DYNAMIC COMPETITIVE EQUILIBRIUM OF A NEOCLASSICAL SYSTEM WITH MONEY AND THE WELFARE IMPLICATION OF INFLATIONARY FINANCE

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1. INTRODUCTION

Most of works which have treated dynamic aggregative models with money or the welfare cost of inflation have been postulating some *ad hoc* properties of the demand functions for money holdings,(1) though a few recent works have based the analysis on more primitive hypotheses.(2) Moreover, most works on the welfare cost of inflation do not pose the problem in a fully dynamic setting. However, if we are to investigate and fully understand the welfare implications of inflation, these approaches are not satisfactory. Therefore, we shall construct a dynamic aggregative model in which the aggregate demands for consumption and money holdings are determined by the utility maximization behavior of each household. The first few sections will be devoted to presenting the definition and existence proof of a dynamic competitive equilibrium. Then, we shall examine its Pareto-optimality property in a relative sense. Finally, we shall investigate whether or not inflationary finance actually causes welfare losses.

2. HOUSEHOLD'S BEHAVIOR

In this section, we shall derive the demand functions (precisely, functionals) of a household for consumption and money holdings from a hypothesis of its utility maximization behavior. It will be assumed that each household is permitted to hold its assets either in the form of money whose holdings yield certain benefits (or utility) or in the form of capital stock which yields some physical returns. For the sake of simplicity, we shall consider an economy with only one good in terms of which all real quantities are to be evaluated. We shall use the following notations:

$K(t, v)$: real capital stock held by household v at time t,
$C(t, v)$: real consumption of household v at time t,
$M(t, v)$: real money holdings of household v at time t,
$Z(t, v)$: nominal money holdings of household v at time t,
$S(t, v)$: real savings in the form of money of household v at time t,

(1) For example, see Bailey [2], Levhari and Patinkin [5], Mundell [6], Phelps [7], Rose [8], Sidrauski [9], and Tobin [11, 12].

(2) Sidrauski [10] and Uzawa [13]. See also Uzawa [14].

$G(t, v)$: real amount of government transfer to household v at time t,
$r(t)$: real rate of return on capital at time t,
$w(t)$: real rate of wage at time t,
$p(t)$: general level of prices at time t,

where "household v" means a household established at time v. It will be assumed that each household supplies one unit of labor force at every point of time. Then, the budget constraints of household v must be expressed as follows:

$$K_1(t, v) = r(t)K(t, v) + w(t) + G(t, v) - C(t, v) - S(t, v)\,,$$
$$M_1(t, v) = S(t, v) - \hat{p}(t)M(t, v)\,,$$

where $\hat{p}(t) = \dot{p}(t)/p(t)$ and subscript i stands for the partial derivative with respect to the ith variable. $K(t, v)$ need not be nonnegative; that is, any household may borrow capital stock for present consumption or money holdings. The amount of borrowing cannot be arbitrarily large, however, if we are to have a meaningful problem. Thus, following Arrow and Kurz,[(3)] we shall define feasibility condition in the following manner:

DEFINITION: A program $\{(C(t, v), S(t, v)) : t \geqq z\}$ is said to be *feasible* for household v if and only if

(a) $M_1(t, v) = S(t, v) - \hat{p}(t)M(t, v)\,,$

(b) $A_1(t, v) = r(t)A(t, v) - C(t, v) - S(t, v)\,,$

(c) $A(t, v) = K(t, v) + P(t, v)\,,$

(d) $P(t, v) = \int_t^\infty (w(s) + G(s, v)) \exp\left(-\int_t^s r(x)\,dx\right) ds\,,$[(4)]

(e) $A(t, v) > 0\,, \quad M(t, v) \geqq 0\,, \quad C(t, v) \geqq 0\,,$[(5)]

(f) $p(z)M(z, v) = Z(z, v) > 0$, given,

$$K(z, v) = \begin{cases} \text{a nonnegative constant if } v \leqq 0\,, \\ 0 \quad \text{if} \quad v > 0\,, \end{cases}$$

where $z = \max(0, v)$.

The problem which household v has to solve is that of finding a feasible program which maximizes its criterion functional

$$\int_z^\infty U(C(s, v), M(s, v))e^{-\delta(s-z)}\,ds\,,$$

(3) Arrow and Kurz [1].

(4) $P(t, v)$ is the present value of the infinite stream from time t of the noninterest income of household v.

(5) The condition $A(t, v) > 0$ means that the maximum admissible value of borrowing is less than the present value of noninterest income stream.

where $z = \max(0, v)$, δ is a positive constant, and U is a strictly concave function with continuous first and second derivatives such that

$$U_C(0, M) = \infty\,, \qquad U_M(C, 0) = \infty\,.^{(6)}$$

If we define the Hamiltonian $H = U(C, M) + q_1(S - \dot{p}M) + q_2(rA - C - S)$, then the necessary conditions for optimality can be written as

$$U_C = q_2\,, \qquad q_1 = q_2\,, \qquad \dot{q}_1 = (\delta + \dot{p})q_1 - U_M\,, \qquad \dot{q}_2 = (\delta - r)q_2\,.$$

At this point, we introduce a simplification by assuming that

$$U(C, M) = \alpha \log C + \beta \log M\,,$$

where α and β are positive constants. When this is done, the necessary conditions above reduce to

$$C(t, v) = C(z, v) \exp \int_z^t (r(x) - \delta)\, dx\,, \quad z = \max(0, v)\,,$$

and

$$M(t, v) = \frac{\beta}{\alpha} \frac{C(t, v)}{r(t) + \dot{p}(t)}\,.$$

These, together with the initial level of money holdings, determine the demand functions of household v for consumption and money holdings, and therefore, by the budget constraint, for material assets (real capital).

