceptable one: every activity uses at least one input and every good can be produced by at least one activity. Such an assumption may be expressed as follows

(*) 1. Every column of A has at least one positive element.

2. Every row of B has at least one positive element.

3. ONE-PERIOD EQUILIBRIUM

Our task is now that of showing how the forces which guide the accumulation mechanism represented by the von Neumann model can determine an equilibrium also outside the state of proportional growth. By considering the equilibrium in a certain period with arbitrary initial endowments of goods, the nature and the way of working of such a mechanism will emerge more clearly.

The *given* initial endowments of goods are represented by the positive vector S.

To keep as closely as possible to the original version of the model, we will maintain all other assumptions made by von Neumann unchanged, excluding the assumption of proportional growth.

Technology is always represented by the two matrices A and B.

The only economic agents which appear explicitly are capitalists. They are in a fairly large number, so that conditions of perfect competition can prevail. As happens in the schemes of classical economists, these capitalists are simultaneously proprietors of the means of production and entrepreneurs. They periodically reinvest all their income and endeavour to maximize expected profits.

Let us suppose we are at the beginning of period t. All decisions must be taken at this moment. During the period, economic agents will do nothing but carry out the production plans decided at the initial moment.

We now consider the relations defining our one-period equilibrium. The amounts of goods used as inputs cannot of course be greater than initial endowments. That is, it must be[2]

$$Ay \leqq S \tag{3}$$

where y is now the vector of absolute activity levels.

Besides, the assumption of perfect competition implies that, if a good is redundant, i.e. if a relation (3) is a strict inequality, then the corresponding price must be zero, that is

$$\text{if } \sum_{j=1}^{m} a_{ij}y_j < S_i \text{ for some } i, \text{ then } p_i = 0\,. \tag{3'}$$

The decisions regarding which activities must be operated depend on the profitability of the activities themselves, which in turn depends on the comparison between costs and revenue. Whilst, however, costs of operating the various ac-

(2) Every variable is intended to be referred to period t. However, to simplify the notation, the time subscript is omitted.

tivities must be sustained at the beginning of period t, when inputs are used, revenue will be obtained only in a unit time, at the beginning of period $t + 1$, when outputs appear. To make possible a comparison between costs and revenue, it is, therefore, necessary *i*) to make some assumptions regarding price expectations and ii) to discount expected revenue. In a very general way, it could be assumed that expected prices are functions (continuous, non-negative and never simultaneously zero) of current prices and of those of preceding periods(3). However, the simplest hypothesis, which is implicit in the von Neumann model and which we shall maintain, is that expectations are *static*, namely that prices expected to obtain at the beginning of the following period are equal to current prices.

The vector of current costs at which the various activities may be operated is pA, while the vector of corresponding discounted expected revenues is, because of the static expectations assumption, $(1/1 + r)pB$, where r is the current rate of interest. In competitive conditions current costs cannot be lower than discounted expected revenues. It must then be

$$pA \geqq \frac{1}{1+r} pB \,. \tag{4}$$

The maximising behaviour of capitalists implies, besides, that, if a relation (4) is a strict inequality, i.e. if current costs in a given activity are greater than discounted expected revenue, then that activity is not operated. That is,

$$\text{if } \sum_{i=1}^{n} a_{ij} p_i > \frac{1}{1+r} \sum_{i=1}^{n} b_{ij} p_i \text{ for some } j, \text{ then } y_j = 0 \,. \tag{4'}$$

The variables to be determined are: m absolute activity levels, $n - 1$ relative prices and the rate of interest. In all, $m + n$ variables, as in the original von Neumann model. There is, however, a difference. In fact, in the von Neumann model, the growth factor and $m - 1$ *relative* activity levels figure; now, on the contrary, such a factor does not figure at all, whilst one finds m ***absolute*** activity levels.

It is possible to give a slightly different interpretation of the accumulation mechanism from that given above. In fact, one has a clearer idea of such a mechanism if, as Walras does, we keep the functions carried out by proprietors of stocks, i.e. capitalists properly called, distinct from those carried out by entrepreneurs, who simply organise production.

At the beginning of the period, capitalists sell to entrepreneurs the goods that they possess and buy the goods that will emerge from the productive process a period later. The reason why capitalists require *at time t* the goods that will be available at time $t + 1$ is explained from the fact that they will be enabled to obtain revenues by selling them again to entrepreneurs.

Since these sales will be effected in the following period, it is necessary to introduce an assumption concerning the prices that will obtain at time $t + 1$. As

(3) This general case of price expectations is considered by the author, in the contest of the Leontief model, in [7].

we have already said, the assumption implicit in the von Neumann model, as in that of Walras, is that expected prices are equal to current prices. Such an assumption very naturally finds a place in a model based on the von Neumann technology, since, at bottom, it means nothing else but that the relative price of a good must be the same either that it be considered as an input (at time t) or as an output (at time $t + 1$).