3. COMPETITIVE EQUILIBRIUM: DEFINITION

Utilizing the results in the previous section, we can now proceed to defining the dynamic competitive equilibrium of a neoclassical economic model. It will be assumed that the number of households established at time t increases at a constant rate n. Then, without loss of generality, we may write the total number of households at time t as

$$L(t) = \int_{-\infty}^t e^{nv}\, dv = (1/n)e^{nt}\,,$$

which can be regarded as the total supply of labor force at time t. Furthermore, quantities per household can be defined as follows:

$$k(t) = ne^{-nt}\int_{-\infty}^t K(t, v)e^{nv}\, dv\,, \qquad c(t) = ne^{-nt}\int_{-\infty}^t C(t, v)e^{nv}\, dv\,,$$

$$m(t) = ne^{-nt}\int_{-\infty}^t M(t, v)e^{nv}\, dv\,, \qquad z(t) = ne^{-nt}\int_{-\infty}^t Z(t, v)e^{nv}\, dv\,,$$

$$s(t) = ne^{-nt}\int_{-\infty}^t S(t, v)e^{nv}\, dv\,, \qquad g(t) = ne^{-nt}\int_{-\infty}^t G(t, v)e^{nv}\, dv\,,$$

$$h(t) = p(t)g(t)\,, \qquad a(t) = nM(t, t)\,, \qquad b(t) = nZ(t, t)\,.$$

(6) Throughout the present paper, it is assumed that the economic units have perfect foresight.

For the sake of simplicity, we shall assume that all households have the same preference. Then clearly

$$(1)\qquad c(t)=\int_0^t \frac{\alpha}{\beta}(r(v)+\hat{p}(t))a(v)\exp\int_v^t(r(x)-n-\delta)\,dxdv + \frac{\alpha}{\beta}(r(0)+\hat{p}(0))m(0)\exp\int_0^t(r(x)-n-\delta)\,dx\,,$$

$$(2)\qquad m(t)=\frac{\beta}{\alpha}\frac{c(t)}{r(t)+\hat{p}(t)}\,.$$

From the budget constraints,

$$(3)\qquad k(t)=\int_0^t\{w(v)+g(v)-c(v)-s(v)\}\exp\int_u^t(r(x)-n)\,dxdv + k(0)\exp\int_0^t(r(x)-n)\,dx\,,$$

$$(4)\qquad m(t)=\int_0^t s(v)\exp\Big(-\int_v^t(\hat{p}(x)+n)\,dx\Big)\,dv + m(0)\exp\Big(-\int_0^t(\hat{p}(x)+n)\,dx\Big) + \int_0^t a(v)\exp\Big(-\int_v^t(\hat{p}(x)+n)\,dx\Big)\,dv\,.$$

Now we turn to the production side of the economy. Suppose that a representative firm acts competitively with the neoclassical aggregate production function $f(k)$ such that $f(0)=0$, $f'(k)>0$, and $f''(k)<0$ for all $k>0$. Since the firm is assumed to maximize profits, the demands for capital and labor are determined so that

$$(5)\qquad r(t)=f'(k(t))\,,$$

$$(6)\qquad w(t)=f(k(t))-k(t)f'(k(t))\,.$$

Finally, according to the definition of the general level of prices,

$$(7)\qquad z(t)=p(t)m(t)\,,\qquad h(t)=p(t)g(t)\,,\qquad b(t)=p(t)a(t)\,,$$

and, from the historical data,

$$(8)\qquad k(0)=k_0\,,\qquad z(0)=z_0\,.$$

We assume that the government can control only the nominal values of the total money supply $z(t)$, general transfers $h(t)$, and transfers to new households $b(t)$. We are ready to set forth our definition of competitive equilibrium.

DEFINITION: The economy is said to be in a *competitive equilbrium relative to the government's policy* $\{(b(t), h(t), z(t)) : t \geqq 0\}$ if and only if equations (1) through (8) hold for all $t \geqq 0$.

It will be useful to rewrite the conditions for competitive equilibrium in the form of a system of differential equations. We can easily show that the economy is in a competitive equilibrium relative to the government's policy $\{(b(t), h(t), z(t)) : t \geqq 0\}$ if and only if

$$\dot{k} = f(k) - nk - c - (\hat{z} + n)m + g + a \,, \tag{9}$$

$$\dot{c} = \{f'(k) - n - \delta + a/m\}c \,, \tag{10}$$

$$\dot{m} = \{\hat{z} + f'(k) - \beta c/\alpha m\}m \,, \tag{11}$$

$$\dot{g} = \{\hat{h} + f'(k) - \beta c/\alpha m\}g \,, \tag{12}$$

$$\dot{a} = \{\hat{b} + f'(k) - \beta c/\alpha m\}a \,, \tag{13}$$

$$k(0) = k_0 \,, \qquad z(0) = z_0 \,, \tag{14}$$

where $\hat{x} = \dot{x}/x$.

4. BALANCED GROWTH EQUILIBRIUM: EXISTENCE AND UNIQUENESS

Before proceeding to characterizing general competitive equilibrium, we examine the possibility of a balanced growth equilibrium relative to a steady policy $(b(t) = b_0 e^{\hat{b}t}, h(t) = h_0 e^{\hat{h}t}, z(t) = z_0 e^{\hat{z}t})$. Stationary solution to the system (9) through (14) must be such that

$$\begin{aligned} f(k^*) - nk^* &= c^* + (\hat{z} + n)m^* - g^* - a^* \,, \\ f'(k^*) &= n + \delta - a^*/m^* \,, \\ f'(k^*) &= \beta c^*/\alpha m^* - \hat{z} \,, \\ f'(k^*) &= \beta c^*/\alpha m^* - \hat{h} \,, \\ f'(k^*) &= \beta c^*/\alpha m^* - \hat{b} \,. \end{aligned}$$

Thus, a balanced growth equilibrium is possible only if $\hat{b} = \hat{h} = \hat{z}$. Hence, by (2), $\hat{p} = \hat{z}$. Therefore, we have

$$\begin{aligned} f'(k^*) + \hat{z} &= (\beta/\alpha z_0)p(0)c^* \,, \\ f'(k^*) &= n + \delta - b_0/z_0 \,, \\ f(k^*) - nk^* &= c^* + \{(\hat{z} + n)z_0 - h_0 - b_0\}/p(0) \,. \end{aligned}$$

(b_0, z_0) is assumed to be given so that $n + \delta - b_0/z_0 > 0$. Then $f'(k^*) > 0$. Further, assume that $f'(\bar{k}) < n + \delta - b_0/z_0 < f'(0)$ where $0 < \bar{k} \leqq \infty$ and $f(\bar{k}) = n\bar{k}$. Then $0 < k^* < \bar{k}$, and hence $f(k^*) - nk^* > 0$. If we assume that the government cannot sell commodities to the private sector, then $(\hat{z} + n)z_0 \geqq b_0 + h_0$.[7] Thus, if $\hat{z} > b_0/z_0 - (n + \delta)$, then $p(0)$ is determined at a unique positive value; and so is c^*. Therefore, we can state the following theorem.

(7) The quantity $\{(\hat{z} + n)z_0 - b_0 - h_0\}/z_0$ can be regarded as the ratio of government expenditure to the stock of real money.

THEOREM 1: *Let* $\hat{b} = \hat{h} = \hat{z} > b_0/z_0 - (n + \delta)$, $(\hat{z} + n)z_0 \geqq b_0 + h_0$ *and* $f'(\bar{k}) < n + \delta - b_0/z_0 < f'(0)$,[8] *where* $0 < \bar{k} \leqq \infty$ *and* $f(\bar{k}) = n\bar{k}$. *Then there exists a unique balanced growth equilibrium* $(k^*, c^*, m^*, g^*, a^*, p^*)$.

5. COMPETITIVE EQUILIBRIUM: ARBITRARY INITIAL CONDITION

In this section, we shall investigate the properties of a general competitive equilibrium relative to a steady policy ($b(t) = b_0e^{\mu t}$, $h(t) = h_0e^{\mu t}$, $z(t) = z_0{}^{\mu t}$) such that $(\mu + n)z_0 > b_0 + h_0$ and $\mu > b_0/z_0 - (n + \delta)$. The first inequality means that there are some government purchases. The conditions of competitive equilibrium can be written as follows:

$$\dot{k} = f(k) - nk - c - \frac{1}{z_0}\{(\mu + n)z_0 - b_0 - h_0\}m \,, \tag{15}$$

$$\dot{c} = \{f'(k) - n - \delta + b_0/z_0\}c \,, \tag{16}$$

$$\dot{m} = \{\mu + f'(k) - \beta c/\alpha m\}m \,, \tag{17}$$

$$k(0) = k_0 \,. \tag{18}$$

Now, define

$$\phi_1(k, m) = f(k) - nk - \frac{1}{z_0}\{(\mu + n)z_0 - b_0 - h_0\}m \,,$$

$$\phi_2(k, m) = \frac{\alpha}{\beta}(\mu + f'(k))m \,,$$

$$D_1 = \{(k, m, c) : c \leqq \phi_1(k, m),\ k \leqq k^*,\ c \leqq \phi_2(k, m)\} \,,$$

$$D_2 = \{(k, m, c) : c \geqq \phi_1(k, m),\ k \geqq k^*,\ c \geqq \phi_2(k, m)\} \,.$$

If (k, m, c) belongs to the boundary of D_1, then at least one of the following equalities hold: $c = \phi_1(k, m)$, $k = k^*$, or $c = \phi_2(k, m)$. Let there be t' such that $c = \phi_1$, $k \leqq k^*$, $c \leqq \phi_2$, and at least one of these two inequalities holds strictly. Then, at this t', $\dot{c} \geqq 0$, $\dot{\phi}_1 \leqq 0$, and at least one of these is a strict inequality. Hence, if $t > t'$ and $t - t'$ is small enough, then $(k(t), m(t), c(t)) \notin D_1$; and if $t < t'$ and $t' - t$ is small enough, then $(k(t), m(t), c(t)) \in D_1$. Similar reasoning shows that any phase point which passes through the boundary point (except for the balanced growth path) of D_1 must be in the interior of D_1 until it reaches the boundary point, and must be in the exterior of D_1 after it has arrived at the boundary point. Similarly, we can prove the same fact for D_2. Thus, if there is a solution to the system (15) through (18) which converges to the balanced growth equilibrium and passes through the interior of either D_1 or D_2, then it must be wholly contained in the set.

(8) The assumption $f'(\bar{k}) < n + \delta - b_0/z_0 < f'(0)$ requires $n + \delta - b_0/z_0 > 0$; the latter is satisfied if the transfer to new households is sufficiently small.