Since goods will be delivered by entrepreneurs to capitalists at time $t + 1$, the price at time t of good i ($i = 1, 2, \dots, n$) available at time $t + 1$ will be $\bar{p}_i = (1/1 + r)p_i$, where r is the rate of interest and p_i is the price at which capitalists expect to sell good i at time $t + 1$ and which is equal, because of the assumption made, to the price at time t of good i available at the same time t.

By this latter interpretation, entrepreneurs obtain revenues at the same time at which they sustain costs. The vector $(1/1 + r)pB$ does now not represent discounted expected revenues but current revenues. The behaviour of entrepreneurs, who seek to maximize current profits, i.e. the difference between current revenues and current costs, and perfect competition among them will bring it about that they "*ne font ni bénéfice ni perte*", i.e. that prices shall satisfy relations (4) with the additional condition (4′) imposed on activity levels.

4. EXISTENCE OF ECONOMICALLY MEANINGFUL SOLUTIONS

As with the original von Neumann model, also with the model represented by relations (3) and (4) and by the additional conditions (3′) and (4′), the existence of economically meaningful solutions "*cannot be proved by any qualitative argument*" [5, p. 1]. To do this, it is necessary to have recourse to mathematical tools analogous to those used by von Neumann. In fact, we will use, in addition to linear-programming properties, the fixed-point theorem of Kakutani which represents a generalisation of the Brouwer theorem.

Let us begin by indicating with the row vector P the second side of (4), that is

$$P = \frac{1}{1 + r} pB . \tag{5}$$

Having done this, it is now possible to eliminate the rate of interest as an independent variable. In fact, since only relative prices are relevant, it is possible to normalise prices, by imposing, without any loss of generality, the condition that $\sum_{j=1}^{m} P_j = 1$. Then, by adding all relations (5) together, one immediately has

$$1 + r = \sum_{i=1}^{n} p_i \sum_{j=1}^{m} b_{ij} . \tag{6}$$

By substituting (5) in (4) and (6) in (5) and by rewriting (3), we have

$$
\begin{aligned}
Ay &\leqq S && \text{a)} \\
pA &\geqq P && \text{b)} \qquad (7) \\
P &= \left(1\Big/\sum_{i=1}^{n} p_i \sum_{j=1}^{m} b_{ij}\right) pB\,. && \text{c)}
\end{aligned}
$$

Our problem now consists in showing that all relations (7), and in addition (3′) and (4′), may be simultaneously satisfied by at least one set of non-negative values of the variables.

We at once notice that (7.a) and (7.b) appear as the constraints of the dual linear-programming problems:

maximize Py subject to (7.a) and $y \geqq 0$

and

minimize pS subject to (7.b) and $p \geqq 0$.

Given a non-negative vector P such that $\sum_{j=1}^{m} P_j = 1$, the hypotheses concerning matrix A and initial endowments of various goods allow us to state that there exist feasible solutions of the two dual problems. Therefore, for a fundamental theorem of linear programming, there exist optimal solutions. These solutions simultaneously satisfy (7.a) and (7.b) and the additional conditions (3′) and (4′). However, in general, when one inserts the p (or the p's) resolving the minimum problem in (7.c), one obtains a P which is different from the initial one.

In order to solve the problem completely, one must demonstrate that there exists at least one P such that at least one of the corresponding optimal p's yields, once inserted in (7.c), the same P.

Let us now provide proof of this. Given a P in the set $V = \{P | \sum_{j=1}^{m} P_j = 1, P \geqq 0\}$, we solve the dual linear-programming problems, which, as we already know, possess optimal solutions. The two sets of optimal y's and p's are non-empty, closed, bounded(4) and convex. Moreover, the set of optimal p's, does not contain the vector $p = 0$, since this vector, as is easily shown, is not feasible. With every p we associate, by means of (7.c), a P and obtain a closed convex(5) set F of P's which is, by construction, in V. Thus we have defined a mapping $F(P)$ from V into itself, which turns out to be upper-semicontinuous. Since V is a non-empty, compact, convex set, all conditions of Kakutani's theorem are sastisfied.

(4) The set of optimal p's is bounded since S is stictly positive.

(5) The set F is convex for the following reasons. It may be imagined as obtained from the convex set D of the optimal solutions p by means of two successive mappings. The first one is the *linear* mapping $z = pB$, which makes the convex set E correspond to the convex set D [1, theorem 5, p. 152]. The second one consists in reproportioning every vector z of set E in such a way that it finds itself in V, that is to say, that the sum of its elements is equal to one: $P = (1/\sum_{j=1}^{m} z_j)z$. For evident geometrical reasons also the set F, obtained from set E through the latter mapping, is convex.

Then, there exists at least one fixed-point, i.e. a point P^* such that $P^* \in F(P^*)$. Such vector and the corresponding vectors y^* and p^*, which solve respectively the maximum and the minimum problems, satisfy all relations (7) and the additional conditions (3′) and (4′).