Next, consider the Taylor expansion of the system (15) through (17) in the neighborhood of the balanced growth equilibrium:

$$\text{(19)} \qquad \begin{bmatrix} \dot{k} \\ \dot{m} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} f'(k^*) - n & -q & -1 \\ m^*f''(k^*) & \dfrac{c^*}{m^*} & -\dfrac{\beta}{\alpha} \\ c^*f''(k^*) & 0 & 0 \end{bmatrix} \begin{bmatrix} k - k^* \\ m - m^* \\ c - c^* \end{bmatrix} + R(k, m, c) ,$$

where $q = \{(\mu + n)z_0 - b_0 - h_0\}/z_0$. Let x be a characteristic root of the coefficient matrix on the right-hand side. Then we get

$$x^3 + a_1x^2 + a_2x + a_3 = 0 ,$$

where

$$a_1 = n - f'(k^*) - \beta c^*/\alpha m^* ,$$
$$a_2 = (f'(k^*) - n)\beta c^*/\alpha m^* + (qm^* + c^*)f''(k^*) ,$$
$$a_3 = -\frac{\beta c^*}{\alpha m^*} f''(k^*)(qm^* + c^*) .$$

Since a_3 is positive, the characteristic equation has a negative real root. Let x_1, x_2, and x_3 be the roots of the equation. Then

$$x_1 + x_2 + x_3 = -a_1 = f'(k^*) - n + \beta c^*/\alpha m^* = 2(n + \delta - b_0/z_0) + \mu - n .$$

Let x_1 be the negative real root whose existence is guaranteed by the positivity of a_3. If the other two roots are real, either both are positive or both are negative. But, if a_1 is assumed to be negative, the latter case is impossible. If they are not real, they must be conjugate complex numbers. Let $x_2 = y + iv$, where i is the imaginary number. Then $x_3 = y - iv$, so that $x_2 + x_3 = 2y$. Hence, if $a_1 < 0$, then we have $y > 0$ again. Therefore, in any case, the real parts of x_2 and x_3 are positive, provided $a_1 < 0$.

Let us assume that $a_1 < 0$ and that the function f is continuously four-times differentiable. The latter implies that the Jacobian matrix of $R(k, m, c)$ exists and converges to zero matrix as (k, m, c) tends to (k^*, m^*, c^*). Hence, we can conclude that for any large t_0 there exists in (k, m, c)-space a real one-dimensional manifold S containing the point (c^*, k^*, m^*) such that any solution $(k(t), m(t), c(t))$ of the system (19) with $(k(t_0), m(t_0), c(t_0))$ on the manifold S satisfies $(k(t), m(t), c(t)) \rightarrow (k^*, m^*, c^*)$ as $t \rightarrow \infty$, and that there exists a K such that any solution $(k(t), m(t), c(t))$ near (k^*, m^*, c^*) but not on S at $t = t_0$ cannot satisfy $||(k(t), m(t), c(t))|| \leq K$ for $t \geqq t_0$.[(9)] Since S is one-dimensional, it contains a unique solution. The solution coincides with the solution to the original system (15) through (17) in the neighborhood of (k^*, m^*, c^*). It can be continued to any positive k and is wholly contained in D_1 and D_2, and

(9) See Theorem 4.1 of Chapter 13 in Coddington and Levinson [4].

hence (18) is satisfied. Furthermore, by the definition of the sets, the solution converges monotonically to (k^*, m^*, c^*). Thus, we can state

THEOREM 2: *Suppose that* $b(t) = b_0 e^{\mu t}$, $h(t) = h_0 e^{\mu t}$, $z(t) = z_0 e^{\mu t}$, $(\mu + n)z_0 > b_0 + h_0$, $\mu > b_0/z_0 - (n + \delta)$, $2(n + \delta - b_0/z_0) + \mu - n > 0$,[10] *and* $f'(\bar{k}) < n + \delta - b_0/z_0 < f'(0)$, *where* $0 < \bar{k} \leqq \infty$ *and* $f(\bar{k}) = n\bar{k}$. *Furthermore, suppose that f is continuously four-times differentiable. Then there exists a unique competitive equilibrium which converges monotonically to the balanced growth equilibrium.*

6. COMPETITIVE EQUILIBRIUM: EFFICIENCY AND PARETO OPTIMALITY

In this section, we wish to examine the normative properties of competitive equilibrium with the aid of the technique devised by Cass and Yaari.[11] It should be noted, however, that we do not treat the *proper* efficiency or Pareto optimality which is defined for technological constraints. We shall examine an efficiency property or Pareto optimality relative to the government's policy as well as technological constraints. Let us introduce some difinitions.

DEFINITION: A policy $\{(b(t), h(t), z(t)) : t \geqq 0\}$ is called a *uniform policy* if and only if $b(0) = b_0$, $h(0) = h_0$, $z(0) = z_0$, and $\hat{b}(t) = \hat{h}(t) = \hat{z}(t) = \mu(t)$, where $\hat{x}(t) = \dot{x}(t)/x(t)$.

DEFINITION: A path $\{(k(t), m(t), c(t)) : t \geqq 0\}$ is said to be *feasible relative to* a uniform policy if and only if $(k(t), m(t), c(t)) \geqq 0$, and $\dot{k}(t) = f(k(t)) - nk(t) - c(t) - (1/z_0)\{(\mu(t) + n)z_0 - b_0 - h_0\}m(t)$, where $c(t)$ is piecewise continuous and $m(t)$ is continuous.

DEFINITION: A path $\{(\bar{k}(t), \bar{m}(t), \bar{c}(t)) : t \geqq 0\}$ feasible relative to a uniform policy is said to be *efficient relative to* the policy if and only if there does not exist another path $\{(k(t), m(t), c(t)) : t \geqq 0\}$ feasible relative to the policy such that $c(t) \geqq \bar{c}(t)$, $m(t) \geqq \bar{m}(t)$ for all $t \geqq 0$ and either $c(t) > \bar{c}(t)$ or $m(t) > \bar{m}(t)$ for some $t > 0$.