The vectors y^* and p^* have at least one element different from zero. This means that at least one activity is operated and that at least one good has positive price, namely it is a scarce good.

From a well known property of linear programming we also have: $P^*y^* = p^*S$. By taking account of (5), it immediately follows

$$1 + r^* = \frac{p^*By^*}{p^*S}\,. \tag{8}$$

In other words, the interest factor is equal to the ratio between the (expected) value of the outputs which will appear at time $t + 1$ and the value (which is always positive) of the inputs available at time t.

5. POSITIVITY OF THE RATE OF INTEREST AND VIABILITY OF TECHNOLOGY

The interest factor, on the basis of (6), is given by $1 + r^* = \sum_{i=1}^{n} p^* \sum_{j=1}^{m} b_{ij}$. Assumption (*.2) implies that for every i the sum $\sum_{j=1}^{m} b_{ij}$ is positive. Hence, by taking into account that at least one element of p^* is positive, we must conclude that the interest factor is positive.

However, we are unable to do the same with the rate of interest r^*. As regards the latter, we can only say that it must be greater than -1. In other words, without further restrictions, it is not possible to exclude a negative r^*. But, one may ask, a negative value of the rate of interest is compatible, from the *economic* point of view, with the hypotheses concerning the way of working of the accumulation mechanism? In fact, since capitalists are guided by the search for maximum profits, is not the accumulation mechanism impeded, if it is to be expected that outputs value at time $t + 1$ is lower than inputs value at time t?

The only case in which it seems possible to state that it is certainly convenient to pursue production, even with prospective loss, is the one in which all n goods are perishable. Then, in fact, if during a certain period productive activity were to be held up, all endowments of goods would perish and, therefore, in the successive period there would be no good. The only way of maintaining goods through time would thus be that of introducing them periodically into production process. And the criterion of maximum profits would be substituted by that of minimum losses.

When, in addition to perishable goods, there exist also non-perishable goods, it could, on the other hand, be convenient not to operate any activity. Indeed, in the event of all goods being non-perishable and, therefore, its being possible to conserve intact through time, this would certainly be convenient(6).

Such a difficulty essentially depends on the fact that it is in no way assured that the productive system is technologically in a position to develop. Assumption (*) only ensures that starting from positive input quantities, it is possible to obtain positive output quantities for all goods. It does not in any way exclude the quantities produced of various goods being lower than those used. Actually, the original assumption of von Neumann ($a_{ij} + b_{ij} > 0$) does not exclude even the case in which $a_{ij} > 0$ and $b_{ij} = 0$ for all i and j: that is, the case in which technology is completely unproductive.

This, of course, is a kind of generality irrelevant from the economic point of view. In fact, only those technologies as are able to supply a positive surplus of all goods, are worthy of consideration. If we confine ourselves to the consideration of the latter kind of technologies, it is then possible to prove that the rate of interest is positive.

Generalising a definition of Gale [3, p. 296], we call a technology (A, B) *productive*, if there exists (at least) one vector $\bar{y}$ of activity levels such that for all goods outputs are *greater* than inputs, that is such that

$$B\bar{y} > A\bar{y} \, . \tag{9}$$

In other words, if an economic system possesses a productive technology, it is technologically *viable* and can, therefore, develop.

We will now show that the equilibrium rate of interest is positive in our one-period model, if technology is productive. By multiplying both member of (4) on the right by y and by taking into account that $p^*B\bar{y}$, because of (9), is positive, we immediately obtain

$$\frac{1}{1 + r^*} \leqq \frac{p^*A\bar{y}}{p^*B\bar{y}} \, . \tag{10}$$

The second side of (10) is no other than the ratio between the inputs and outputs values at equilibrium prices in the case in which activities are operated at the levels $\bar{y}$. From the fact that these outputs are, by definition, greater then the corresponding inputs, it follows that the denominator is higher than the numerator and that, therefore, the above ratio is lower than unit. It immediately results that $1/1 + r^*$, being lower than or equal to this ratio, is lower than the unit and that, therefore, r^* is positive.

6. CONCLUSION

We have shown how the elements of the accumulation mechanism implicit in the von Neumann model are sufficient to determine economically meaningful

(6) We notice that if the technology (A, B) takes explicitly account of the possibility of maintaining all n goods unchanged through time, then it is easily shown that the interest factor cannot be lower than one and, therefore, the rate of interest cannot be negative.

equilibria even when taking into account the *historically given* structure of initial endowments of goods. The succession of one-period equilibria, starting from the given initial conditions (endowments of goods at time zero), describes the path of the system through time. This path, which may be not unique, depends on initial conditions and is in general different from the proportional growth state.

Von Neumann's original version now appears as the model which considers the proportional growth state corresponding to the one-period model examined by us.

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