DEFINITION: A set of individual programs $\{(C(t, v), M(t, v)) : v \leqq t\}$ is called a *distribution* of $\{(c(t), m(t)) : t \geqq 0\}$ if and only if

$$\int_{-\infty}^{t} C(t, v)e^{nv}\,dv = (1/n)e^{nt}c(t)\,, \qquad C(t, v) \geqq 0\,, \qquad \text{for all} \quad v \leqq t\,,$$

$$\int_{-\infty}^{t} M(t, v)e^{nv}\,dv = (1/n)e^{nt}m(t)\,, \qquad M(t, v) \geqq 0\,, \qquad \text{for all} \quad v \leqq t\,.$$

DEFINITION: A path $\{(\bar{k}(t), \bar{m}(t), \bar{c}(t)) : t \geqq 0\}$ which is feasible relative to

(10) It can be seen that the assumption $2(n + \delta - b_0/z_0) + \mu - n > 0$ is implied by $\delta > n$ and the other assumptions of the theorem; in fact, $2(n + \delta - b_0/z_0) + \mu - n = \mu + n + 2\delta - 2b_0/z_0 > 2\delta + (h_0 - b_0)/z_0 > h_0/z_0 + \delta - n$, where we assume $h_0 \geqq 0$.

(11) Cass and Yaari [3].

a uniform policy and has a distribution $\{(\bar{C}(t, v), \bar{M}(t, v)) : v \leqq t\}$ is said to be *Pareto optimal relative to* the policy if and only if there does not exist another path $\{(k(t), m(t), c(t)) : t \geqq 0\}$ which is feasible relative to the policy and has a distribution $\{(C(t, v), M(t, v)) : v \leqq t\}$ such that

$$\int_z^s \{U(C(t, v), M(t, v)) - U(\bar{C}(t, v), \bar{M}(t, v))\}e^{-\delta(t-z)}\,dt \geqq 0$$

for all s and v such that $s \geqq \max(0, v)$ and

$$\lim_{s\to\infty} \int_z^s \{U(C(t, v), M(t, v)) - U(\bar{C}(t, v), \bar{M}(t, v))\}e^{-\delta(t-z)}\,dt > 0$$

for some v, where $z = \max(0, v)$.

Utilizing the proof by Cass and Yaari(12) with slight modification, we can prove the following two lemmas.

LEMMA 1: *If a competiteve equilibrium* $\{(\bar{k}(t), \bar{m}(t), \bar{c}(t), \bar{C}(t, v), \bar{M}(t, v), \bar{r}(t), \bar{p}(t)) : t \geqq 0, v \leqq t\}$ *relative to a uniform policy is not Pareto optimal relative to the policy, then there is another path* $\{(k(t), m(t), c(t)) : t \geqq 0\}$ *feasible relative to the policy such that*

$$\int_0^\infty \{(c(t) - \bar{c}(t)) + (\bar{r}(t) + \bar{p}(t))(m(t) - \bar{m}(t))\} \times \exp\left(-\int_0^t (\bar{r}(x) - n)\,dx\right) dt > 0\,.$$

LEMMA 2: *Let* $\{(\bar{k}(t), \bar{m}(t), \bar{c}(t), \bar{r}(t), \bar{p}(t)) : t \geqq 0\}$ *be a competitive equilibrium relative to a uniform policy such that* $\bar{r}(t) + \bar{p}(t) = \{(\mu(t) + n)z_0 - b_0 - h_0\}/z_0$. *Then, it is inefficient relative to the policy if there is another path* $\{(k(t), m(t), c(t)) : t \geqq 0\}$ *feasible relative to the policy such that*

$$\liminf_{T\to\infty} \int_0^T \{(c(t) - \bar{c}(t)) + (\bar{r}(t) + \bar{p}(t))(m(t) - \bar{m}(t))\} \times \exp\left(-\int_0^t (\bar{r}(x) - n)\,dx\right) dt > 0\,.$$

These lemmas immediately imply

THEOREM 3: *If* $\{(k(t), m(t), c(t), r(t), p(t)) : t \geqq 0\}$ *is a competitive equilibrium relative to a uniform policy such that* $r(t) + \hat{p}(t) = \{(\mu(t) + n)z_0 - b_0 - h_0\}/z_0$ *and is not Pareto optimal relative to the policy, then it is inefficient relative to the policy.*

Let us notice that the converse of Lemma 2 evidently holds. Then it can be easily seen that the proof of Theorem 2 in Cass and Yaari establishes

THEOREM 4: *Let* $\{(k(t), m(t), c(t), r(t), p(t)) : t \geqq 0\}$ *be a competitive equilibrium*

(12) Cass and Yaari [3], pp. 249-251, 264-265.

relative to a uniform policy such that $r(t) + \dot{p}(t) = \{(\mu(t) + n)z_0 - b_0 - h_0\}/z_0$. *Then, it is efficient relative to the policy if there is a finite number K such that*

$$\liminf_{t\to\infty} \exp\left(-\int_0^t (r(x) - n)\,dx\right) < K\,.^{(13)}$$

Now, consider the competitive equilibrium relative to the policy in Theorem 4. It is described by

$$\dot{k} = f(k) - nk - \frac{\alpha + \beta}{\alpha} c\,, \tag{20}$$

$$\dot{c} = \{f'(k) - n - \delta + b_0/z_0\}c\,, \tag{21}$$

$$\dot{m} = \{f'(k) - n + (b_0 + h_0)/z_0\}m\,. \tag{22}$$

$$k(0) = k_0\,. \tag{23}$$

It is easy to show that this system has a unique *quasi-balanced growth equilibrium* for which capital per household and consumption per household remain constant. Furthermore, if a tax policy such as $h_0 = -\delta z_0$ is adopted, then the system has a balanced growth equilibrium which is not unique except for capital per household and consumption per household. Formally,

THEOREM 5: *Let* $f'(\bar{k}) < n + \delta - b_0/z_0 < f'(0)$, *where* $0 < \bar{k} \leqq \infty$ *and* $f(\bar{k}) = n\bar{k}$. *Then there exists a unique quasi-balanced growth equilibrium relative to a uniform policy such that* $r(t) + \dot{p}(t) = \{(\mu(t) + n)z_0 - b_0 - h_0\}/z_0$. *Furthermore, if* $h_0 = -\delta z_0$, *then there exists a balanced growth equilibrium relative to the policy.*

We can also easily prove

THEOREM 6: *Let* $f'(\bar{k}) < n + \delta - b_0/z_0 < f'(0)$, *where* $0 < \bar{k} \leqq \infty$ *and* $f(\bar{k}) = n\bar{k}$. *Then there exists a competitive equilibrium relative to a uniform policy such that* $r(t) + \dot{p}(t) = \{(\mu(t) + n)z_0 - b_0 - h_0\}/z_0$. *The equilibrium is unique as concerns* $c(t)$ *and* $k(t)$, *and converges monotonically to the unique quasi-balanced growth equilibrium.*

In view of Theorems 3, 4, and 6, we can state

THEOREM 7: *Let* $f'(\bar{k}) < n + \delta - b_0/z_0 < f'(0)$, *where* $0 < \bar{k} \leqq \infty$ *and* $f(\bar{k}) = n\bar{k}$. *Then the competitive equilibrium relative to a uniform policy such that* $r(t) + \dot{p}(t) = \{(\mu(t) + n)z_0 - b_0 - h_0\}/z_0$ *is Pareto optimal relative to the policy, provided one of the following two conditions holds*:

(i) $\delta > b_0/z_0$, (ii) $\delta \geqq b_0/z_0$ *and* $f'(k_0) > n + \delta - b_0/z_0$.[14]

(13) Cass and Yaari [3], pp. 265-267.

(14) Since $r(t) + \dot{p}(t)$ can be regarded as the money rate of interest, the condition $r(t) + \dot{p}(t) = \{(\mu(t) + n)z_0 - b_0 - h_0\}/z_0$ means that the government expenditure is equal to the money interest of the stock of real money.

7. THE WELFARE IMPLICATIONS OF INFLATIONARY FINANCE

In order to examine the welfare implications of inflationary finance, however, we cannot use the notion of Pareto optimality *relative to* a given policy. It is rather preferable to use the notion of Pareto optimality in the usual sense: that defined relative to only the technological constraint. But, in the present section, we simplify the analysis by supposing a welfare function with individualistic basis. Let the welfare function be defined by

$$W(\{c(t)\}_0^\infty, \{m(t)\}_0^\infty)$$
$$= \max\left\{\lim_{T\to\infty}\int_{-\infty}^{T}\int_{v}^{T} U(C(t, v), M(t, v))e^{-\delta(t-v)}\,dt\,e^{(n-\delta)v}\,dv :\right.$$
$$\left.\int_{-\infty}^{t} C(t, v)e^{nv}\,dv = (1/n)e^{nt}c(t),\ \int_{-\infty}^{t} M(t, v)e^{nv}\,dv = (1/n)e^{nt}m(t)\right\},$$

where $z = \max(0, v)$. Then, by the assumption $U(C, M) = \alpha\log C + \beta\log M$, it follows that

$$W(\{c(t)\}_0^\infty, \{m(t)\}_0^\infty) = \frac{1}{n}\int_0^\infty\{\alpha\log c(t) + \beta\log m(t)\}e^{-(\delta-n)t}\,dt\,.$$

Thus, our problem can be formulated so as to find a policy which maximizes this welfare function subject to the conditions of competitive equilibrium. Here we consider the case where the government can choose only among uniform policies. The problem is:

$$\text{Maximize}\quad \int_0^\infty\{\alpha\log c(t) + \beta\log m(t)\}e^{-(\delta-n)t}\,dt \quad (\delta > n)$$

subject to

$$\dot{k}(t) = f(k(t)) - nk(t) - c(t) - \frac{1}{z_0}\{(\mu(t) + n)z_0 - b_0 - h_0\}m(t)\,,$$
$$\dot{c}(t) = \{f'(k(t)) - n - \delta + b_0/z_0\}c(t)\,,$$
$$\dot{m}(t) = \left\{\mu(t) + f'(k(t)) - \frac{\beta c(t)}{\alpha m(t)}\right\}m(t)\,,$$
$$k(t) \geqq 0\,,\quad c(t) \geqq 0\,,\quad m(t) \geqq 0\,,\quad (\mu(t) + n)z_0 \geqq b_0 + h_0\,,$$
$$k(0) = k_0\,.$$

Define the Hamiltonian

$$H = \alpha\log c + \beta\log m + q_1\left\{f(k) - nk - c - \frac{1}{z_0}((\mu + n)z_0 - b_0 - h_0)m\right\}$$
$$+ q_2\{f'(k) - n - \delta + b_0/z_0\}c + q_3\{\mu + f'(k) - \beta c/\alpha m\}m\,.$$

Since $\mu(t)$ maximizes H at every point of time, it must be that

$$q_3 \leqq q_1 \qquad \text{with equality if} \quad (\mu + n)z_0 > b_0 + h_0\,.$$

Furthermore, the auxiliary variables satisfy the differential equations

$$\dot{q}_1 = (\delta - f'(k))q_1 - cf''(k)q_2 - mf''(k)q_3 ,$$
$$\dot{q}_2 = q_1 + (2\delta - f'(k) - b_0/z_0)q_2 + (\beta/\alpha)q_3 - \alpha/c ,$$
$$\dot{q}_3 = \frac{1}{z_0}\{(\mu + n)z_0 - b_0 - h_0\}q_1 + (\delta - n - \mu - f'(k))q_3 - \beta/m .$$

Let us consider a stationary solution to the problem. Suppose that $(\mu^* + n)z_0 > b_0 + h_0$. Then, $q_1^* = q_3^*$, so that

$$\left\{f(k^*) - nk^* + \frac{\delta - n}{f''(k^*)}\frac{b_0 - nz_0}{z_0}\right\}\mu^*$$
$$= (f(k^*) - nk^*)\left\{\frac{b_0}{z_0} - \frac{\alpha}{\alpha + \beta}\frac{h_0 + \delta z_0}{z_0} - 2n\right\}$$
$$+ \frac{nz_0 - b_0}{z_0}\frac{\delta - n}{f''(k^*)}\left\{\frac{nz_0 - b_0}{z_0} + \frac{\alpha\beta}{\alpha + \beta} - \frac{\beta}{\alpha + \beta}\frac{h_0}{z_0}\right\} .$$

If $b_0 \leqq nz_0$, then the substitution of $(b_0 + h_0)/z_0 - n$ for μ^* in the equation above yields

$$\left\{\frac{2\alpha + \beta}{\alpha}\frac{h_0}{z_0} + \frac{\alpha + \beta}{\alpha}n + \delta\right\}(f(k^*) - nk^*) < \frac{\delta - n}{f''(k^*)}\frac{nz_0 - b_0}{z_0}\left(\delta + \frac{h_0}{z_0}\right),$$

which is a contradiction. Thus, if $b_0 \leqq nz_0$, then $(\mu^* + n)z_0 = b_0 + h_0$. This establishes

THEOREM 8: *Let $f'(\bar{k}) < n + \delta - b_0/z_0 < f'(0)$, where $0 < \bar{k} \leqq \infty$ and $f(\bar{k}) = n\bar{k}$. If $b_0 \leqq nz_0$, then the optimal uniform rate of change in the money supply corresponding to the optimal steady-state path is given by $\mu^* = (b_0 + h_0)/z_0 - n$.*[15]

8. CONCLUSIONS

In the present paper, competitive equilibrium is defined for an economic model where (i) the aggregate demands for consumption, physical assets, and money are determined by the behavior of each household which maximizes a discounted sum of instantaneous utility of consumption and money holdings, (ii) the aggregate demand for physical assets (which is the supply of capital) is identically equal to the demand for capital by firms, and (iii) the aggregate production is carried out by the representative firm which acts competitively with the neoclassical aggregate production function. Moreover, the equilibrium is defined as one relative to the government's policy of money supply.

Given mild assumptions on the property of the production function, the economy has a unique competitive equilibrium converging monotonically to a unique balanced growth equilibrium, provided the policy is steady, uniform,

(15) The assumption $b_0 \leqq nz_0$ means that the transfer to a new household is not greater than the stock of nominal money per household.

and such that the rate of change in the money supply is sufficiently large. If the government expenditure is identically equated to the money interest of the stock of real money, then the competitive equilibrium is efficient if and only if it is Pareto optimal; and furthermore, if the subjective rate of discount is sufficiently high, then the competitive equilibrium is really Pareto optimal relative to the given policy.

Suppose that the government can choose only among uniform policies. If the level of transfer to a new household does not exceed the stock of nominal money per household, then the optimal policy is to keep the level of government expenditure at zero, provided the optimal path is a steady-state one. Since the government expenditure does not yield any benefit to the private sector in the form of public goods in our model, this is a reasonable conclusion. In this case, if the general transfer takes a positive value, it is not necessarily impossible that the optimal rate of change in the money supply is positive. If so, the optimal rate of inflation is positive on the steady state. This implies that inflation does not always cause welfare costs.

APPENDIX

PROOF OF LEMMA 1: Since the competitive equilibrium is not Pareto optimal, there is a path $\{(c(t), m(t)) : t \geqq 0\}$ feasible relative to the policy with a distribution $\{(C(t, v), M(t, v)) : v \leqq t\}$ such that

$$\begin{aligned} 0 < \lim_{T\to\infty} \int_{-\infty}^{T} \int_{z}^{T} &\{U(C(t, v), M(t, v)) \\ &- U(\bar{C}(t, v), \bar{M}(t, v))\} e^{-\delta(t-z)}\, dt\, \bar{C}(z, v) e^{nv} \exp\Big(-\int_0^z \bar{r}(x)\, dx\Big) dv \\ = \int_0^\infty \int_{-\infty}^{t} &\{\alpha \log C(t, v) + \beta \log M(t, v) - \alpha \log \bar{C}(t, v) - \beta \log \bar{M}(t, v)\} \\ &\times \bar{C}(t, v) e^{-n(t-v)}\, dv \left\{\exp\Big(-\int_0^t (\bar{r}(x) - n)\, dx\Big)\right\} dt\,, \end{aligned}$$

where $z = \max(0, v)$.

Let $\hat{C}(t, v)$ be the solution of the problem to maximize

$$\int_{-\infty}^{t} (\log C(t, v)) \bar{C}(t, v) e^{-n(t-v)}\, dv$$

subject to

$$\int_{-\infty}^{t} C(t, v) e^{nv}\, dv = (1/n) e^{nt} c(t)\,, \qquad C(t, v) \geqq 0\,.$$

Then,

$$\frac{c(t)}{\hat{C}(t, v)} = \frac{\bar{c}(t)}{\bar{C}(t, v)}\,.$$

Let

$$\hat{M}(t, v) = \frac{m(t)}{\bar{c}(t)} \bar{C}(t, v) .$$

Then clearly,

$$0 < \int_0^\infty \int_{-\infty}^t \{\alpha \log \hat{C}(t, v) + \beta \log \hat{M}(t, v) - \alpha \log \bar{C}(t, v) - \beta \log \bar{M}(t, v)\}$$
$$\times \bar{C}(t, v) e^{-n(t-v)} dv \left\{\exp\left(-\int_0^t (\bar{r}(x) - n)\, dx\right)\right\} dt$$
$$= \frac{1}{n} \int_0^\infty \{\alpha(\log c(t) - \log \bar{c}(t)) + \beta(\log m(t) - \log \bar{m}(t))\} \bar{c}(t)$$
$$\times \exp\left(-\int_0^t (\bar{r}(x) - n)\, dx\right) dt$$
$$\leqq \frac{\alpha}{n} \int_0^\infty \{(c(t) - \bar{c}(t)) + (\bar{r}(t) + \bar{p}(t))(m(t) - \bar{m}(t))\}$$
$$\times \exp\left(-\int_0^t (\bar{r}(x) - n)\, dx\right) dt .$$

PROOF OF LEMMA 2: By hypothesis, there is a $\bar{T}$ such that, for all $T > \bar{T}$,

$$\int_0^T \{(c(t) - \bar{c}(t)) + (\bar{r}(t) + \bar{p}(t))(m(t) - \bar{m}(t))\} \exp\left(-\int_0^t (\bar{r}(x) - n)\, dx\right) dt > 0 .$$

The left-hand side is equal to

$$(\bar{k}(T) - k(T)) \exp\left(-\int_0^T (\bar{r}(x) - n)\, dx\right)$$
$$- \int_0^T \{f(\bar{k}(t)) - f(k(t)) - (\bar{k}(t) - k(t)) f'(\bar{k}(t))\} \exp\left(-\int_0^t (\bar{r}(x) - n)\, dx\right) dt ,$$

since $\bar{r}(t) + \bar{p}(t) = \{(\mu(t) + n)z_0 - b_0 - h_0\}/z_0$. Hence, by the continuity of $k(t)$ and the strict concavity of f, we have

$$\bar{k}(T) - k(T) > v(T) > 0 \qquad \text{for all} \quad T > \bar{T} ,$$

where

$$v(T) = \int_0^T \{f(\bar{k}(t)) - f(k(t)) - (\bar{k}(t) - k(t)) f'(\bar{k}(t))\} \exp \int_t^T (\bar{r}(x) - n)\, dx dt .$$

Let

$$\dot{y}(T) = f(y(T)) - n(y(T) + \bar{m}(T)) - (\mu(T) - (b_0 + h_0)/z_0)\bar{m}(T)$$
$$- (\bar{c}(T) + c)$$

for $\bar{T} \leqq T \leqq T'$, where

$$y(\bar{T}) = k(\bar{T}) ,$$
$$T' = \min\{T : y(T) = \bar{k}(T) - v(T)\} ,$$

c is an arbitrary positive number.

Let

$$k^*(T) = \bar{k}(T),\ c^*(T) = \bar{c}(T),\ m^*(T) = \bar{m}(T) \qquad \text{for} \quad 0 < T < \bar{T},$$

$$k^*(T) = y(T),\ c^*(T) = \bar{c}(T) + c,\ m^* = \bar{m}(T) \qquad \text{for} \quad \bar{T} \leqq T \leqq T',$$

$$\begin{aligned} k^*(T) = \bar{k}(T) - v(T),\ c^*(T) &= f(k^*(T)) - n(k^*(T) + m^*(T)) \\ &- (\mu(T) - (b_0 + h_0)/z_0)m^*(T) - \dot{k}^*(T),\ m^*(T) = \bar{m}(T) \\ &\text{for} \quad T \geqq T'. \end{aligned}$$

Then, since

$$\dot{v}(T) = f(\bar{k}(T)) - f(k(T)) - (\bar{k}(T) - k(T))f'(\bar{k}(T)) + (\bar{r}(T) - n)v(T),$$

it follows that, for $T \geqq T'$,

$$\begin{aligned} c^*(T) &= f(k^*(T)) - f(k(T)) - (k^*(T) - k(T))f'(\bar{k}(T)) + \bar{c}(T) \\ &> f(k^*(T)) - f(k(T)) - (k^*(T) - k(T))f'(k^*(T)) + \bar{c}(T) \\ &> \bar{c}(T). \end{aligned}$$

Hence, the path $\{(k^*(t), m^*(t), c^*(t)) : t \geqq 0\}$ is feasible and $m^*(t) = \bar{m}(t)$, $c^*(t) \geqq \bar{c}(t)$ for all $t \geqq 0$ with $c^*(t) > \bar{c}(t)$ for all $t > \bar{T}$. Thus, the path $\{(\bar{k}(t), \bar{m}(t), \bar{c}(t)) : t \geqq 0\}$ is inefficient relative to the policy.

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